OPERATORS WITH SINGULAR CONTINUOUS SPECTRUM, VII. EXAMPLES WITH BORDERLINE TIME DECAY

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Abstract. We construct one-dimensional potentials $V(x)$ so that if $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R})$, then H has purely singular spectrum; but for a dense set D, $\varphi \in D$ implies that $|(\varphi, e^{-itH}\varphi)| \leq C_\varphi |t|^{-1/2} \ln(|t|)$ for $|t| > 2$. This implies the spectral measures have Hausdorff dimension one and also, following an idea of Malozemov-Molchanov, provides counterexamples to the direct extension of the theorem of Simon-Spencer on one-dimensional infinity high barriers.

§1. Introduction

This is a continuation of my series of papers (some joint) exploring singular continuous spectrum especially in suitable Schrödinger operators and Jacobi matrices [3,15,4,8,2,19,17,
5,7,16]. Our main goal here is to construct potentials $V(x)$ on $\mathbb{R}$ so that if $H = -\frac{d^2}{dx^2} + V(x)$, then $\sigma(H) = [0,\infty)$, $\sigma_{\rm ac}(H) = \sigma_{\rm pp}(H) = \emptyset$, and there is a dense set $D \subset L^2(\mathbb{R})$ so that if $\varphi \in D$, then

$$|(\varphi, e^{itH}\varphi)| \leq C_\varphi t^{-1/2} \ln(|t|) \tag{1.1}$$

for $|t| > 2$. (We say $|t| > 2$ because of the behavior of $\ln(|t|)$ for $|t| \leq 1$; note all matrix elements are bounded by 1, so control in $|t| \leq 2$ is trivial.)

(1.1) is interesting because the stated bound on $F_\varphi(t) \equiv (\varphi, e^{-itH}\varphi)$ is just at the borderline for operators with singular continuous spectrum. Indeed, if $t^{-1/2}$ in (1.1) were replaced by $t^{-\alpha}$ for any $\alpha > \frac{1}{2}$, then $F_\varphi(t)$ would be in L^2 and so the spectral measures $d\mu_\varphi(E) = F(E)\, dE$ for $F \in L^2$; that is, $d\mu_\varphi$ would be a.c. and so $\sigma_{\rm ac}(H) \neq \emptyset$.

As an indication of the borderline nature of (1.1), we note that by Falconer [6], (1.1) implies $d\mu_\varphi$ is a measure carried on set of Hausdorff dimension 1 in the sense that it gives zero weight to any set of Hausdorff dimension strictly less than 1.

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The potentials V are sparse potentials in the sense that they are mainly zero. They are examples of the type already studied in [19]. We will have examples where $V \to 0$ at infinity but also examples where $\overline{\lim}_{x\to\pm\infty} V(x) = \infty$. The latter are of some interest because of an idea of Malozemov-Molchanov [13], which was the starting point of our work here.

This idea is related to results of Simon-Spencer [18]. To describe it, we need some notions. Call a barrier a compact subset $B \subset \mathbb{R}^n$ so that $\mathbb{R}^n \backslash B$ has exactly one bounded component and so that $\mathbb{R}^n \backslash B$ has two components if $n \geq 2$ and three if $n = 1$. If B_1 and B_2 are barriers, we say B_2 surrounds B_1 if B_1 is contained in the bounded component of $\mathbb{R}^n \backslash B_2$.

By the width of a barrier B, we mean the distance between the bounded component of $\mathbb{R}^n \backslash B$ and the unbounded component (in case $n = 1$, the union of the two unbounded components). By the diameter of B, we mean $\max\{|x-y| \mid x, y \in B\}$.

We say a potential V on $\mathbb{R}^n$ has a sequence of high barriers if

(1) V is globally bounded from below and locally bounded.
(ii) There is a sequence $B_1, B_2, \ldots$ of barriers so B_{k+1} surrounds B_k.
(iii) The width of each barrier is at least 1.
(iv) There exists $a_k \to \infty$ so $V(x) \geq a_k$ if $x \in B_k$.

