Global Large Time Self-similarity of a Thermal-Diffusive Combustion System with Critical Nonlinearity

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Abstract

We study the initial value problem of the thermal-diffusive combustion system: $u_{1,t} = u_{1,x,x} - u_1 u_2^2, u_{2,t} = du_{2,xx} + u_1 u_2^2, x \in R^1$, for non-negative spatially decaying initial data of arbitrary size and for any positive constant d. We show that if the initial data decays to zero sufficiently fast at infinity, then the solution (u_1, u_2) converges to a self-similar solution of the reduced system: $u_{1,t} = u_{1,xx} - u_1 u_2^2, u_{2,t} = du_{2,xx}$, in the large time limit. In particular, u_1 decays to zero like $\mathcal{O}(t^{-\frac{1}{2}-\delta})$, where $\delta > 0$ is an anomalous exponent depending on the initial data, and u_2 decays to zero with normal rate $\mathcal{O}(t^{-\frac{1}{2}})$. The idea of the proof is to combine the a priori estimates for the decay of global solutions with the renormalization group (RG) method for establishing the self-similarity of the solutions in the large time limit.

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1 Introduction.

In this paper, we study the initial value problem of the following thermal-diffusive combustion system:

$$u_{1,t} = u_{1,xx} - u_1 u_2^2 \tag{1.1}$$
$$u_{2,t} = d u_{2,xx} + u_1 u_2^2, \qquad x \in R^1, \tag{1.2}$$

with non-negative initial data $(u_1, u_2)|_{t=0} = (a_1(x), a_2(x)) \in (L^1(R^1) \cap L^\infty(R^1))^2$, of arbitrary size, where the positive constant d is the Lewis number. We are interested in the large time behavior of solutions of (1.1)-(1.2).

The system (1.1)-(1.2) on a bounded domain is well-studied in the literature, see [1], [8], [10], [11] and references therein. In case of homogeneous Dirichlet or Neumann boundary conditions, the large time behavior of solutions is that (u_1, u_2) converges uniformly to a constant vector (c_1, c_2) such that $c_1 \cdot c_2 = 0$, see K. Masuda [11].

More recently, the system (1.1)-(1.2) on the line R^1 has been proposed and investigated as a model for cubic autocatalytic chemical reactions of the type $A + 2B \to 3B$, with rate proportional to $u_1 u_2^2$, where u_1 and u_2 are concentrations of the reactant A, and the autocatalyst B. We refer to the interesting papers by J. Billingham and D. Needham [4], [5], for details. In [4] and [5], the authors established the existence of traveling front solutions rigorously by shooting and phase plane methods; moreover, they studied the long time asymptotics of solutions by formal methods and numerical computations for a class of front initial data, i.e. data such that $a_1 + a_2$ has nonvanishing limits as $x \to \infty$.

Motivated by thermal-diffusive models with Arrhenius reactions, [12], [2] etc. Berlyand and Xin [3] considered system (1.1)-(1.2) for a class of small initial data in $(L^1 \cap L^\infty(R^1))^2$ and showed that $u_i (i = 1, 2)$ are bounded from above and below by self-similar upper and lower solutions. The results of [3] imply that u_1 decays to zero in time with an algebraic rate faster than $t^{-\frac{1}{2}-\delta}$, for some $\delta > 0$, and u_2 decays to zero like $\mathcal{O}(t^{-\frac{1}{2}})$.

In the present work, we prove the exact large time self-similar asymptotics with no restriction on the size of initial data as long as the data has sufficiently fast spatial decay. Our main result is the following. We consider the system (1.1)-(1.2) with initial data $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the Banach space of continuous functions on R^1 with the norm

$$\|f\| = \sup_{x \in R^1} |f(x)|(1 + |x|)^q, \quad \text{with } q > 1 \text{ fixed below.} \tag{1.3}$$

Let $\phi = \phi(x)$ be the Gaussian:

$$\phi(x) = \frac{1}{\sqrt{4\pi d}} e^{-\frac{x^2}{4d}}. \tag{1.4}$$

Given $A \geq 0$, let ψ_A be the principal eigenfunction (ground state) of the differential operator:

$$\mathcal{L}_A = -\frac{d^2}{dx^2} - \frac{1}{2} x \frac{d}{dx} - \frac{1}{2} + A^2 \phi^2(x), \tag{1.5}$$

on $L^2(R^1, d\mu)$, with $d\mu(x) = e^{\frac{x^2}{4}}dx$. The corresponding eigenvalue is denoted by $E_A \geq 0$ (and $E_A > 0$ for $A > 0$). We normalize ψ_A by $\int \psi_A^2(x)d\mu(x) = 1$. Our main result is:

Theorem 1 *(Global Large Time Self-Similarity). Consider initial data $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, $a_i \not\equiv 0, a_i \geq 0, i = 1, 2$. Let $A = \int_{R^1} a_1(x) + a_2(x)dx$, the total mass of the system, which is conserved in time. Then system (1.1)-(1.2) has a unique global classical solution $(u_1(x,t), u_2(x,t)) \in \mathcal{B}\times\mathcal{B}$ for $\forall t \geq 0$. Moreover, there exists a $q(A)$ such that, if $q \geq q(A)$ in (1.3), there is positive number B depending continuously on (a_1, a_2) such that:*

$$\|t^{\frac{1}{2}+E_A}u_1(\sqrt{t}\cdot, t) - B\psi_A(\cdot)\| \underset{t\uparrow\infty}{\longrightarrow} 0, \tag{1.6}$$

$$\|t^{\frac{1}{2}}u_2(\sqrt{t}\cdot, t) - A\phi(\cdot)\| \underset{t\uparrow\infty}{\longrightarrow} 0. \tag{1.7}$$

Remark 1.1 All the results of Sections 2 and 3 hold for any $q > 1$. We need the decay at infinity of a_i to be fast enough only to obtain the exact decay rate in (1.6). For A large, E_A will be large, and the decay in (1.6) may be much faster than the diffusive one. Alternatively, we could consider data $a_i \in \mathcal{B}_{\exp}$ where $\mathcal{B}_{\exp}$ is defined through the norm

$$\|f\|_{\exp} = \sup_x |f(x)|e^{\gamma|x|}$$

for some $\gamma > 0$. Then, the conclusions of Theorem 1 hold for any A.

Remark 1.2 The rate of convergence in (1.6) and (1.7) to zero is actually $O(t^{-\eta})$, for some $\eta > 0$, see (4.28, 4.29). The convergence in (1.6) and (1.7) implies that

$$u_1(x,t) \sim \frac{B\psi_A(\frac{x}{\sqrt{t}})}{t^{\frac{1}{2}+E_A}} + h.o.t. \quad \text{and}$$

$$u_2(x,t) \sim \frac{A}{t^{\frac{1}{2}}}\phi(\frac{x}{\sqrt{t}}) + h.o.t.,$$

as $t \to \infty$, where the leading terms are just the two parameter self-similar solutions to the reduced system. The anomalous exponent E_A occurs as a result of the interactions of nonlinearities of opposite signs. Furthermore, E_A can be computed or estimated as the ground state energy of operator $\mathcal{L}_A$ depending only on the Lewis number d and the total mass of the system. A nonperturbative upper bound is $\frac{A^2}{4\pi\sqrt{2d+1}}$, while, for A small,

$$E_A = \frac{A^2}{4\pi\sqrt{2d+1}} - O(A^4),$$

see [3] for details. Actually, it is more natural physically to normalize $A = 1$, which amounts to putting a coupling constant A^2 in front of the reaction terms in (1.1,1.2), and the anomalous exponent E_A depends then on the strength of that coupling constant.

