What Is Localization?

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Abstract

We examine various issues relevant to localization in the Anderson model. We show there is more to localization than exponentially localized states by presenting an example with such states but where $\langle x(t)^2 \rangle / t^{2-\delta}$ is unbounded for any $\delta > 0$. We show that the recently discovered instability of localization under rank one perturbations is only a weak instability.

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Localization in random media is basic to a variety of physical situations. We wish to report here on a number of rigorous mathematical results that shed light on the phenomenon of localization in the Anderson model. Mathematically complete proofs of our results will appear elsewhere[(1)]. Our goal here is to describe the ideas behind the results.

Throughout, we'll consider the Anderson model, that is, the Hamiltonian H_ω on $\ell^2(Z^d)$ (namely, on the d-dimensional cubic lattice)

$$(H_\omega u)(n) = \sum_{|j|=1} u(n+j) + V_\omega(n)u(n) \tag{1}$$

where the potentials, V_ω, are identically distributed independent random variables with distribution[(2)]

$$\frac{1}{2\eta}\,\chi_{[-\eta,\eta]}(x)\,dx$$

with $\chi_{[-\eta,\eta]}$ the characteristic function of the interval $[-\eta, \eta]$.

Many of the claimed proofs of localization show that for almost all ω, an Anderson model Hamiltonian H_ω has a complete set of normalized eigenvectors[(3)] $\{\varphi_{\omega,m}\}_{m=1}^{\infty}$ obeying

$$|\varphi_{\omega,m}(n)| \le C_{\omega,m} e^{-A|n-n_{\omega,m}|} \tag{2}$$

where A is fixed, the $n_{\omega,m}$'s are some centers of localization, and the $C_{\omega,m}$'s are constants depending on ω and m.

Our first result is an example that shows that mere "exponential localization" of eigenfunctions in the form (2) need not have very strong consequences for the dynamics. We can construct a non-random potential V in one dimension so that

(i) H has a complete set of normalized eigenvectors obeying Eqn. (2).

(ii) Let $\langle x^2\rangle(t)$ denote $\langle e^{-itH}\delta_0, x^2 e^{-itH}\delta_0\rangle$; then for any $\delta > 0$, $\langle x^2(t)\rangle / t^{2-\delta}$ is unbounded as $t \to \pm\infty$[(4)].

The potential V for this example is

$$V(n) = 3\cos(2\pi\alpha n + \theta) + \lambda\delta_{n0},$$

which we consider on the positive half of the lattice ($n \ge 0$), with a Dirichlet (or any other) boundary condition at the origin. The 3 in front of the cosine can be replaced by any number larger than 2, and is chosen so that when $\lambda = 0$, the problem has a positive Lyapunov exponent[(5)]. The α is an irrational, which is specially chosen so that for suitable time scales $T_n \to \infty$, V is so close to periodic that we can show $\langle x^2(T_n)\rangle$ is large compared to $T_n^{2-\delta}$. The local perturbation $\lambda\delta_{n0}$ pushes the spectrum to be pure point and forces Eqn. (2) to hold.

While V is very far from random, it illustrates that Eqn. (2) is not enough to restrict dynamics. The main failing in (2) is the total freedom given to the constants

$C_{\omega,m}$. Indeed, when one thinks of "localization", one usually thinks of the eigenvectors as being confined, at least roughly, within some typical length-scale. If the $C_{\omega,m}$'s are allowed to grow arbitrarily as m changes, it means that eigenvectors are allowed to be "extended" over arbitrarily large length-scales. We have shown that a correct condition, which does give correspondence between eigenvector localization and dynamical localization, is what we call semi-uniformly localized eigenvectors (SULE): There are sites $n_{\omega,m}$ so that for each $\epsilon > 0$, there is $C_{\omega,\epsilon}$ for which

$$|\varphi_{\omega,m}(n)| \leq C_{\omega,\epsilon} e^{\epsilon|n_{\omega,m}|} e^{-A|n-n_{\omega,m}|} \tag{3}$$