Then Simon-Spencer proved:

Theorem 1.1 [18]. *If $n = 1$, $H = -\frac{d^2}{dx^2} + V(x)$, and V has a sequence of high barriers, then $\sigma_{\mathrm{ac}}(H) = \emptyset$.*

Malozemov-Molchanov [13] have studied extensions of this result to higher dimensions, which require some relations between the size of a_k and diameter of B_k. It is clearly expected that the result does not extend without restriction to $n \geq 2$ but it is unclear how to make counterexamples. Malozemov-Molchanov noted that there exist purely singular measures $d\nu$ on $\mathbb{R}$ so that the convolution $d\nu * d\nu$ is absolutely continuous. Moreover, if V_1 is a potential on $\mathbb{R}$ with such a spectral measure $d\nu$ and

$$V(x,y) = V_1(x) + V_1(y)$$

is a potential V on $\mathbb{R}^2$, then $-\Delta + V$ has $d\nu * d\nu$ as spectral measure (specifically, if $\varphi(x)$ has spectral measure $d\nu$, then $\tilde{\varphi}(x,y) = \varphi(x)\varphi(y)$ has spectral measure $d\nu * d\nu$). Finally, if V_1 has a sequence of high barriers, so does V.

Our examples in obeying (1.1) will let us implement this strategy and so prove:

Theorem 1.2. *If $n \geq 2$, there exist potentials V with a sequence of high barriers so that $-\Delta + V$ has purely absolutely continuous spectrum. If $n \geq 3$, there are such V's for which the spectrum is purely transient.*

We'll discuss transient and recurrent spectrum further below. It was in thinking of how to implement this Malozemov-Molchanov strategy that I was led to think of time decay and (1.1).

The potentials V which implement (1.1) will be chosen even, so we may as well consider half-line problems with Dirichlet or Neumann boundary conditions at $x = 0$. The half-line potentials will have the form

$$V(x) = \sum_{n=1}^{\infty} V_n(x - C_n) \tag{1.2}$$

where V_n is a potential of compact support and the C_n's are sufficiently large. In principle, our constructions let us determine how large the C_n's must be, but since the main point of this construction is existence, we won't completely track the restrictions on C_n.

Section 4 is the technical core of the paper where we prove a critical lemma about half-line potentials $V(x)$ of the form

$$V_L(x) = V_\infty(x) + W(x - L) \tag{1.3}$$

with V_∞, W bounded non-negative of compact support. We obtain some uniform in L bounds on the time decay of $|(\varphi, e^{-itH}\varphi)|$. This lemma is used in Section 2 to make the construction of V obeying (1.1). The application to Theorem 1.2 is found in Section 3. Finally, Section 5 contains some remarks about how big the C_n's in (1.2) need to be.

While Section 4 is somewhat technical, it is technicality with an elegant physical interpretation and technology we expect will be useful in other contexts.

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§2. The Construction Modulo the Main Technical Lemma

In this section, we'll construct potentials V on $\mathbb{R}$ so that $-\frac{d^2}{dx^2} + V(x)$ has purely singular continuous spectrum, but (1.1) holds for a dense set of φ's. The construction will depend on a lemma only proven in Section 4.

Our V's will obey

$$V(-x) = V(x),$$

so $-\frac{d^2}{dx^2} + V(x)$ is a direct sum of two operators, unitarily equivalent to the half line with Neumann and Dirichlet boundary conditions. We'll prove the result for the Neumann boundary condition case. The argument for the Dirichlet boundary condition case is similar: One replaces the Neumann m-function $m_N(E)$ by $m_D(E) = -m_N(E)^{-1}$ and the "vector" $\delta(x)$ by $\delta'(x)$ ($\delta(x)$ lies in $\mathcal{H}_{-1}$ for the Neumann case but $\delta'(x)$ is only in $\mathcal{H}_{-2}$ (e.g., [9] but this doesn't change the analysis in any essential way).

Suppose V is bounded below and let H be the Neumann b.c. operator $-\frac{d^2}{dx^2} + V(x)$ on $L^2(0,\infty)$. Let $\mathcal{H}_s$ be the usual scale of spaces associated to H [14] (so, e.g., $\mathcal{H}_{+1}$ is the form domain of H). Then $\delta(x)$, the delta function at 0, lies in $\mathcal{H}_{-1}$; so, in particular, $f(H)\delta \in L^2$ for any function $f \in C_0^\infty(\mathbb{R})$.