Remark 1.3 In order to understand the heuristics of (1.6), (1.7), consider a more general problem:

$$u_{1,t} = u_{1,xx} - u_1u_2^m \tag{1.8}$$

$$u_{2,t} \;\;=\;\; du_{2,xx} + u_1 u_2^m \tag{1.9}$$

for $m \geq 1$. For $m > 2$, as explained in [3], we can use the RG method of [6] to prove that both u_1 and u_2 go diffusively to zero. For $1 \leq m < 2$, one can use the maximum principle, as in Lemma 2.3 and equation (3.5) below, to bound from above u_1 by $\bar{u}_1$, which is the solution of

$$\bar{u}_{1,t} = \bar{u}_{1,xx} - \mathcal{O}(t^{-m/2}\phi^m(\frac{x}{\sqrt{t}}))\bar{u}_1 \tag{1.10}$$

Then using the Feynman-Kac formula, we get

$$u_1(x,t) \leq \exp(-\mathcal{O}(t^{1-\frac{m}{2}})) \tag{1.11}$$

for $|x| \leq \mathcal{O}(\sqrt{t})$. For $|x| \geq \mathcal{O}(\sqrt{t})$, one gets a diffusive behaviour, depending on the rate of decay, as $x \to \pm\infty$, of the initial data. Then, inserting the fast decay (1.11) of u_1 in (1.9), one shows that the effect of the nonlinear term in (1.9) is small and that u_2 diffuses to zero. Clearly the borderline case $m = 2$ is the most delicate and the most interesting one. Instead of (1.11), one gets $\exp(-\mathcal{O}(\log t))$ which gives rise, after some analysis, to (1.6).

The rest of the paper is organized as follows. In section 2, we derive a priori estimates on the solutions of the system (1.1)-(1.2) based on the work of K. Masuda [11] for finite domains. Quite a few estimates are different here due to the unboundedness of R^1. The a priori estimates imply the existence of global smooth solutions. In section 3, we derive decay estimates for the solutions using the maximum principle and a simple renormalization group (RG, see [6]) idea to show that u_1 goes to zero like $\mathcal{O}(t^{-\frac{1}{2}-\delta})$, for some $\delta > 0$. We use this information to prove that the nonlinearity is irrelevant (in the RG sense) in (1.2) and that $\|u_2\|_\infty \leq O(t^{-\frac{1}{2}})$ as $t \to \infty$. In section 4, we use the results of sections 2 and 3, and the renormalization group method to prove the convergence to a self-similar solution and thus complete the proof of the main theorem.

2 A priori estimates and global bounds.

The goal of this section is to prove

Proposition 1 *The system (1.1, 1.2) has a unique classical solution satisfying*

$$\|u_i\|_{L^p} \leq C(a_1, a_2), \;\; i = 1, 2, \;\; 1 \leq p \leq +\infty, \tag{2.1}$$

where the constant C depends only on the initial data $(a_1, a_2) \in (L^1(R^1) \cap L^\infty(R^1))^2$.

Remark 2.1 Although some of the arguments below follow those of Masuda [11], we provide them for completeness. Here and below, we use C to denote a generic constant that may vary from place to place. Moreover we write, as above, $C(\cdot)$ to indicate the only variables on which the constant may depend.

First, we have the obvious

Lemma 2.1 The solution (u_1, u_2) satisfies the L^1 estimates:

$$\begin{aligned} \|u_1 + u_2\|_{L^1(R^1)} = \|a_1 + a_2\|_{L^1(R^1)} \quad , \quad & \|u_2\|_{L^1(R^1)} \geq \|a_2\|_{L^1(R^1)}, \\ \|u_1\|_{L^1(R^1)} \leq \|a_1\|_{L^1(R^1)} \quad , \quad & \int_0^\infty \int_{R^1} u_1 u_2^2 dx d\tau < +\infty. \end{aligned} \tag{2.2}$$

Proof Integrating (1.1)-(1.2) over R^1, assuming spatial decay at infinity, we get:

$$\|u_1\|_{L^1}(t) = \|a_1\|_{L^1} - \int_0^t \|u_1 u_2^2\|_{L^1}(\tau) d\tau, \tag{2.3}$$

$$\|u_2\|_{L^1}(t) = \|a_2\|_{L^1} + \int_0^t \|u_1 u_2^2\|_{L^1}(\tau) d\tau. \tag{2.4}$$

Combining (2.3)-(2.4) gives (2.2). $\square$

Lemma 2.2 The function $g_p(u_2) \equiv u_2^p$ satisfies, for $p \geq 2$,

$$0 \leq g_p'(u_2) \leq ((\frac{p}{p-1}) g_p(u_2) g_p''(u_2))^{\frac{1}{2}}. \tag{2.5}$$

Proof Direct calculation. $\square$

Using the classical parabolic maximum principle, we have,

Lemma 2.3

$$\begin{aligned} &(1) \quad 0 < u_1(x,t) \leq \|a_1\|_\infty, \quad \forall\, t > 0; && (2.6) \\ &(2) \quad 0 < \underline{u}_2(x,t) \leq u_2(x,t), \quad \forall\, t > 0, \quad \text{where } \underline{u}_2 \text{ is a solution of:} && \\ &\qquad \underline{u}_{2,t} = d\underline{u}_{2,xx}, \quad \underline{u}_2|_{t=0} = a_2(x); && (2.7) \\ &(3) \quad u_1(x,t) \leq \bar{u_1}(x,t), \quad \forall t \geq 0, \quad \text{where } \bar{u_1} \text{ solves:} && \\ &\qquad \bar{u_1}_t = \bar{u}_{1,xx} - \bar{u_1} \cdot \underline{u}_2^2, \quad \bar{u_1}|_{t=0} = a_1(x). && (2.8) \end{aligned}$$

Remark 2.2 Using this Lemma, one immediately proves (2.1) for u_1, since, by (2.8), u_1 is a fortiori bounded by the solution of the heat equation with the same initial data.

Lemma 2.4 The solutions (u_1, u_2) of (1.1)-(1.2) satisfy the L^p bounds:

$$\|u_i\|_{L^p} \leq C(a_1, a_2, p) < +\infty, \quad i = 1, 2, \quad 1 \leq p < +\infty, \quad \text{and } p = \text{ integer}. \tag{2.9}$$

where $C(a_1, a_2, p)$ is a constant depending only on the initial data and p.

Proof Due to Remark 2.2, we have only to prove the bounds for u_2. We use standard local existence of classical solutions for parabolic equations, and, therefore, we freely integrate by parts below. Our goal will be to prove bounds uniform in time. We shall show that

$$\begin{aligned}
&\int_{R^1} u_2^p dx + \int_0^t \int_{R^1} (u_{2,x}^2 u_2^{p-2} + u_{1,x}^2 u_2^p + u_1 u_2^{p+2}) dx d\tau \\
\leq\ & C(a_1, a_2, p)(1 + \int_0^t \int_{R^1} u_1 u_2^{p+1} dx d\tau) \qquad (2.10)
\end{aligned}$$

for all $p \geq 2$ (p integer). Besides, we shall show, for $p = 1$,

$$\int_0^t \int_{R^1} (u_1 u_2^3 + u_{1,x}^2 u_2) dx d\tau \ \leq\ C(a_1, a_2)(1 + \int_0^t \int_{R^1} u_1 u_2^2 dx d\tau) \qquad (2.11)$$

Using (2.2) to bound $\int_0^t \int_{R^1} u_1 u_2^2 dx d\tau$, and using induction in p, we get that all the terms on the left hand side of (2.10) are bounded, for all $p \geq 1$, p integer ($u_2 \in L^1$ by (2.2)). In particular, this implies the claims of the Lemma.

