QUANTIZATION OF A CLASS OF PIECEWISE AFFINE TRANSFORMATIONS ON THE TORUS

S. De Bièvre [1], M. Degli Esposti [2], R. Giachetti [3],

[1] UFR de Mathématiques et Laboratoire de Physique Théorique et Mathématique Université Paris VII - Denis Diderot 75251 Paris Cedex 05 France,
debievre@mathp7.jussieu.fr

[2] Dipartimento di Matematica, Università di Bologna,
Porta di Piazza San Donato 5, 40127 Bologna, Italy,
desposti@dm.unibo.it

[3] Dipartimento di Fisica, Università di Firenze,
Largo E. Fermi 2, 50125 Firenze, Italy,
giachetti@fi.infn.it

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Abstract. We present a unified framework for the quantization of a family of discrete dynamical systems of varying degrees of "chaoticity". The systems to be quantized are piecewise affine maps on the two-torus, viewed as phase space, and include the automorphisms, translations and skew translations. We then treat some discontinuous transformations such as the Baker map and the sawtooth-like maps. Our approach extends some ideas from geometric quantization and it is both conceptually and calculationally simple.

1. Introduction.

Interest in the quantization of discrete dynamical systems on compact phase spaces comes from the desire to understand the possible signature of classical chaotic dynamics in quantum mechanics. Recall for example that it is expected and in some cases proved that the asymptotic properties ($\hbar \to 0$) of the eigenfunctions of quantized systems depend on the degree of "chaoticity" of the corresponding classical ones

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(see, for instance, [Sar] and references therein). The torus forms an excellent testing ground for these ideas. Indeed, the simplest ergodic systems are the irrational translations on the torus, whereas the simplest hyperbolic dynamical systems are certain area-preserving maps [AA,CFS]. Among these, the best known are the toral automorphisms, the Baker transformation and some discontinuous maps such as the sawtooth map considered in [Ch,LW,V,Li]. It has been shown there that their singularities do not destroy the ergodicity and mixing properties one expects for hyperbolic maps.

One way in which the classical singularities will show up at the quantum level is as follows. For the linear automorphisms the classical and the quantum evolution are identical, as in the harmonic oscillator. The singularities will destroy this property, so that, to control the semiclassical behaviour of the eigenfunctions a non trivial Egorov theorem will be needed. Similarly, the statistics of the eigenvalues of the quantum propagator should be more generic than in the linear case, where they are determined by purely arithmetic properties. Clearly, before being able to address this kind of problems, one needs to develop a quantization for the systems considered. Since none of the above examples is obtained by evaluating a smooth Hamiltonian flow on the torus at discrete times, the usual quantization schemes all fail and a direct attack is needed.

In this paper we will show how to extend the most elementary part of geometric quantization [Bl,GuSt, Ko, Sn, Wo] beyond its natural context in order to construct a unified and simple framework for the quantization of all of the above systems. Some of them had not been quantized before, such as the translations and certain piecewise affine hyperbolic maps. It will turn out that the unitary matrices describing the quantum evolution of each of those systems can be computed straightforwardly and with relatively little effort in this way.

The toral automorphisms and the Baker transformation were quantized respectively in [HB,DE, DGI] and in [BV] and they have been studied intensely ever since, both numerically and analytically [Ke1, Ke2, Ke3, DGI, Eck, Sa]. The methods of quantization used in these papers look very different from each other. Our approach reproduces the same results in those cases.

In order to get a more precise flavour of the ideas to be developed, recall that in classical mechanics the dynamics of a system is obtained by integrating a Hamiltonian vector field X_H on a symplectic manifold (M,ω). Here $H \in C^\infty(M)$ and X_H is defined by $X_H \rfloor \omega = dH$. In quantum mechanics, the dynamics is given by a unitary flow U_t on a Hilbert space $\mathcal{H}_\hbar$. A *quantization* is a set of rules allowing to associate to (M, ω) a Hilbert space $\mathcal{H}_\hbar$ and to each function f on M in a suitable class $\mathcal{C}$, a self-adjoint operator $\widehat{f}$ on $\mathcal{H}_\hbar$. One then says that $U_t = \exp[\,(-i/\hbar)\widehat{H}t\,]$ is the quantization of the classical flow of X_H. Typical requirements [Be] are that the map $f \mapsto \widehat{f}$ is linear, injective, unital, *i.e.* that it satisfies $\widehat{1} = Id_{\mathcal{H}_\hbar}$, and that it is compatible with the natural involutions, $(\widehat{f}\,)^* = \widehat{\overline{f}}$. Moreover, one requires the classical limit condition $(1/i\hbar)\,[\,\widehat{f}, \widehat{g}\,] \underset{\hbar\to 0}{\longrightarrow} \widehat{\{f, g\}}$.