The technical lemma we will prove in Section 4 is

Theorem 2.1. *Suppose V_L has the form (1.3) with V_∞, W fixed bounded non-negative functions of compact support. Let $f, g \in C_0^\infty(0)$ with support in $(0,\infty)$. Then*

(i) $\lim_{L\to\infty} (f(H_L)\delta, e^{-itH_L} g(H_L)\delta) = (f(H)\delta, e^{-itH} g(H)\delta)$ *uniformly for t in compact subsets of $(-\infty,\infty)$.*

(ii) *There exist C independent of L and t so that*

$$|(f(H_L)\delta, e^{-itH_L} g(H_L)\delta)| \le Ct^{-1/2} \tag{2.1.}$$

Remark. This is in essence a diffusion bound. For each fixed L, eventually $(f(H_L)\delta, e^{-itH_L}g(H_L)\delta)$ decays faster than any power of t. However, suppose $f = g$ is supported very near energy $E = k^2$. Then at time $t = \pm L/k$ we should expect a bump in $(f(H_L)\delta, e^{-itH_L}g(H_L)\delta)$ due to return of a reflected wave (the distance traveled there and back is $2L$ but since the free energy is p^2, not $\frac{1}{2}p^2$, the velocity is near $2k$). Because of diffusion, this reflected bump will decay but only as $t^{-1/2}$ for this particular t. Similarly, there will be multiple reflection bumps at times $t = \pm nL/k$. Our proof in Section 4 will essentially invoke a rigorous multi-reflection expansion.

A sequence V_n non-negative potentials of compact support will be called trapping if

$$-\frac{d^2}{dx^2} + \sum_{n=1}^{\infty}[V_n(x - L_n) + V_n(-x - L_n)) \tag{2.2}$$

has no a.c. spectrum if the L_n's are sufficiently large. Trapping potentials are constructed in Simon-Spencer [18] and Last-Simon [11]. They are of three types:

1) High barriers: What we have called sequence of high barriers where $V_n(x) \geq a_n$ on $(-\frac{1}{2}, \frac{1}{2})$ and $a_n \to \infty$.
2) Long random barriers: If $V_n(x)$ is the sample of a random potential on the interval $\left(\frac{n(n-1)}{2}, \frac{n(n+1)}{2}\right)$, there is no a.c. spectrum so long as L_n is large enough.
3) Very long decaying barriers: If V_n is the sample of $|x|^{-\alpha}W(x)$ (with $W(x)$ random and $\alpha < \frac{1}{2}$) on (a_{n-1}, a_n) and a_n is large enough, then for L_n large, there is no a.c. spectrum.

Potentials of type 1,2 are discussed in [18]. [11] has a method that handles all these cases. In all cases, the L_n's need only be so large that the support of $V_n(x - L_n)$ is to the right of the support of $V_{n-1}(x - L_{n-1})$. Our main theorem in this paper is

Theorem 2.2. *Let $\{V_n\}$ be a sequence of trapping potentials and let V be defined by* (2.2)*. Then the L_n's can be chosen so that*

(i) *$H = -\frac{d^2}{dx^2} + V(x)$ has purely singular continous spectrum.*
(ii) *For a dense set $D \subset L^2(\mathbb{R})$, and all $\varphi, \psi \in D$,*

$$|(\varphi, e^{-itH}\psi)| \leq C_{\varphi,\psi} \ln(|t|)/|t|^{1/2}$$

for $|t| \geq 2$.

Proof. Without loss, we'll restrict to the half-line Neumann problem as explained. We'll make the argument for $\varphi = f(H)\delta$ for a single f and then explain the modifications needed to get a dense set of φ.

Theorem 2.1 implies that

$$\lim_{L\to\infty} \sup_{|t|>2} \left[(\ln|t|)^{-1}|t|^{1/2}|(f(H_L)\delta, e^{-itH_L}f(H_L)\delta) - (f(H_\infty)\delta, e^{-itH_\infty}f(H_\infty)\delta)|\right] = 0.$$

Thus in adding in V_n, we can choose L_n so the change in

$$\sup_{|t|>2} \left[(\ln|t|)^{-1}|t|^{1/2}(f(H^{(n)})\delta, e^{-itH^{(n)}}f(H^{(n)})\delta)\right] \tag{2.3}$$

from the same quantity for $n-1$ is at most $\frac{1}{2^n}$. Here

$$H^{(n)} = -\frac{d^2}{dx^2} + \sum_{m=1}^{n} V_m(x - L_m)$$

(on $(L^2(0,\infty))$.