To prove (2.10, 2.11), we multiply (1.2) by $g_p'(u_2)$, we integrate over $R_1^+ \times R_1$, and we get, after integrating by parts:

$$\begin{aligned}
\int_{R^1} g_p(u_2) dx \ =\ & \int_{R^1} g_p(a_2) dx - d \int_0^t \int_{R^1} u_{2,x}^2 g_p''(u_2) dx d\tau \\
+\ & \int_0^t \int_{R^1} u_1 u_2^2 g_p'(u_2) dx d\tau, \qquad (2.12)
\end{aligned}$$

here $g_p(u_2) = u_2^p, \ \ p \geq 1$.
Consider the identity ($p \geq 1$),

$$\begin{aligned}
& \frac{d}{dt} \int_{R^1} (u_1^2 + u_1) g_p(u_2) dx = \int_{R^1} (1 + 2u_1)(u_{1,xx} - u_1 u_2^2) g_p(u_2) dx \\
& + \int_{R^1} (u_1 + u_1^2) g_p'(u_2)(d u_{2,xx} + u_1 u_2^2) dx \\
=\ & - \int_{R^1} (1 + 2u_1) u_{1,x} u_{2,x} g_p'(u_2) dx - 2 \int_{R^1} u_{1,x}^2 g_p(u_2) dx - d \int_{R^1} (1 + 2u_1) u_{1,x} u_{2,x} g_p'(u_2) dx \\
-\ & d \int_{R^1} (u_1 + u_1^2) g_p''(u_2) u_{2,x}^2 dx - \int_{R^1} (1 + 2u_1) u_1 u_2^2 g_p(u_2) dx + \int_{R_1} (u_1 + u_1^2) g_p'(u_2) u_1 u_2^2 dx \\
\equiv\ & I + II + III + IV + V + VI. \qquad (2.13)
\end{aligned}$$

We estimate for $p \geq 2$, using (2.6), (2.5):

$$\begin{aligned}
& I + II + III \leq (1 + 2\|a_1\|_\infty)(1 + d) \int_{R^1} |u_{1,x} u_{2,x}| g_p'(u_2) dx - 2 \int_{R^1} u_{1,x}^2 g_p(u_2) dx \\
\leq\ & (1 + 2\|a_1\|_\infty)(1 + d) \int_{R^1} |u_{1,x} u_{2,x}| (g_p(u_2) g_p''(u_2) (\frac{p}{p-1}))^{\frac{1}{2}} dx \\
& -2 \int_{R^1} u_{1,x}^2 g_p(u_2) dx \\
\leq\ & \frac{1}{2}(1 + 2\|a_1\|_\infty)(1 + d)(\frac{p}{p-1})^{\frac{1}{2}} (\epsilon \int_{R^1} u_{1,x}^2 g_p(u_2) dx + \epsilon^{-1} \int_{R^1} u_{2,x}^2 g_p''(u_2) dx) \\
& -2 \int_{R^1} u_{1,x}^2 g_p(u_2) dx \qquad (2.14)
\end{aligned}$$

Picking

$$\epsilon = \frac{2}{(1+2\|a_1\|_\infty)(1+d)}(\frac{p-1}{p})^{\frac{1}{2}}$$

in (2.14), we continue:

$$\begin{aligned} & I + II + III \\ \leq \ & (\frac{p}{p-1})\epsilon^{-1}(1+2||a_1\|_\infty)^2(1+d)^2 \int_{R^1} u_{2,x}^2 g_p''(u_2)dx - \int_{R^1} u_{1,x}^2 g_p(u_2)dx \quad (2.15) \end{aligned}$$

In addition, we have:

$$\begin{aligned} IV &\leq 0, & (2.16) \\ V &\leq -\int_{R^1} u_1 u_2^2 g_p(u_2)dx, & (2.17) \\ VI &\leq (\|a_1\|_\infty + \|a_1\|_\infty^2) \int_{R^1} g_p'(u_2)u_1 u_2^2 dx. & (2.18) \end{aligned}$$

Integrating (2.13) from zero to t yields:

$$\begin{aligned} & 0 \leq (\int_{R^1}(u_1+u_1^2)g_p(u_2)dx)(t) \leq \int_{R^1}(a_1+a_1^2)g_p(a_2)dx \\ & +C(p,a_1)\int_0^t \int_{R^1} u_{2,x}^2 g_p''(u_2)dxd\tau \\ + \ & (\|a_1\|_\infty + \|a_1\|_\infty^2)\int_0^t \int_{R^1} g_p'(u_2)u_1 u_2^2 dxd\tau \\ - \ & \int_0^t \int_{R^1} u_1 u_2^2 g_p(u_2)dxd\tau - \int_{R^1} u_{1,x}^2 g_p(u_2)dxd\tau. \quad (2.19) \end{aligned}$$

Combining (2.12) and (2.19) gives (2.10) for $p \geq 2$.

For $p = 1$, we proceed from (2.13) as follows:

$$\begin{aligned} & \frac{d}{dt}\int_{R^1}(u_1+u_1^2)u_2 dx = -\int_{R^1}(1+2u_1)u_{1,x}u_{2,x}dx - 2\int_{R^1} u_{1,x}^2 u_2 dx \\ & -d\int_{R^1}(1+2u_1)u_{1,x}u_{2,x}dx - \int_{R^1}(1+2u_1)u_1 u_2^3 dx \\ & +\int_{R^1}(u_1+u_1^2)u_1 u_2^2 dx \\ \leq \ & (1+2\|a_1\|_\infty)\int_{R^1}|u_{1,x}u_{2,x}|dx - 2\int_{R^1} u_{1,x}^2 u_2 dx \\ & +d(1+2\|a_1\|_\infty)\int_{R^1}|u_{1,x}u_{2,x}|dx - \int_{R^1} u_1 u_2^3 dx \\ & +(\|a_1\|_\infty + \|a_1\|_\infty^2)\int_{R^1} u_1 u_2^2 dx \\ \leq \ & \frac{1}{2}(1+d)(1+2\|a_1\|_\infty)[\epsilon^{-1}\int_{R^1} u_{1,x}^2 dx + \epsilon\int_{R^1} u_{2,x}^2 dx] \\ & -2\int_{R^1} u_{1,x}^2 u_2 dx - \int_{R^1} u_1 u_2^3 dx \\ & +(\|a_1\|_\infty + \|a_1\|_\infty^2)\int_{R^1} u_1 u_2^2 dx, \quad (2.20) \end{aligned}$$

for any $\epsilon > 0$. Now, integrate (2.20) from 0 to t, to get (for ϵ small enough):

$$\begin{aligned}\int_0^t \int_{R^1} (2u_1u_2^3 + u_{1,x}^2 u_2) dx d\tau &\leq 2d \int_0^t \int_{R^1} u_{2,x}^2 dx d\tau \\ + \quad C(a_1, a_2)(1 + \int_0^t \int_{R^1} (u_{1,x}^2 + u_1u_2^2) dx d\tau). &\qquad (2.21)\end{aligned}$$

Now, use (2.12) with $p = 2$ to bound

$$2d \int_0^t \int_{R^1} u_{2,x}^2 dx d\tau \leq \int_{R^1} a_2^2 dx + \int_0^t \int_{R^1} u_1u_2^3 dx d\tau. \qquad (2.22)$$

Finally, observe that, multiplying (1.1) by u_1, and integrating by parts, we get:

$$\frac{d}{dt} \int_{R^1} u_1^2 dx = -\int_{R^1} u_{1,x}^2 dx - \int_{R^1} u_1^2 u_2^2 dx \qquad (2.23)$$

from which we immediately obtain:

$$\int_0^t \int_{R^1} u_{1,x}^2 dx d\tau \leq \int_{R^1} a_1^2 dx \qquad (2.24)$$

uniformly in t. Combining (2.21, 2.22, 2.24), we get (2.11). This completes the proof of the lemma. □

Remark 2.3 In Masuda [11], $g_p(u_2) = (1 + u_2)^p$, and lemma 2.4 is proved for fractional p's by starting the induction from (2.12), $p \in (0, 1)$ instead of combining (2.20) and (2.19). However, for unbounded domains like R^1, such argument fails since $g_p(u_2) \notin L^1(R^1)$. Lemma 2.4 still holds for fractional powers p, and we will show that with the help of L^∞ bounds which we establish below using (2.10).

Lemma 2.5 The solutions (u_1, u_2) of (1.1)-(1.2) obey the estimates

$$\|u_{1,x}\|_2 + \|u_{2,x}\|_2 \leq C(a_1, a_2, a_{1,x}, a_{2,x}) \qquad (2.25)$$

where $C(a_1, a_2, a_{1,x}, a_{2,x})$ is a positive constant depending on $\|a_i\|_1, \|a_i\|_\infty, \|a_{i,x}\|_2$.