When the classical evolution is not a flow, but a discrete map, this scheme is clearly not sufficient. We extend here some of the simplest ideas of geometric quantization beyond their natural range of applicability to obtain a unified framework for the quantization of a reasonably large class of area preserving maps on the torus. We will show that, in spite of its reputation, the essence of geometric quantization is intuitive, simple and well suited for such generalizations. For that purpose, we first present in Section 2 a revisited version of the geometric quantization on $T^*\mathbb{R}$, just to demonstrate how it permits to *reformulate* quantum mechanics for systems having $T^*\mathbb{R}$ as phase space and to quantize linear flows. At several points, we shall use physical or intuitive arguments to motivate parts of the construction that are usually justified in terms of very general geometrical objects. We then apply this approach to the quantization of toral automorphisms in Section 3: the resulting quantum propagators are identical to the ones obtained elsewhere by other methods [HB,DE]. In the final Section 4 we shall obtain the quantization of translations, skew-translations as well as of a class of *piecewise linear* hyperbolic maps such as the Baker transformation and other maps studied, for instance, in [Ch,LW,Li,V]. Those maps do not preserve the natural geometric structures associated with the torus, and therefore geometric quantization as such does not apply to them. The proposed extension, however, will provide a definite answer.

2. Geometric quantization on $T^*\mathbb{R}$.

As usual we call (q, p) the coordinates of $T^*\mathbb{R} \cong \mathbb{R}^2$ and choose the standard symplectic form $\omega = dq \wedge dp$ that gives the canonical Poisson bracket $\{q, p\} = 1$. Our goal is to realize the space of the quantum states $\mathcal{H}_\hbar$ as a subspace of $\mathcal{S}'(\mathbb{R}^2)$, equipped with a suitable Hilbert space structure, and to establish a correspondence between classical and quantum observables, so as to be able to describe the physical properties of the quantum system. To this purpose we recall a first result, the validity of which is easily checked by a direct computation: there exists a map $f \in C^\infty(\mathbb{R}^2) \to \widehat{f} \in L(\mathcal{S}'(\mathbb{R}^2), \mathcal{S}'(\mathbb{R}^2))$, which is linear, unital and satisfies the classical limit condition. This map is explicitly given by

$$\widehat{f} = -i\hbar \nabla_{X_f} + f, \tag{2.1}$$

where $X_f = (\partial_p f)\,\partial_q - (\partial_q f)\,\partial_p$ is the Hamiltonian vector field associated to f and $\nabla_X = X - (i/\hbar)\, X \rfloor \theta$ denotes the covariant derivative with respect to the connection form $\theta = \frac{1}{2}(pdq - qdp)$. Note that the use of ∇_X guarantees the local gauge invariance of the construction (see [Wo, Sn] for details). It is moreover worth remarking that, if $\hat{f}$ in (2.1) is replaced by $-i\hbar X_f$, then the unital property fails to hold, thereby violating the uncertainty principle. In particular we have $\widehat{q} = i\hbar\,\partial_p + q/2$ and $\widehat{p} = -i\hbar\,\partial_q + p/2$, so that, indeed, the canonical commutation relation $[\widehat{q}, \widehat{p}] = i\hbar$ is satisfied. The correspondence between f and $\widehat{f}$ given in (2.1) is referred to as *prequantization* [Ko].