Since $H^{(n)} \to H$ in strong resolvent sense, we have the result for H by taking $n \to \infty$ and noting $\sum\limits_{n=1}^{\infty} 2^{-n} < \infty$. To get a dense set of vectors, choose f_k, C^∞ functions on $(0,\infty)$ so the f_k's are dense in $\|\cdot\|_\infty$ norm in the continuous functions on $[0,\infty)$ vanishing at zero and infinity. Then $\{f_k(H)\delta\}$ is a dense set in $L^2(0,\infty)$. At step n, arrange for the change in (2.3) to be no more than 2^{-n} for $f = f_k$ with $k = 1, \dots, n$. Then each of

$$|(f_k(H)\delta, e^{-itH} f_k(H)\delta)| \le C_k |t|^{-1/2} \ln(|t|)$$

for $|t| > 2$. $\square$

Note. $\ln(|t|)$ plays no special role in the proof or statement of the theorem. It could be replaced by any function $\ell(|t|)$ so long as $\lim\limits_{\alpha\to\infty} \ell(\alpha) = \infty$.

Corollary 2.3. *For any potential V of the form given in Theorem 2.2, H has singular continuous spectrum of Hausdorff dimension 1 in the sense that its spectral measures E_Δ have $E_S = 0$ if S is a Borel set of Hausdorff dimension $\alpha < 1$.*

Proof. Follows from Falconer [6], pg. 67.

Corollary 2.4. *For any potential V of the form of Theorem 2.2, we have*

$$\lim_{|t|\to\infty} \frac{1}{t^{2-\epsilon}} \|xe^{-itH}\delta_0\|^2 = \infty$$

for any $\epsilon > 0$.

Proof. Follows from the results of Last [10].

§3. High Barriers in Dimension Two or More

In this section, we carry through the strategy of Malozemov-Molchanov described in the introduction.

For this section, we'll fix once and for all a function V_1 on $\mathbb{R}$ so that

(i) $V_1(-x) = V_1(x)$
(ii) There is $a_n \to \infty$ so $V_1(x) \ge n$ on $[a_n, a_n + 1]$.
(iii) $\sigma(H_1) = [0,\infty)$ and is purely singular continuous where $H_1 = -\frac{d^2}{dx^2} + V_1(x)$.
(iv) For a dense set $D_1 \subset L^2(\mathbb{R})$, $|(\varphi, e^{-itH_1}\psi)| \le C_{\varphi,\psi}|t|^{-1/2}\ln(|t|)$ for $|t| \ge 2$ and any $\varphi, \psi \in D_1$.

On $\mathbb{R}^n$ define

$$V_n(x_1, x_2, \dots, x_n) = V_1(x_1) + V_1(x_2) + \cdots + V_1(x_n)$$

and on $L^2(\mathbb{R}^n)$,

$$H_n = -\Delta + V_n.$$

Theorem 3.1. (a) *If $n \geq 2$, there is a dense set D_n in $L^2(\mathbb{R}^n)$ so that for $\varphi, \psi \in D_n$, $(\varphi, e^{-itH_n}\psi) \in L^p$ for all $p > 1$.*

(b) *If $n \geq 3$, there is a dense set D_n in $L^2(\mathbb{R}^n)$ so that for $\varphi, \psi \in D_n$, $(\varphi, e^{-itH}\psi) \in L^1 \cap L^\infty$.*

Proof. If $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_n$, $\psi = \psi_1 \otimes \cdots \otimes \psi_n$ with $\varphi_i, \psi_i \in D_1$, then $(\varphi, e^{-itH_n}\psi) = \prod_{j=1}^{n} (\varphi_j, e^{-itH_1}\psi_j)$ so for $|t| \geq 2$,

$$|(\varphi, e^{-itH_n}\psi)| \leq C_{\varphi,\psi}|t|^{-n/2}(\ln|t|)^n.$$

Since it is also bounded, we have the L^p results. Linear combinations of those φ's are dense. $\square$