Proof Multiplying (1.1) by $u_{1,xx}$, (1.2) by $u_{2,xx}$, and integrating over R^1 gives:

$$-\frac{1}{2}\frac{d}{dt}\|u_{1,x}\|_2^2 = \int_{R^1} |u_{1,xx}|^2 dx - \int_{R^1} u_1u_2^2 u_{1,xx} dx$$

or

$$\frac{1}{2}\frac{d}{dt}\|u_{1,x}\|_2^2 = -\int_{R^1} |u_{1,xx}|^2 dx - \int_{R^1} u_2^2 u_{1,x}^2 dx - 2\int_{R^1} u_2u_1u_{1,x}u_{2,x} dx \qquad (2.26)$$

Similarly, we have:

$$\frac{1}{2}\frac{d}{dt}\|u_{2,x}\|_2^2 = -d\int_{R^1} |u_{2,xx}|^2 dx + 2\int_{R^1} u_1u_2u_{2,x}^2 + \int u_2^2 u_{1,x}u_{2,x} dx. \qquad (2.27)$$

Adding (2.26) and (2.27) gives:

$$\frac{1}{2}\frac{d}{dt}(\|u_{1,x}\|_2^2+\|u_{2,x}\|_2^2) = -\int_{R^1} u_{1,xx}^2 dx - d\int_{R^1} u_{2,xx}^2 dx - \int_{R^1} u_2^2 u_{1,x}^2 dx$$
$$-2\int u_1u_2u_{1,x}u_{2,x}dx + 2\int u_1u_2u_{2,x}^2 dx + \int u_2^2 u_{1,x}u_{2,x}dx. \tag{2.28}$$

Now, integrate (2.28) from 0 to t, and bound the resulting terms on the right hand side as follows. The first three terms are negative, and the last three terms can be bounded, using $\|u_1\|_\infty \leq \|a_1\|_\infty$ (see (2.6)) and the Cauchy-Schwarz' inequality:

$$|\int_0^t\int_{R^1} u_1u_2u_{1,x}u_{2,x}dxd\tau| \leq \|a_1\|_\infty(\int_0^t\int_{R^1} u_{1,x}^2u_2^2dxd\tau\int_0^t\int_{R^1} u_{2,x}^2dxd\tau)^{1/2} \tag{2.29}$$

$$|\int_0^t\int_{R^1} u_1u_2u_{2,x}^2dxd\tau| \leq \|a_1\|_\infty\int_0^t\int_{R^1} u_2u_{2,x}^2dxd\tau, \tag{2.30}$$

and

$$|\int_0^t\int_{R^1} u_2^2u_{1,x}u_{2,x}dxd\tau| \leq (\int_0^t\int_{R^1} u_2^2u_{1,x}^2dxd\tau\int_0^t\int_{R^1} u_2^2u_{2,x}^2dxd\tau)^{1/2}. \tag{2.31}$$

Now, the terms on the right hand side of (2.29, 2.30, 2.31) are uniformly bounded in t, because all the terms in the left hand side of (2.10) are bounded, for $p \geq 2$. So, we get

$$\|u_{1,x}\|_2^2(t) + \|u_{2,x}\|_2^2(t) \leq C(a_1,a_2,a_{1,x},a_{2,x}), \forall\ t \geq 0. \tag{2.32}$$

□

By local existence of classical solutions in $C([0,T^*);(L^2(R^1))^2)\cap C((0,T^*);\,(H^2(R^1))^2)$ of system (1.1)-(1.2), and parabolic regularity: $\exists t_1 > 0$, such that

$$\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2(t_1) \leq C(t_1)(\|a_1\|_2^2+\|a_2\|_2^2),\ \ t_1 < T^*$$

If we replace 0 by t_1 in the proof of Lemma 2.5, then (2.25) in fact implies

Corollary 2.1

$$\|u_{1,x}\|_2 + \|u_{2,x}\|_2 \leq C(a_1,a_2) \tag{2.33}$$

where the constant C depends only on $\|a_i\|_1$ and $\|a_i\|_\infty, i = 1,2$.

Corollary 2.2 *By Sobolev imbedding, (2.9) for $p = 2$ and (2.33) imply:*

$$\|u_2\|_\infty \leq C(a_1,a_2). \tag{2.34}$$

Thus Proposition 1 follows from:

Corollary 2.3 *Combining the estimates in Lemma 2.4, (2.6), (2.34) and standard local existence of classical solutions, we have shown that global smooth and bounded solutions exist for system (1.1)-(1.2) in $(L^1(R^1)\cap L^\infty(R^1))^2$.*

3 Decay estimates

To exhibit the decay of the solutions of (1.1)-(1.2), let us introduce the scaled solutions

$$\tilde{u}_1(x,t) = \sqrt{t}u_i(\sqrt{t}x, t)$$

for $i = 1, 2$. From now on, we consider nonnegative initial data $(a_1(x), a_2(x)) \in \mathcal{B} \times \mathcal{B} \subseteq (L^1(R^1) \cap L^\infty(R^1))^2$ for $q > 1$. The purpose of this section is to prove

Proposition 2 *The solution (u_1, u_2) of (1.1) constructed in Proposition 1 satisfies the bounds*

$$\|\tilde{u}_1\|(t) \leq C(\|a_1\|, \|a_2\|)(1+t)^{-\delta} \tag{3.1}$$
$$\|\tilde{u}_2\|(t) \leq C(\|a_1\|, \|a_2\|) \tag{3.2}$$

where $\delta = \delta(\|a_2\|_{L^1}) > 0$ and $\|\cdot\|$ is, for all $t > 0$, the norm (1.3).

Remark 3.1 Note that, in particular, (3.1) and (3.2) imply

$$\begin{aligned} \|u_1\|_\infty(t) \leq C(\|a_1\|, \|a_2\|)(1+t)^{-\frac{1}{2}-\delta}, \\ \|u_2\|_\infty(t) \leq C(\|a_1\|, \|a_2\|)(1+t)^{-\frac{1}{2}} \end{aligned} \tag{3.3}$$

Remark 3.2 The bound (3.1) was essentially derived in [3], using (2.8) and (2.7), but with a different norm.

Proof Using (2.8), (2.7), it is enough to prove (3.1) with u_1 replaced by $\bar{u}_1$. Also, since $\underline{u_2}$ solves the heat equation, we have:

$$\underline{u_2}(x,t) = \frac{A}{\sqrt{t}}\phi(\frac{x}{\sqrt{t}}) + h.o.t. \tag{3.4}$$

as $t \to \infty$, with ϕ given by (1.4) and $A = \int_{R^1} a_2(x)dx$.

The higher order terms are easy to control, so we can consider, instead of (2.8),

$$\bar{u}_{1,t} = \bar{u}_{1,xx} - \frac{A^2}{t}\phi^2(\frac{x}{\sqrt{t}})\bar{u}_1 \tag{3.5}$$

By a simple change of variables, $\xi = \frac{x}{\sqrt{t}}, \tau = \log t$, we get, for $t \geq 1$,

$$\sqrt{t}\bar{u}_1(x,t) = (e^{-\tau\mathcal{L}_A}\bar{u}_1(\cdot, 1))(\xi) \tag{3.6}$$

with $\mathcal{L}_A$ given by (1.5). Then, (3.1) follows from the first bound on the semigroup $e^{-\tau\mathcal{L}_A}$ given in Lemma 4.1.b.
To prove (3.2), we need

Lemma 3.1 *(Sharp Decay of u_2 in L^∞ norm). There exists a constant C depending only on $\|a_i\|$, $i = 1, 2$ such that*

$$\|u_2\|_\infty(t) \le \frac{C}{(1+t)^{\frac{1}{2}}}, \quad \forall t \ge 0. \tag{3.7}$$

Consider then equation (1.2): $u_{2,t} = du_{2,xx} + (u_1u_2)u_2$. By the first inequality in (3.3) (which follows from (3.1)), and Lemma 3.1, $u_2 \le \bar{u_2}$, where $\bar{u_2}$ solves:

$$\begin{aligned} \bar{u}_{2,t} &= d\bar{u}_{2,xx} + C(1+t)^{-1-\delta}\bar{u}_2 \\ \bar{u}_2|_{t=0} &= a_2. \end{aligned} \tag{3.8}$$

But (3.8) is a linear heat equation, from which (3.2) follows easily since $\int_0^\infty (1+t)^{-1-\delta}dt < \infty$. □

We are left with the

Proof of Lemma 3.1 Write equation (1.2) in integral form:

$$\begin{aligned} u_2(t,x) &= \int_{R^1} \frac{1}{\sqrt{4\pi dt}} e^{-\frac{x^2}{4dt}} a_2(x-y)dy + \\ & \int_0^t \int_{R^1} (4\pi ds)^{-\frac{1}{2}} \exp(-\frac{y^2}{4ds})(u_1u_2^2)(x-y,t-s)dyds. \end{aligned} \tag{3.9}$$