We now need some conditions to choose the subspace $\mathcal{H}_\hbar$ of $\mathcal{S}'(\mathbb{R}^2)$ and the Hilbert space structure it has to carry for it to correspond to the quantum Hilbert space of states. Note first that the equation $i\hbar\partial_t\psi_t = \widehat{f}\psi_t$ is easily solved on $\mathcal{S}'(\mathbb{R}^2)$. Writing $\psi_t = \exp[-(i/\hbar)\widehat{f}t]\,\psi$, one has

$$\begin{aligned}&\big(\exp[\frac{i}{\hbar}\,\widehat{f}t\,]\,\psi\big)(q,p) = \\ &\qquad \exp[-\frac{i}{\hbar}\int_0^t ds\,\big(\frac{1}{2}(p(s)\dot{q}(s) - q(s)\dot{p}(s)) - f(q(s),p(s))\big)\,]\,\psi(q(t),p(t))\,.\end{aligned}$$

where $(q(s), p(s))$ is the solution of the Hamilton equations $\dot{q} = \partial_p f$, $\dot{p} = -\partial_q f$, with initial conditions (q, p). Note that the prequantized flow $\exp[\,(i/\hbar)\widehat{f}t\,]$ makes sense also when $\psi \in \mathcal{S}'(\mathbb{R}^2)$.

The idea is then to try to pick $\mathcal{H}_\hbar$ in such a way that $\exp[\,(i/\hbar)\widehat{f}t\,]$ is a unitary one-parameter group for a suitable large class $\mathcal{C}$ of functions f. This allows then for the interpretation of $\widehat{f}$ as the quantized observable.

An a priori obvious choice would be $L^2(\mathbb{R}^2, \frac{dqdp}{2\pi\hbar})$. It is nevertheless not suitable as the quantum Hilbert space. Indeed it is easily seen that the spectra of $\widehat{q}$ and $\widehat{p}$ are not simple: actually, the generalized eigenspaces are infinite dimensional, which is in contradiction with standard quantum mechanics on $L^2(\mathbb{R})$. Otherwise stated, $\widehat{q}$ (or $\widehat{p}$) is not a complete set of commuting observables on $L^2(\mathbb{R}^2, \frac{dqdp}{2\pi\hbar})$, or, equivalently, $\widehat{q}$

and $\widehat{p}$ do not generate an irreducible algebra. To put this more precisely, recall that the Heisenberg group is the group $H = \mathbb{R}^3$ (as a set) equipped with the group law $(a,b,\phi)(a',b',\phi') = (a+a', b+b', \phi+\phi'+\frac{1}{2}(ab'-a'b))$. H acts on $\mathbb{R}^2$ by $(a,b,\phi)(q,p) = (q+a, p+b)$. The prequantized operators $\hat{q}$, $\hat{p}$ generate a unitary representation of H on $L^2(\mathbb{R}^2, \frac{dqdp}{2\pi\hbar})$ given explicitly by

$$[U(a,b,\phi)\,\psi](q,p) = \exp[-\frac{i}{\hbar}\phi]\ \exp[-\frac{i}{2\hbar}(ap-bq)]\ \psi(q-a,p-b). \tag{2.2}$$

This representation is not irreducible on $L^2(\mathbb{R}^2, \frac{dqdp}{2\pi\hbar})$.

There is a second problem with (2.1) which is worthwhile mentioning. It is easy to see that, if $H(q,p) = p^2/2 + V(q)$, then $\widehat{H} \neq \widehat{p}^{\,2}/2 + V(\widehat{q})$. It is then clear that the correspondence (2.1) is far from reproducing the Schrödinger equation.

Some conditions have to be imposed on the quantum Hilbert space $\mathcal{H}_\hbar \subset \mathcal{S}'(\mathbb{R}^2)$ in order to avoid the previous difficulties. For the irreducibility of the algebra generated by $\hat{q}$ and $\hat{p}$ we should require

(i) $U(a,b,\phi)$ restricts to a unitary irreducible representation of H on $\mathcal{H}_\hbar$.

To reproduce the Schrödinger equation we should impose:

(ii) $\exists n_0 \in \mathbb{N}^*$, and a dense subspace D of $\mathcal{H}_\hbar$ so that $\widehat{p}$, $\widehat{p}^{\,2}$, and $\widehat{q}^{\,n}$ $(1 \leq n \leq n_0)$ are essentially self-adjoint on D and $\widehat{p}^{\,2} = \widehat{p^2}$, $\widehat{q}^{\,n} = \widehat{q^n}$ on D.