(3) says that the constants $C_{\omega,m}$ of (2) are allowed to grow at a rate which is less than exponential in the distance of the $n_{\omega,m}$'s from the origin. SULE is closely related to a dynamical condition, which we call semi-uniform dynamical localization (SUDL)

$$\sup_t |e^{-itH_\omega}(n,\ell)| \leq \tilde{C}_{\omega,\epsilon} e^{\epsilon|\ell|} e^{-\tilde{A}|n-\ell|} \tag{4}$$

We have proven that (3) implies (4) with A arbitrarily close to $\tilde{A}$, and that if H_ω has simple eigenvalues[6], then (4) implies (3) with $A = \frac{1}{2}\tilde{A}$[7]. (4) is sufficient to show that $\langle x^2(t)\rangle$ (or any other positive moment of x) is bounded. By standard probability arguments, (4) is implied by

$$\mathrm{E}\Big(\sup_t |e^{-itH_\omega}(n,\ell)|\Big) \leq C e^{-\tilde{A}|n-\ell|} \tag{5}$$

where $\mathrm{E}(\cdot)$ denotes expectation over realizations. (5) has been proven by Delyon et al.[8] in the one-dimensional case and by Aizenman[9] in multidimensional cases at large coupling[10].

A priori, one may wish to consider a more restrictive condition than (3), which is to consider (2), but with C_ω independent of m instead of $C_{\omega,m}$. We call this condition uniformly localized eigenvectors (ULE). Indeed, ULE is related to the dynamical condition (which we call uniform dynamical localization (UDL))

$$\sup_t |e^{-itH_\omega}(n,\ell)| \leq \tilde{C}_\omega e^{-\tilde{A}|n-\ell|} \tag{6}$$

in essentially the same way that SULE is related to (4). The problem is that ULE does not occur: We have shown[1],[11] that ULE can't occur for a large class of models, and, in particular, it can't occur for the Anderson model in any dimension. It is an open question, in fact, whether there is *any* Schrödinger operator with ULE. The Maryland model[12], which has an unbounded quasiperiodic potential, exhibits a weak form of ULE in the sense that for any finite energy interval, there exists a uniform constant for all eigenvectors with energies in this interval. The Almost Mathieu operator[13], however, does not have ULE[1],[11]. For the Anderson model, or any other random Hamiltonian obeying Eqn. (5), one can actually show stronger uniformity than what is given by Eqn. (3). That is, one can get an explicit (dimension dependent) polynomial bound on the growth of the $C_{\omega,m}$'s. Our definition of SULE by (3) attempts to get, at least roughly, a minimal uniformity requirement that would still have a two-way relationship with corresponding dynamical localization.

Our second set of results concerns the following discovery of Gordon[14] and del Rio et al.[15]: Let H_ω be an Anderson Hamiltonian in the localized regime and let[16] $H_\omega(\lambda) = H_\omega + \lambda|0\rangle\langle 0|$. Then for a set S of couplings λ, which is dense and locally uncountable[17], $H_\omega(\lambda)$ has purely singular continuous spectrum[18]. In particular, for $\lambda \in S$, $\langle x^2(t)\rangle$ is unbounded[19]. So, the strong dynamical localization discussed above can be destroyed by an arbitrarily small perturbation of the potential at a single point: a disturbing fact.

We have found[1] that this instability is a mild one in the following senses:

(i) For *all* λ, $\langle x^2(t)\rangle \leq C(\ln|t|)^2$ for t large.

(ii) The spectral measures in the singular continuous case are supported on a set of zero Hausdorff dimension[20]. In fact, this follows from (i) by a result of Last[21], which is based on ideas originally due to Guarneri[22].

(iii) S is contained in a set of coupling constants $\tilde{S}$ so that $\tilde{S}$ has zero Hausdorff dimension, and so that if $\lambda \notin \tilde{S}$, $H_\omega(\lambda)$ has pure point spectrum.

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Notes

1. R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, *Operators with singular continuous spectrum, IV. Hausdorff dimension, rank one perturbations, and localization*, to be submitted to J. d'Analyse Math.