Taking L^∞ norm in (3.9) yields

$$\|u_2\|_\infty(t) \le \frac{C(\|a_2\|)}{(1+t)^{\frac{1}{2}}} + C\int_0^t s^{-\frac{1}{2}} \|u_1u_2^2\|_{L^1}(t-s)ds \tag{3.10}$$

Now, use $\|u_1u_2^2\|_{L^1} \le \|u_1\|_\infty \|u_2^2\|_{L^1}$, $\|u_1\|_\infty(t) \le C(\|a_1\|, \|a_2\|)(1+t)^{-1/2-\delta}$ (which is the first inequality in (3.3)),

$$\|u_2^2\|_{L^1}(t) \le \|u_2\|_\infty(t)\|u_2\|_{L^1}(t) \le C(\|a_1\|, \|a_2\|), \tag{3.11}$$

(which follows from Proposition 1) and

$$\int_0^t s^{-\frac{1}{2}}(1+(t-s))^{-1/2-\delta}ds \le C(1+t)^{-\delta} \tag{3.12}$$

to get

$$\|u_2\|_\infty(t) \le C(\|a_1\|, \|a_2\|)(1+t)^{-\delta} \tag{3.13}$$

(for $\delta \le 1/2$, or else, (3.7) is already proven). Now, use (3.13) to improve (3.11) into $\|u_2^2\|_{L^1}(t) \le C(\|a_1\|, \|a_2\|)(1+t)^{-\delta}$, which, inserted into (3.10) yields

$$\|u_2\|_\infty(t) \le C(\|a_1\|, \|a_2\|)(1+t)^{-2\delta},$$

and we can iterate up to 1/2, which proves (3.7). □

4 Self-similarity

In this section we apply the Renormalization Group method to improve Proposition 2 and finish the proof of the Theorem. We prove:

Proposition 3 *Under the assumptions of Theorem 1, there exists $\epsilon > 0$ such that, if*

$$\|a_1\|\|a_2\| < \epsilon, \tag{4.1}$$

the claims of Theorem 1 hold.

Proof of Theorem 1 By Proposition 2, we can find a $T < \infty$ such that the functions $u_{iT}(x,t) = \sqrt{T}u_i(\sqrt{T}x, Tt)$ satisfy

$$\|u_{1T}(\cdot,1)\|\|u_{2T}(\cdot,1)\| < \epsilon, \tag{4.2}$$

where T depends on the initial data (a_1, a_2). Moreover, u_{iT} solve the equations (1.1) and (1.2), and thus, by Proposition 3 and (4.2), u_{iT} and hence u_i, will have the asymptotics claimed in the Theorem. $\square$

We will now set up an inductive scheme for the proof of Proposition 3. We define, for $L > 1$,

$$u_i^{(n)}(x,t) = L^n u_i(L^n x, L^{2n}t), \quad t \in [1, L^2]. \tag{4.3}$$

Then $u_i^{(n)}$ satisfy the equations (1.1) with initial data

$$a_i^{(n)}(x) = L^n u_i(L^n x, L^{2n}). \tag{4.4}$$

We will study $a_i^{(n)}$ inductively in n i.e. we will consider the RG map $(a_1, a_2) \to (a_1', a_2')$ defined in $\mathcal{B} \times \mathcal{B}$ where

$$a_i'(x) = L u_i(Lx, L^2) \tag{4.5}$$

and (u_1, u_2) solve (1.1)-(1.2) with initial data (a_1, a_2). We first prove a Lemma for the linearization of this map when (1.1)-(1.2) is linearized around the expected asymptotics (1.6) and (1.7).

Hence we consider the equations

$$\begin{aligned} v_{1t} &= v_{1xx} - A^2 t^{-1}\phi^2(\frac{x}{\sqrt{t}})v_1 \\ v_{2t} &= dv_{2xx} \end{aligned} \tag{4.6}$$

for $t \in [1, L^2]$ and $v_i(x,1) = a_i(x)$. By the change of variables $\xi = \frac{x}{\sqrt{t}}, \tau = \log t$ one gets

$$\begin{aligned} Lv_1(Lx, L^2) &= (L^{-2\mathcal{L}_A} a_1)(x) \\ Lv_2(Lx, L^2) &= (L^{-2\mathcal{L}} a_2)(x) \end{aligned} \tag{4.7}$$

where

$$\mathcal{L} = -d\frac{d^2}{dx^2} - \frac{1}{2}x\frac{d}{dx} - \frac{1}{2} \tag{4.8}$$

and $\mathcal{L}_A$ is given by (1.5). Recall also that ψ_A is the principal eigenvalue (ground state) of $\mathcal{L}_A$. We collect some properties of $\mathcal{L}_A$ and $\mathcal{L}$ in

Lemma 4.1 *$\mathcal{L}_A$ and $\mathcal{L}$ have the following properties*

a) $\mathcal{L}_A\psi_A = E_A\psi_A, \quad E_A > 0 \quad$ *if* $A > 0$.

b) Let $f \in \mathcal{B}$. There exist $\delta > 0$ and $\tau_0 < \infty$ such that, for $\tau \geq \tau_0$,

$$\|e^{-\tau\mathcal{L}_A}f\| \leq e^{-\tau\delta}\|f\|.$$

Moreover, there exists $q(A)$ such that, if $f \in \mathcal{B}$, with $(\psi_A, f) = 0$ ($(\cdot,\cdot)$ being the scalar product in $\mathcal{H} = L^2(\mathbf{R}, d\mu)$, $d\mu = e^{\frac{x^2}{4}}dx$), and with $q > q(A)$ in (1.3),

$$\|e^{-\tau\mathcal{L}_A}f\| \leq e^{-\tau(E_A+\delta)}\|f\|$$

c) Let P_A be the orthogonal projection in $\mathcal{H}$ on ψ_A. The quantities $|E_A - E_{A'}|, |1 - (\psi_A, \psi_{A'})|, \|P_A - P_{A'}\|$ (operator norm in $\mathcal{B}$) are bounded by $C(K)|A - A'|$ and $\|P_A\| \leq C(K)$, for $0 \leq A, A' \leq K$.

d) $e^{-\tau\mathcal{L}}\phi = \phi$.

e) Let $f \in \mathcal{B}, \int f dx = 0$. Then, there $\delta > 0$ and $\tau_0 < \infty$ such that, for $\tau \geq \tau_0$,

$$\|e^{-\tau\mathcal{L}}f\| \leq e^{-\tau\delta}\|f\|.$$

The proof of Lemma 4.1 is based on [7] (see also [14]), and will be given in the Appendix, where we also show that the scalar product (ψ_A, f) is well-defined for $f \in \mathcal{B}$.

Returning to the proof of Proposition 3, we write the RG map, defined by (4.5), as

$$a_1' = L^{-2\mathcal{L}_A}a_1 + Ln_1(Lx, L^2) \tag{4.9}$$
$$a_2' = L^{-2\mathcal{L}}a_2 + Ln_2(Lx, L^2) \tag{4.10}$$

where

$$n_1(x,t) = -\int_1^t ds \int dy G_A(t,s,x,y)(u_1u_2^2(y,s) - u_1A^2s^{-1}\phi^2(\frac{y}{\sqrt{s}})) \tag{4.11}$$
$$n_2(x,t) = \int_1^t ds \int dy G(t-s, x-y)u_1u_2^2(y,s) \tag{4.12}$$

and G_A is the fundamental solution of the v_1 equation in (4.6) and G is the kernel of $e^{d(t-s)\Delta}$, where we write Δ for $\frac{d^2}{dx^2}$. Denote by s_L the scaling $(s_Lf)(x) = Lf(Lx)$ and by $G_A(t,s)$ the operator corresponding the kernel $G_A(t,s,x,y)$. Then we have

Lemma 4.2

a) $\|s_L\| \leq L$

b) $\|G_A(t,s)\| \leq e^{c(t-s)}$; $\quad \|e^{d(t-s)\Delta}\| \leq e^{c(t-s)}$, $\quad$ *for* $c < \infty$.