Corollary 3.2. *If $n \geq 2$, H_n has purely a.c. spectrum.*

Proof. If $d\mu$ is a measure and $F_\mu(t) \equiv \int e^{-iEt} d\mu(E)$ is in L^p with $p < 2$, then by the Hausdorff-Young inequality, $d\mu(E) = g(E)\, d(E)$ with $g \in L^q$ $(q = p/p-1)$. $\square$

In [1], Avron-Simon introduced the notion of transient and recurrent a.c. spectrum. $\varphi \in \mathcal{H}_{\rm ac}(A)$ is transient if it is a limit of φ_n's where each $(\varphi_n, e^{-itH}\varphi_n)$ decays faster than any inverse polynomial in t. If $\mathcal{H}_{\rm tac}$ is the set of such φ's, then $\mathcal{H}_{\rm rac} = \mathcal{H}_{\rm ac} \cap \mathcal{H}_{\rm tac}^\perp$ is called the set of recurrent a.c. vectors. It is proven in [1] that if $F(t) = (\varphi, e^{-itH}\varphi)$ lies in L^1, then φ is in $\mathcal{H}_{\rm tac}$. Thus,

Corollary 3.3. *If $n \geq 3$, H_n has purely transient a.c. spectrum.*

Thus, if $n = 2$, it is possible that there is a weakened form of the result of Simon-Spencer [18], that is,

Open Question. Are there examples of $n = 2$ with a sequence of barriers with transient a.c. spectrum or is any a.c. spectrum in such cases of necessity recurrent?

§4. The Main Technical Lemma

Our goal in this section is to prove Theorem 2.1. Since $-\frac{d^2}{dx^2} + V_L$ converges to $-\frac{d^2}{dx^2} + V_\infty$ in strong resolvent sense, and δ is in the common form domain, (i) is elementary but also follows from the discussion below.

Our analysis depends on the Weyl-Titchmarsh theory of spectral measures for the Neumann problem (see [12]); explicitly, we'll use the form:

Proposition 4.1. *Suppose V is bounded and non-negative with compact support in $[0,\infty)$ and $H = -\frac{d^2}{dx^2} + V(x)$ with $u'(0) = 0$ boundary conditions. For any $E > 0$, let $k = \sqrt{E}$ and let $u_+(x,E)$ be the solution of $-u'' + Vu = Eu$ which is equal to e^{ikx} for x large. Define $m(E) = -u_+(0,E)/u_+'(0,E)$, the Neumann m-function. Then for f, a smooth function of compact support,*

$$(\delta, f(H)\delta) = \frac{1}{\pi}\int f(E)[\operatorname{Im} m(E)]\, dE. \tag{4.1}$$

Because of (4.1), we'll need to estimate integrals of the form:

Lemma 4.2. *Let g be a C^∞ function of compact support on $(0,\infty)$, and let*

$$Q(y,t) = \int_0^\infty e^{iky-ik^2t} g(k^2)\, d(k^2).$$

Then

$$|Q(y,t)| \le Ct^{-1/2}\left[\int_0^\infty \{|g(k^2)|^2 + k^2|g'(k^2)|^2\} k^2\, dk\right]^{1/2}.$$

Proof. Let H_0 be the operator $-\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$. Let $h(y)$ be the function $Q(y,0)$. Then, using the explicit integral kernel of H_0:

$$\begin{aligned} Q(y,t) &= (e^{-itH_0}h)(y) \\ &= (4\pi t)^{-1/2}\int e^{i|x-y|^2/4t} h(y)\, dy \end{aligned}$$

so

$$\begin{aligned} |Q(y,t)| &\le (4\pi t)^{-1/2}\int |h(y)|\, dy \\ &\le (4\pi t)^{-1/2}\left(\int |h(y)|^2(1+y^2)\, dy\right)^{1/2}\left[\int (1+y^2)^{-1}\, dy\right]^{1/2} \end{aligned}$$