2. The results hold for much more general distributions; see ref. 1.

3. Or complete in some energy range.

4. For $\delta = 0$, it is known that the limit is zero in the region of pure point spectrum. See B. Simon, Commun. Math. Phys. **134**, 209–212 (1990).

5. S. Aubrey and G. Andre, Ann. Israel Phys. Soc. **3**, 133–164 (1980); J. Avron and B. Simon, Duke Math. J. **50**, 369–391 (1983).

6. This is known in the Anderson localization regime; see B. Simon, Rev. Math. Phys. **6**, 1183–1185 (1994).

7. The idea behind the proof that (4) implies (3) is to use simplicity of the spectrum and (4) to show that

$$|\varphi_{\omega,m}(n)|\,|\varphi_{\omega,m}(\ell)| \leq \tilde{C}_{\omega,\epsilon} e^{\epsilon|\ell|} e^{-\tilde{A}|n-\ell|} \; .$$

Take $n_{\omega,m}$ to be a point where $\varphi_{\omega,m}(\,\cdot\,)$ takes its maximum value, then

$$|\varphi_{\omega,m}(n)|^2 \leq |\varphi_{\omega,m}(n)|\,|\varphi_{\omega,m}(n_{\omega,m})| \leq \tilde{C}_{\omega,\epsilon} e^{\epsilon|n_{\omega,m}|} e^{-\tilde{A}|n-n_{\omega,m}|} \; .$$

8. F. Delyon, H. Kunz, and B. Souillard, J. Phys. **A 16**, 25 (1983).

9. M. Aizenman, Rev. Math. Phys. **6**, 1163–1182 (1994). For related results, also see F. Martinelli and E. Scoppola, *Introduction to the mathematical theory of Anderson localization*, Rivista del Nuovo Cimento **10**, N. 10 (1987).

10. If the potential is unbounded (e.g., Gaussian), there are also results at small coupling and large energy.

11. S. Jitomirskaya, *Singular continuous spectrum and uniform localization for ergodic Schrödinger operators*, preprint.

12. The Maryland model is the discrete Schrödinger operator in one dimension with potential $V(n) = \lambda \tan(2\pi\alpha n + \theta)$. See D.R. Grempel, S. Fishman, and R.E. Prange, Phys. Rev. Lett. **49**, 833 (1982); R.E. Prange, D.R. Grempel, and S. Fishman, Phys. Rev. **B 29**, 6500–6512 (1984); A. Figotin and L. Pastur, Commun. Math. Phys. **95**, 401–425 (1984); B. Simon, Ann. Phys. **159**, 157–183 (1985).

13. The Almost Mathieu operator is the discrete Schrödinger operator in one dimension with potential $V(n) = \lambda \cos(2\pi\alpha n + \theta)$. For large λ, it is known to exhibit localization in the sense of Eqn. (2); see Ya. Sinai, J. Statist. Phys. **46**, 861–909 (1987); J. Fröhlich, T. Spencer, and P. Wittwer, Commun. Math. Phys. **132**, 5–25 (1990); S. Jitomirskaya, Commun. Math. Phys. **165**, 49–57 (1994).

14. A. Gordon, Commun. Math. Phys. **164**, 489–505 (1994).

15. R. del Rio, N. Makarov, and B. Simon, Commun. Math. Phys. **165**, 59–67 (1994).

16. $H_\omega(\lambda)$ has the potential modified at zero by a small amount λ. For a general theory on such perturbations, see B. Simon, *Spectral analysis of rank one perturbations and applications*, Lecture notes given at the Vancouver Summer School in Mathematical Physics, August 1993.

17. Technically, S is a dense G_δ.

18. Singular continuous spectrum is sometimes called "critical states" in physics literature.

19. It follows from the RAGE theorem, see M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III. Scattering Theory*, London: Academic Press 1979, that if $\langle x^2(t) \rangle$ is bounded, then the spectrum is pure point. Also see ref. 21.

20. For a discussion of Hausdorff dimension, see K.J. Falconer, *Fractal Geometry*, Chichester: Wiley 1990.

21. Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*, preprint.

22. I. Guarneri, Europhys. Lett. **10**, 95–100 (1989).