Proof a) follows from

$$\sup_x L \;\; |f(Lx)|(1+|x|)^q \;\; \leq L\|f\|$$

and b) from

$$0 \leq G_A(t,s,x,y) \leq G_0(t-s,x-y)$$

(which itself follows from the Feynman-Kac formula and $A^2\phi^2 \geq 0$), and the explicit Gaussian kernel of G_0. The kernel of $e^{d(t-s)\Delta}$ is similar. $\square$

Let us now specify the A in (4.9) and (4.11) (which is not the same as the one in Theorem 1). We write

$$a_1(x) = B\psi_A(x) + b_1(x) \tag{4.13}$$
$$a_2(x) = A\phi(x) + b_2(x) \tag{4.14}$$

with

$$B = (\psi_A, a_1),$$
$$A = \int a_2 dx.$$

Remembering that ψ_A is normalized by $(\psi_A,\psi_A) = 1$, and ϕ by $\int \phi(x)dx = 1$, we see that

$$(\psi_A, b_1) = 0,$$
$$\int b_2 dx = 0.$$

Write (a_1', a_2') similarily, with primes. The main estimate then is

Lemma 4.3 Given $L \geq L_0 = e^{2\tau_0}$, with τ_0 as in Lemma 4.1, there is an $\epsilon_0(L) > 0$ such that if $\|a_1\|\,\|a_2\| < \epsilon \leq \epsilon_0(L)$, we have

a) $|A' - A| \leq C(L)\epsilon\|a_2\|$

b) $\|b_2'\| \leq L^{-2\delta}\|b_2\| + C(L)\epsilon\|a_2\|$

c) $|B' - L^{-2E_A}B| \leq C(L)\epsilon[\epsilon(1+\|a_2\|) + \|b_2\|]$

d) $\|b_1'\| \leq L^{-2(E_A+\delta)}\|b_1\| + C(L)\epsilon[\epsilon(1+\|a_2\|) + \|b_2\|]$

where $C(L)$ is an L-dependent constant.

Proof We solve first u_2 from the equation

$$u_2(x,t) = e^{d(t-1)\Delta} a_2 + n_2(x,t) \equiv u_{20} + n_2, \tag{4.15}$$

with n_2 given by (4.12), by the contraction mapping principle. Consider the ball

$$B_R = \{u_2 : |||u_2||| \equiv \sup_{t\in[1,L^2]} \|u_2(\cdot,t)\| \leq R\|a_2\|\}.$$

For $u_2 \in B_R$ we bound n_2 by using (2.6), i.e.

$$\|u_1(\cdot,s)\|_\infty \leq \|e^{d(s-1)\Delta} a_1\|_\infty \leq \|a_1\|_\infty \leq C\|a_1\|$$

and Lemma 4.2.b, to get

$$|||n_2||| \leq R^2 C(L)\|a_1\|\,\|a_2\|^2 \leq R^2 C(L)\epsilon\|a_2\|. \tag{4.16}$$

Since by Lemma 4.2.b again,

$$|||u_{20}||| \leq C(L)\|a_2\|,$$

we see that the right hand side of (4.15) maps B_R into itself if $R = R(L)$ is large enough and $\epsilon < \epsilon(L)$ is small enough. It is easy to see that the right hand side of (4.15) is a contraction in B_R, so that we get a solution in B_R. By Lemma 4.2.a then,

$$\|Ln_2(L\cdot, L^2)\| \leq C(L)\epsilon\|a_2\| \tag{4.17}$$

and since

$$\begin{aligned} A' &= A + \int Ln_2(Lx, L^2)dx \\ b_2' &= L^{-2\mathcal{L}} b_2 + Ln_2(Lx, L^2) + (A - A')\phi, \end{aligned}$$

a) and b) follow from (4.17), and Lemma 4.1.e.

For a_1', consider n_1 in (4.11), and write

$$\begin{aligned} w(y,s) &\equiv u_2^2(y,s) - A^2 s^{-1}\phi^2(\frac{y}{\sqrt{s}}) \\ &= (u_2(y,s) + As^{-1/2}\phi(\frac{y}{\sqrt{s}}))((e^{d(s-1)\Delta} b_2)(x) + n_2(x,s)), \end{aligned} \tag{4.18}$$

using (4.15), (4.14) and $e^{d(s-1)\Delta}\phi(y) = s^{-\frac{1}{2}}\phi(\frac{y}{\sqrt{s}})$. Thus, by (4.16) and Lemma 4.2.b,

$$|||w||| \leq C(L)(\|a_2\| + A)(\|b_2\| + \epsilon\|a_2\|) \tag{4.19}$$

and, since $A \leq C\|a_2\|$, $s_L n_1 = Ln_1(L\cdot, L^2)$ is bounded by

$$\|s_L n_1\| \leq C(L)\epsilon(\|b_2\| + \epsilon\|a_2\|). \tag{4.20}$$

(use $\|u_1\|_\infty \leq C\|a_1\|$ and $\|a_1\|\,\|a_2\| \leq \epsilon$).
Since, from (4.9)

$$B' = (\psi_{A'}, a_1') = (\psi_{A'}, L^{-2\mathcal{L}_A} a_1) + (\psi_{A'}, s_L n_1)$$

and, from (4.13)

$$(\psi_{A'}, L^{-2\mathcal{L}_A} a_1) = BL^{-2E_A}(\psi_{A'}, \psi_A) + (\psi_{A'}, (P_{A'} - P_A)L^{-2\mathcal{L}_A} b_1)$$

(we used $P_A b_1 = 0$), we get, using Lemma 4.1.c and (4.20),

$$|B' - L^{-2E_A}B| \;\leq\; C|A - A'|(B + \|b_1\|) + C(L)\epsilon(\|b_2\| + \epsilon\|a_2\|)$$

where the constant C in $C|A - A'|$ is independent of A, A', because A here is uniformly bounded (by Lemma 2.1). Then, using part a) above, we get

$$\begin{aligned} |B' - L^{-2E_A}B| &\leq C(L)\epsilon[(B + \|b_1\|)\|a_2\| + \|b_2\| + \epsilon\|a_2\|] \\ &\leq C(L)\epsilon[\epsilon(1 + \|a_2\|) + \|b_2\|)] \end{aligned} \tag{4.21}$$

(since $B + \|b_1\| \leq C\|a_1\|$ and $\|a_1\|\,\|a_2\| \leq \epsilon$), i.e. we prove c).

Finally, for b_1', write (use (4.9), (4.13))

$$\begin{aligned} b_1' = (1 - P_{A'})a_1' &= BL^{-2E_A}(P_A - P_{A'})\psi_A + L^{-2\mathcal{L}_A} b_1 + (P_A - P_{A'})L^{-2\mathcal{L}_A} b_1 \\ &\quad + (1 - P_{A'})s_L n_1 \end{aligned}$$

(using again $P_A b_1 = 0$). Now, Lemma 4.1.b, c and (4.20) imply

$$\|b_1'\| \leq L^{-2(E_A+\delta)}\|b_1\| + C(L)\epsilon[\epsilon(1 + \|a_2\|) + \|b_2\|]$$

which is d). $\square$

For later purposes, we derive a lower bound for B. Recalling the definition (4.18), write

$$u_{1t} = (\Delta - A'^2 t^{-1}\phi^2(\frac{x}{\sqrt{t}}))u_1 - (w(x,t) + (A^2 - A'^2)t^{-1}\phi^2(\frac{x}{\sqrt{t}}))u_1.$$