by the Schwartz inequality, so by the Plancherel theorem,

$$|Q(y,t)| \le (2t)^{-1/2}\left[\int |f(k)|^2 + |f'(k)|^2\, dk\right]^{1/2}$$

where $f(k) = 2kg(k^2)$ and we are done. $\square$

For the remainder of this section, we'll fix V_∞ and W and always take L so large that $W_L(\,\cdot\,) \equiv W(\cdot - L)$ has its support to the right of the support of V_∞. Thus, there are $a < b < c$ so $\operatorname{supp}(V_\infty) \subset [0,a)$, $\operatorname{supp}[W_L] \subset (b,c)$. In the regions (a,b) and (c,∞), any solution of $-u'' + V_L u = E_L$ is a linear combination of $e^{\pm ikx}$ where $k \equiv \sqrt{E}$. For u_+, we have it equal to e^{ikx} on (c,∞) and it will be $\frac{1}{t_L}e^{ikx} + \frac{r_L}{t_L}e^{-ikx}$ on (a,b). Hence, by general principles, $|t_L|^2 + |r_L|^2 = 1$ and $t_L \neq 0$. Since m only involves a ratio, we can instead look at $\tilde{u}_+ = t_L e^{ikx}$ on (c,∞) and $= e^{ikx} + r_L e^{-ikx}$ on (a,b).

Given any complex number r, we can solve on $[0,a]$ for the function $= e^{ikx} + re^{-ikx}$ near a and then let

$$M(r;E)$$

be the value of $-u(0)/u'(0)$ for the corresponding solution. As indicated, $M(r;E)$ depends on the value of the reflection coefficient r (r is distinct from x, of course; beware of the possible confusion) and energy E. It is also a function of V_∞ but not of W. If we choose $r = r_L(E)$ which *is* dependent on W (and L and E), then, of course, $m_L(E) \equiv M(r_L(E), E)$ is the m for V_L. And, of course, $m_\infty(E) \equiv M(r=0,E)$ is the m for V_∞.

Theorem 4.3. *For each fixed $E > 0$, $M(r, E)$ is anlytic in the complex disc $\{r \mid |r| < 1\}$. Similarly, $\frac{\partial M}{\partial E}$ is analytic there too. Moreover, both functions are uniformly bounded as E run through compact subsets of $(0, \infty)$ and r through compact subsets of $\{r \mid |r| < 1\}$. In particular, for any $R_0 < 1$ and compact $K \subset (0, \infty)$, there is a C with*

$$M(r, E) = \sum_{n=0}^{\infty} a_n(E) r^n$$

$$|a_n(E)| \leq C R_0^{-n} \tag{4.2a}$$

$$\left| \frac{da_n}{dE} \right| \leq C R_0^{-n} \tag{4.2b}$$

if $E \in K$.

Remark. The proof actually shows more, as we'll note in the next section; namely, $|a_n(E)|$
$\leq C$, $\left|\frac{da_n}{dE}\right| \leq C(n+1)$.

Proof. Let $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be the transfer matrix from a to zero, that is,

$$\begin{pmatrix} w(0) \\ w'(0) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} w(a) \\ w'(a) \end{pmatrix} \qquad \text{for solutions of } -w'' + (V - E)w = 0.$$

Then $M(E, r)$ is the fractional linear transformation

$$M(r) = -\frac{\alpha(\omega + r\omega^{-1}) + ik\beta(\omega - r\omega^{-1})}{\gamma(\omega + r\omega^{-1}) + ik\delta(\omega - r\omega^{-1})}$$

where $\omega = e^{ika}$. The denominator vanishes exactly if $r_0 = \omega^2(-\gamma + ik\delta)/(\gamma + ik\delta)$. Notice that since γ, δ are real, $|r_0| = 1$, so as claimed, M is analytic in $|r| < 1$. The uniform bounds on M follow by noting that $\alpha, \beta, \gamma, \delta$ are uniformly bounded. Similarly, $\frac{\partial M}{\partial E}$ has a second order pole on the unit circle and we get its uniform bounds. □

1. A convenient way to write M is in terms the zeros r_0 and r_1 of the denominator and numerator of M. As in the proof, $|r_0| = |r_1| = 1$ and

$$M(r) = M(0) \left(\frac{\bar{r}_1}{\bar{r}_0} \right) \frac{r - r_1}{r - r_0}$$

so, in fact $|a_n(E)| \leq 2|M(0)|$ (just expand the geometric series and multiply out). Similarly, we can control $\left|\frac{da_n}{dE}\right|$.