Note that this is equivalent to requiring the correct form of the Schrödinger equation for all polynomial potentials of order at most n_0. We are however already asking too much if we take $n_0 \geq 2$, as we now show.

Proposition 2.1. *If $\psi \in \mathcal{S}'(\mathbb{R}^2)$ satisfies $\widehat{p}^{\,2}\psi = \widehat{p^2}\psi$ and $\widehat{q}^{\,2}\psi = \widehat{q^2}\psi$, then $\psi = 0$.*

The proof of this proposition is a simple calculation that we omit. In conclusion, the requirements $(i-ii)$ can not be satisfied on any non trivial subspace of $\mathcal{S}'(\mathbb{R}^2)$. Hence we can not even quantize in the proposed manner Hamiltonians with quadratic, let alone general polynomial potentials. The best we can still hope to do is to impose (i) and a weakened version of (ii), as we now explain.

Given $w \in \mathbb{R}^2$, with $w = (w_1, w_2)$, let $v \in \mathbb{R}^2$ such that $\omega(w,v) = 1$, we define the subspace

$$\mathcal{D}_w = \{\psi \in \mathcal{S}'(\mathbb{R}^2) \mid \nabla_w \psi = 0\}, \tag{2.3}$$

where X_{h_w} is the Hamiltonian vector field associated to $h_w(x) = {}^T\!w\,x = w_1 q + w_2 p$ and $\nabla_w := \nabla_{X_{h_w}}$. Here and in the following $x \equiv (q,p)$. We then have

Lemma 2.1. *Let $w \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$ such that $\omega(w,v) = 1$. Then $\psi \in \mathcal{D}_w$ if and only if there exists a tempered distribution f_v on the line such that*

$$\psi(x) = f_v(h_w(x)) \, \exp[\, -\frac{i}{2\hbar} \, h_w(x) h_v(x) \,]. \tag{2.4}$$

Proof. We have $\nabla_w = (w_2 \partial_q - w_1 \partial_p) - (i/2\hbar) h_w(x)$. Consider the map $(q,p) \mapsto (y_1, y_2) = (h_w(x), h_v(x))$ which is linear and with determinant equal to unity. $\nabla_w \psi = 0$ becomes $\partial_{y_2} \eta(y_1,y_2) = -(i/2\hbar)\, y_1 \, \eta(y_1,y_2)$, with $\eta(y_1,y_2) = \psi(q,p)$. Its general solution is $\eta(y_1,y_2) = f_v(y_1) \, \exp[\, -(i/2\hbar)\, y_1 y_2 \,]$, thus proving the lemma. $\square$

Remark. If $v' \in \mathbb{R}^2$ satisfies $\omega(w,v') = 1$, then $v' = v + rw$, with $r \in \mathbb{R}$. It is easy to see that $f_{v'}(y) = \exp[\,(i/2\hbar) r y^2\,] \, f_v(y)$. We will therefore omit the indication of the dependence of f on v. We then have the following Lemma.

Lemma 2.2. *Let $\psi \in \mathcal{S}'(\mathbb{R}^2)$ and $w \in \mathbb{R}^2$. Then the following are equivalent:*

1. $\widehat{h}_w^{\,2} \psi = \widehat{h_w^2} \psi$;
2. $\widehat{h}_w^{\,n} \psi = \widehat{h_w^n} \psi$, *for all $n \in \mathbb{N}$;*
3. $\psi \in \mathcal{A}_w := \{\eta \in \mathcal{S}'(\mathbb{R}^2) \mid (\nabla_{X_w})^2 \eta = 0\}$;
4. *Let $v \in \mathbb{R}^2$ such that $\omega(w,v) = 1$. Then there exist $\varphi_0, \varphi_1 \in \mathcal{S}'(\mathbb{R})$ such that*

$$\psi(x) = \big(h_v(x) \, \varphi_0(h_w(x)) + \varphi_1(h_w(x)) \big) \, \exp[\, -\frac{i}{2\hbar} \, h_w(x) h_v(x) \,].$$

Moreover, if $u \in \mathbb{R}^2$, then $\widehat{h}_u \mathcal{A}_w \subset \mathcal{A}_w$.