Using the Feynman-Kac formula, we deduce

$$a_1'(x) \geq (L^{-2\mathcal{L}_{A'}} a_1)(x) e^{-C(L)(\||w\|| + |A^2 - A'^2|)}$$

and thus a lower bound

$$B' \geq L^{-2E_{A'}} B e^{-C(L)(\||w\|| + |A^2 - A'^2|)}. \tag{4.22}$$

Proof of Proposition 3 We decompose a_i^n as in (4.13), (4.14) and derive bounds for A_n, B_n and b_i^n using Lemma 4.3. Set

$$nE_n = \sum_{m=0}^{n-1} E_{A_m}.$$

Note that $A_n \geq A$, so that $E_{A_n} \geq E_A > 0$. Let $\eta < \min(\delta, E_A)$. Then there exists a constant $C(L)$ (depending possibly on L but not on n) such that

$$
\begin{aligned}
0 \leq A_n &\leq C(L)\|a_2\| \\
\|b_2^n\| &\leq C(L)L^{-2n\eta}\|a_2\| \\
0 \leq B_n &\leq C(L)L^{-2nE_n}\|a_1\| \\
\|b_1^n\| &\leq C(L)L^{-2n(E_n+\eta)}(\|a_1\| + \epsilon\|a_2\|) \\
\epsilon_n &= \|a_1^n\|\|a_2^n\| \leq C(L)L^{-2nE_n}\epsilon.
\end{aligned} \tag{4.23}
$$

The bounds (4.23) hold by definition for $n = 0$, and the induction follows from Lemma 4.3: the bound on ϵ_n follows from the first four bounds in (4.23), and it can, in turn, be inserted in Lemma 4.3 to iterate those bounds. For B_n, we iterate $B_n \leq C(L)(1 - L^{-n\eta})L^{-2nE_n}\|a_1\|$ (which implies (4.23)), in order to control the right hand side in Lemma 4.3 c. Furthermore, the bound on ϵ_n and Lemma 4.3 a imply that

$$|A_{n+1} - A_n| \leq C(L)L^{-2nE_n}\epsilon\|a_2\| \tag{4.24}$$

and thus $A_n \to A^*$, for some A^*; moreover,

$$A^* = \int (a_1 + a_2)dx,$$

because $\int(a_1 + a_2)dx$ is conserved (by Lemma 2.1) and $\int a_1^n dx \to 0$, by (4.13), (4.23). Since E_A is continuous in A, by Lemma 4.1 c,

$$E_{A_n} \to E^* = E_{A^*}, \quad E_n \to E^* \tag{4.25}$$

From Lemma 4.3.c and (4.23), we get that

$$|B_{n+1} - L^{-2E_{A_n}}B_n| \leq C(L)\epsilon L^{-2n(E_n+\eta)}[\epsilon + \|a_2\|] \tag{4.26}$$

and by (4.25), there exists a B^* such that

$$B_n L^{2nE_n} \to B^* \tag{4.27}$$

By (4.22) and (4.19), (4.23), (4.24),

$$B_{n+1} \geq L^{-2E_{A_{n+1}}}B_n e^{-C(L)(\|a_2\|^2 L^{-2n\eta} + \epsilon\|a_2\|e^{-2nE_n})}$$

so, $B^* > 0$. Equations (4.23), (4.24) and (4.27) may be rewritten, using (4.4), (4.13), (4.14), as:

$$\|\sqrt{t}u_2(\sqrt{t}\cdot, t) - A^*\phi\| \leq Ct^{-\eta}\|a_2\| \tag{4.28}$$

$$\|t^{\frac{1}{2}+E_{A^*}}u_1(\sqrt{t}\cdot, t) - B^*\psi_{A^*}\| \leq Ct^{-\eta}\epsilon(\epsilon + \|a_2\|) \tag{4.29}$$

for times $t = L^{2n}, L > L_0$. For $t \in [L^{2n}, L^{2n+2}]$ we use similar estimates for the n_i in (4.11) and (4.12), and, dropping the $*$, we get (1.6, 1.7). $\square$

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Appendix 1: Proof of Lemma 4.1.

First, observe that $\mathcal{L}_A$, acting on its domain in $L^2(R^1, d\mu)$, is conjugated to a perturbation of the Hamiltonian of the harmonic oscillator:

$$e^{\frac{x^2}{8}} \mathcal{L}_A e^{-\frac{x^2}{8}} = H_A \equiv -\frac{d^2}{dx^2} + \frac{x^2}{16} - \frac{1}{4} + A^2 \phi^2(x), \tag{A.1}$$

acting in $L^2(R^1, dx)$. Hence, $\mathcal{L}_A$ has a compact resolvent, a pure point spectrum and, using the Feynman-Kac formula and the Perron-Frobenius theorem [9], a non-degenerate lowest eigenvalue. The same conclusions hold for $\mathcal{L}$.

To prove a), let us differentiate

$$\mathcal{L}_A \psi_A = E_A \psi_A \tag{A.2}$$

with respect to A^2. We get:

$$\phi^2 \psi_A + \mathcal{L}_A \psi'_A = E'_A \psi_A + E_A \psi'_A. \tag{A.3}$$

Now, we take the scalar product of (A.3) with ψ_A, and use $(\psi_A, \psi_A) = 1$ (which implies $(\psi_A, \psi'_A) = 0$), to get

$$E'_A = (\psi_A, \phi^2 \psi_A). \tag{A.4}$$

Since $\phi > 0$, $E'_A > 0$, and, for $A = 0$, we have $\psi_0 = \frac{e^{-\frac{x^2}{4}}}{\sqrt{4\pi}}$, and $E_0 = 0$. Therefore, $E_A > 0$ for $A > 0$.

To prove b), we discuss only the second claim, since the first one is similar but easier (and holds for any $q > 1$). Observe that, since E_A is non-degenerate and f is orthogonal to ψ_A, the bound would be trivial if we took the norms in $\mathcal{H}$. But functions in $\mathcal{H}$ have essentially a Gaussian decay at infinity, while those in $\mathcal{B}$ have a polynomial decay. To go from a contraction in $\mathcal{H}$ to a contraction in $\mathcal{B}$, we use an idea of [7]. First notice that, since $A^2\phi^2 \geq 0$, the Feynman-Kac formula gives

$$e^{-\tau \mathcal{L}_A}(x, y) \leq e^{-\tau \mathcal{L}_0}(x, y) \tag{A.5}$$

and $e^{-\tau\mathcal{L}_0}(x,y)$ is explicitly given by Mehler's formula [13]:

$$(e^{-\tau\mathcal{L}_0})(x,y) = (4\pi(1-e^{-\tau}))^{-\frac{1}{2}} \exp\left(-\frac{(x-e^{-\tau/2}y)^2}{4(1-e^{-\tau})}\right) \tag{A.6}$$

Hence, if a function v satisfies

$$|v(x)| \leq C(1+|x|)^{-q}, \tag{A.7}$$

for some constant C, we have

$$|(e^{-\tau\mathcal{L}_0}v)(x)| \leq C' e^{\frac{\tau}{2}}(1+|x|e^{\frac{\tau}{2}})^{-q} \tag{A.8}$$

for $|x| \geq 2\sqrt{q\tau}$ and another constant C'. Hence, the operator $e^{-\tau\mathcal{L}_0}$ contracts, for $|x|$ and τ large, any function that decays as in (A.7) with $q > 1$. By (A.5), we see that $\mathcal{L}_A$ behaves similarly. So, to prove b), we shall use the contraction in $\mathcal{H}$ for x small and (A.8) for x large. However, we need here q large, depending on E_A, hence on A. For the other bounds in Lemma 4.1, any $q > 1$ suffices.

Besides, let ϕ_n be the nth Hermite function which is an eigenvector of H_0 in (A.1) (they are of the form $P_n(x)e^{-\frac{x^2}{8}}$, where P_n is a polynomial of degree n). One can show that, for any $C > 0$, for some even $n = n(A)$ and for any $|x|$ large enough,

$$(H_A - E_A)(C\phi_n - e^{\frac{x^2}{8}}\psi_A) > 0.$$

Indeed, $(H_A - E_A)e^{\frac{x^2}{8}}\psi_A = 0$ by (A.1, A.2) and $H_A\phi_n \geq \frac{n}{2}\phi_n > 0$ (because $\phi_n > 0$ for n even and $|x|$ large). Using the maximum principle, the inequality $\frac{x^2}{16} - \frac{1}{4} - E_A > 0$, for $|x|$ large, and the fact that there exists a large $|x|$ so that $C\phi_n - e^{\frac{x^2}{8}}\psi_A > 0$, for a sufficiently big C, one concludes that ψ_A is bounded by:

$$0 \leq \psi_A(x) \leq C(A)(1+|x|^n)e^{-\frac{x^2}{4}} \tag{A.9}$$

for some $n = n(A)$, which implies that the scalar product (ψ_A, f) for $f \in \mathcal{B}$ is well-defined, if $q = q(A)$ is large enough.