2. That the zero of the denominator has $|r_0| = 1$ just happens in the proof. But one can understand it from two factors. First, every r with $|r| < 1$ occurs with some W_L as we run through all possible W's. Thus, since m is finite for any $V_\infty + W_L$ of compact support, M must be analytic in $|r| < 1$. Moreover, $M(\bar{r}^{-1}) = \overline{M(r)}$, so we have analyticity also in $|r| > 1$. □

Proof of Theorem (2.1). Let $r(E)$ be the reflection coefficient on the whole line for $-\frac{d^2}{dx^2} + W(x)$. Then by translation covariance, $r_L(E)$, the reflection coefficient for $W(x-L)$, is

$$r_L(k^2) = e^{2ikL} r(k^2).$$

Thus, in terms of the expansion above equation (4.2):

$$S(L,t) \equiv (f(H_L)\delta, e^{-itH_L} g(H_L)\delta) = \sum_{n=-\infty}^{\infty} S_n(L,t)$$

where

$$S_n(L,t) = \begin{cases} \frac{1}{2i}\int \overline{f(k^2)}\, g(k^2) e^{-ik^2t+2ikLn} a_n(k^2) r(k^2)^n \, d(k^2); & n \geq 1 \\ \int \overline{f(k^2)}\, g(k^2) e^{-ik^2t} \mathrm{Im}\, m_\infty(k^2)\, dk^2; & n = 0 \\ -\frac{1}{2i}\int \overline{f(k^2)}\, g(k^2) e^{-ik^2t-2ikLn} \overline{a_n(k^2)}\, \overline{r(k^2)}^n \, d(k^2); & n \leq 1 \end{cases}$$

where m_∞ is the Neumann m-function for V_∞. Since $\mathrm{supp}(\bar{f}g) \subset (0,\infty)$, we know that on that support $\sup |r(k^2)|$ is some $\alpha < 1$. So in (4.2), take $R_0 > \alpha$ and use Lemma 4.2 to be able to sum up the $t^{-1/2}$ contributions and so obtain the theorem. □

§5. Towards Explicit Estimates of the L_n

Our goal here is to explain why for the $\ln(t)/t^{1/2}$ bound we believe that one needs to take $L_n \sim \exp(\exp(Cn^{3/2}))$ for the case where, say, $V_n = n\chi_{(-1/2,1/2)}$. If we only wanted $t^{-1/2+\epsilon}$ behavior for fixed ϵ, these same considerations would only require $L_n \sim \exp(C_\epsilon n^{3/2})$ (consistent with the behavior needed in [19]).

As noted in the remark after Theorem 4.3, we have $|a_n(E)| \leq 2|M(0)|$, $\left|\frac{da_n}{dE}(E)\right| \leq \left|2M(0)n\frac{dr_0}{dE}\right|$. Thus, if

$$A = \inf(1-|r|)$$

on the support in question, $|M(r)| \leq 2|M(0)|A^{-1}$ and $\left|\frac{dM}{dr}\right| \leq 2QA^{-2}$ with Q bounded by $\left|M(0)\frac{dM}{dE}(0)\right|$. Because of the definition of the transfer matrix, in adding bump n, the transfer matrix for V_∞ is of order $\prod_{j=1}^{n} e^{C\sqrt{j}} \sim \exp(C_1 n^{3/2})$, so M and $\frac{dM}{dE}$ are bounded by $\exp(C_1 n^{3/2})$. On the other hand, $|r|$ for nth bump is of order $1 - e^{-\sqrt{n}}$ by tunneling estimates, so the A^{-1} term in $|M(0)|A^{-1}$ is much smaller than the $|M(0)|$ bound. Thus, the change in $(f, e^{-itH}g)$ is of order

$$\exp(C_1 n^{3/2}) t^{-1/2}$$

and only for t's of order at least $L_n^{1-\delta}$ for any $\delta > 0$. Thus, to get a $\ln(t)/t^{1/2}$ bound, we need only arrange

$$\ln(L_n)^{1/2} \geq n^{+2} \exp(C_1 n^{3/2}),$$

and to get $t^{-1/2+\epsilon}$, we can have

$$L_n^\epsilon \geq n^2 \exp(C_1 n^{3/2}),$$

as claimed at the start of the section.

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