To prove b), it is convenient to introduce the characteristic functions

$$\chi_s = \chi(|x| \leq \rho)$$

$$\chi_\ell = \chi(|x| > \rho)$$

where ρ will be chosen suitably below. The properties of $\mathcal{L}_A$ that we need are summarized in the following

Lemma A.1 There exist constants $C < \infty, c > 0$, such that

i) For $g \in \mathcal{B}$,

$$\|e^{-\tau\mathcal{L}_A}g\| \leq Ce^{\frac{\tau}{2}}\|g\|. \tag{A.10}$$

ii) For $g \in L^2(R^1, d\mu)$,

$$\|e^{-\mathcal{L}_A} g\| \leq C\|g\|_2, \tag{A.11}$$

where $\|\cdot\|_2$ is the norm in $L^2(R^1, d\mu)$.

iii) For g such that $\chi_s g \in L^2(R^1, d\mu)$,

$$\|\chi_\ell e^{-\rho\mathcal{L}_A} \chi_s g\| \leq e^{-\frac{\rho^2}{5}} \|\chi_s g\|_2, \tag{A.12}$$

for ρ large enough.

iv) For $g \in \mathcal{B}$,

$$\|\chi_\ell e^{-\rho\mathcal{L}_A} g\| \leq e^{-cq\rho}\|g\|, \tag{A.13}$$

for ρ large enough, and $q > 1$.

Now, take $f \in \mathcal{B}, (\psi_A, f) = 0, \|f\| = 1$. We set $\tau_n = n\rho$, and, using the Lemma, we prove inductively that there exists a $\delta > 0$ such that $v(\tau_n) = e^{-\tau_n \mathcal{L}_A} f$ satisfies, for ρ large,

$$\|\chi_s v(\tau_n)\|_2 + \|\chi_s v(\tau_n)\| \leq e^{\frac{\rho^2}{6}} e^{-\beta n}, \tag{A.14}$$

and

$$\|\chi_\ell v(\tau_n)\| \leq e^{-\beta n}. \tag{A.15}$$

with $\beta = (E_A + \delta)\rho$. Part b) of Lemma 4.1. follows from (A.14), (A.15) by taking a smaller δ, in order to bound the constants, for $\tau \geq \tau_0$, with τ_0 large (for times not of the form $\tau = n\rho$, use (A.10)). The bounds (A.14), (A.15) hold for $n = 0$, for ρ large enough, using $\|f\| = 1$ and the obvious inequality

$$\|\chi_s f\|_2 \leq e^{\frac{\rho^2}{8}} \|f\|. \tag{A.16}$$

So, let us assume (A.14), (A.15) for some $n \geq 0$ and prove it for $n+1$. Let $v = v(\tau_n)$ and write

$$v = \chi_s v + \chi_\ell v \equiv v_s + v_\ell.$$

For all n, $(v(\tau_n), \psi_A) = 0$, so that $|(v_s, \psi_A)| = |(v_l, \psi_A)| \leq C(A)\rho^{-(q-1)}$, where we use (A.9) to derive the last inequality. Then, we get

$$\begin{aligned} \|e^{-\rho\mathcal{L}_A} v_s\|_2 &\leq (C(A)\rho^{-(q-1)} e^{-\rho E_A} + e^{-\rho E'_A})\|v_s\|_2 \\ \leq\ & e^{-\rho(E_A + 2\delta)}\|v_s\|_2 \leq \tfrac{1}{4} e^{\frac{\rho^2}{6}} e^{-\beta(n+1)} \end{aligned} \tag{A.17}$$

where E'_A is the second lowest eigenvalue of $\mathcal{L}_A$ and, in the second inequality, we choose δ small, $q > \delta\rho$ and ρ large. For the third inequality, we used (A.14) and ρ large, so that $e^{-\delta\rho} \leq \frac{1}{4}$. Combining (A.11) and (A.17), we get

$$\|e^{-\rho\mathcal{L}_A} v_s\| \leq C\|e^{-(\rho-1)\mathcal{L}_A} v_s\|_2 \leq Ce^{-(\rho-1)(E_A+2\delta)} e^{\frac{\rho^2}{6}} e^{-\beta n} \leq \frac{1}{4} e^{\frac{\rho^2}{6}} e^{-\beta(n+1)} \tag{A.18}$$

again, for $e^{-\delta\rho}$ small enough. Finally, from (A.10) and (A.15), we have

$$\|e^{-\rho\mathcal{L}_A}v_\ell\| \leq Ce^{\frac{\rho}{2}}e^{-\beta n} \tag{A.19}$$

and, from this and (A.16), we get

$$\|\chi_s e^{-\rho\mathcal{L}_A}v_\ell\|_2 \leq Ce^{\frac{\rho^2}{8}}e^{\frac{\rho}{2}}e^{-\beta n}. \tag{A.20}$$

Combining (A.17)-(A.20), one gets (A.14), with n replaced by $n+1$ for ρ large enough. On the other hand, (A.15), with n replaced by $n+1$, follows immediately from (A.14), (A.15) and (A.12), (A.13), taking $cq > E+\delta$. We choose the constants as follows: take δ small and ρ large and $q > \delta\rho > \frac{E+\delta}{c}$.

Turning to c), we observe that (A.4) and (1.4) imply that $E'_A \leq (4\pi d)^{-1}$. Next, (A.3) implies that

$$\psi'_A = (\mathcal{L}_A - E_A)^{-1}(E'_A - \phi^2)\psi_A \tag{A.21}$$

where

$$(\mathcal{L}_A - E_A)^{-1} = \int_0^\infty e^{-\tau(\mathcal{L}_A - E_A)}d\tau \tag{A.22}$$

is a bounded operator on the subspace $\{f \in \mathcal{B} | (\psi_A, f) = 0\}$, because of b) above. Also, (A.4) means that $(\psi_A, (E'_A - \phi^2)\psi_A) = 0$. Hence, the norm of the right hand side of (A.21) is bounded, and we have

$$\|\psi_A - \psi'_A\| \leq C(A, A')|A^2 - A'^2| \leq C(K)|A - A'| \tag{A.23}$$

for $A, A' \leq K$. Since (A.9) shows that P_A is well-defined and bounded in $\mathcal{B}$, point c) is proven. Point d) is an explicit computation ($\mathcal{L}\phi = 0$), and the proof of e) is similar to the one of point b), since $\int f dx = 0$ means that $(f, \phi) = 0$ where $(\cdot,\cdot)$ is the scalar product in $L^2(R^1, e^{\frac{x^2}{4d}}dx)$ and ϕ, given by (1.4), is the principal eigenvalue of $\mathcal{L}$. □

We are left with the

Proof of Lemma A1. Part (i) follows immediately from (A.5) and (A.6).

For (ii), we use the Cauchy-Schwarz inequality applied to

$$(e^{-\mathcal{L}_A}g)(x) = \int e^{-\mathcal{L}_A}(x,y)e^{-\frac{y^2}{8}}e^{\frac{y^2}{8}}g(y)dy$$

and the bound

$$\sup_x (1+|x|)^q \left(\int (e^{-\mathcal{L}_A}(x,y))^2 e^{-\frac{y^2}{4}}dy\right)^{1/2} < \infty, \tag{A.24}$$

which follows from (A.5) and (A.6).

For (iii) proceed as in (ii) by using Cauchy-Schwarz' inequality, but replace (A.24) by

$$\sup_{|x|>\rho}(1+|x|)^q \left(\int |e^{-\rho\mathcal{L}_A}(x,y)|^2 \chi(|y| \leq \rho)e^{-\frac{y^2}{4}}dy\right)^{\frac{1}{2}} \leq e^{-\frac{\rho^2}{5}} \tag{A.25}$$

which again follows from (A.5) and (A.6) (we can replace $\frac{1}{5}$ in (A.25) by $\frac{1}{4} - \epsilon$ for any $\epsilon > 0$, if ρ is large enough).

Finally, (iv) follows from (A.5) and (A.8). (Since it is enough to have $q > \delta\rho$ we have $|x| > \rho \geq 2\sqrt{q\rho}$ for δ small and we can use (A.8)). □

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