Proof. A direct calculation shows $\widehat{h_w^n} = -i\hbar n \, h_w^{n-1} \nabla_w + h_w^n$. Using $[\nabla_w, h_w(x)] = 0$ to compare $\widehat{h_w^n}$ to $(\widehat{h_w})^n$, and the previous lemma, the result follows easily. $\square$

The lemma suggests to weaken (ii) by imposing, $(\widehat{h_w})^n = \widehat{h_w^n}$, for some choice of w. This would imply $D \subset \mathcal{A}_w$. Now it is not hard to see that the eigenvalues of $\hat{q}$ and $\hat{p}$ on $\mathcal{A}_w$ are doubly degenerate. In order to satisfy (i) it would be natural to pick D in a subspace of $\mathcal{A}_w$ on which this degeneracy is lifted. It is easy to describe all subspaces of $\mathcal{A}_w$ that are, like $\mathcal{A}_w$ itself, invariant under all $\widehat{h}_u$, and on which the eigenvalues of all $\widehat{h}_u$ are non-degenerate. Although there seems to be no physical criteria permitting to select one such subspace, $\mathcal{D}_w$ (see (2.3)) satisfies the above requirements and it is customary in geometric quantization to construct $\mathcal{H}_w$ as a subspace of $\mathcal{D}_w$ because of its geometric appeal. The condition $\nabla_w \psi = 0$ is called a *polarization condition* in this context. Note that we can identify $\mathcal{D}_w$ with $\mathcal{S}'(\mathbb{R})$ and that $\widehat{h}_w$ then acts as a multiplication operator while $\widehat{h}_v$ as a derivative operator. A calculation as in the proof of Proposition 2.1 shows that if $u \in \mathbb{R}^2$ is not a multiple of w, then $\widehat{h_u^2}\,\mathcal{D}_w \cap \mathcal{D}_w = \{0\}$, thus excluding a priori the quantization of quadratic Hamiltonians, as already pointed out.

Let us now briefly show how one can nevertheless correctly describe the quantization of quadratic Hamiltonians within the framework of geometric quantization (see [GuSt, Wo] for details). Recall that for a quadratic polynomial $f(q,p) = (\lambda/2)q^2 + \mu qp + (\nu/2)p^2$ the flow of X_f is linear and can be written as ${}^T(q(t),p(t)) = A(t)\,{}^T(q,p)$, with $A(t) \in SL(2,\mathbb{R})$ (^{T} denotes the transpose). The prequantized flow then reads

$$\big(\exp[\,\frac{i}{\hbar}\,\widehat{f}t\,]\,\psi\big)(q,p) = \psi(A(t)\begin{pmatrix} q \\ p \end{pmatrix}) \;:= (U(A^{-1}(t))\psi)(q,p)$$

and the map $A \mapsto U(A)$ gives a unitary representation of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R}^2, \frac{dqdp}{2\pi\hbar})$. We now observe that $U(A)$ satisfies $U(A)\mathcal{D}_w = \mathcal{D}_{\,{}^TA^{-1}w}$. We will explain below that it is possible to equip a suitable subspace $\mathcal{H}_w$ of $\mathcal{D}_w$ with a Hilbert space structure and then to identify the Hilbert spaces for different values of w by means of unitary maps $P_{zw} : \mathcal{H}_w \to \mathcal{H}_z$. This is a particular case of a general construction which allows to compare Hilbert spaces corresponding to different real or complex polarizations (BKS kernels [Wo,GuSt, Sn]). The quantized linear transformation $V(A)$ is then defined by $V(A) = D_\hbar P_{w,{}^TA^{-1}w} \circ U(A) : \mathcal{H}_w \to \mathcal{H}_w$ (see (2.8)).

We start by showing how to equip suitable subspaces $\mathcal{H}_w$ of the $\mathcal{D}_w$ with a Hilbert space structure. Note first that (2.4) implies that if $\psi_1, \psi_2 \in \mathcal{D}_w$, then $\bar{\psi}_1\psi_2$ is a function of $h_w(x)$. Moreover

$$[\,\overline{U(a,b,\phi)\psi_1}\,U(a,b,\phi)\psi_2\,](q,p) = \bar{f}_1 f_2(h_w(x) - aw_1 - bw_2).$$

This suggests defining a Hilbert subspace $\mathcal{H}_w$ of $\mathcal{D}_w$ by

$$\mathcal{H}_w = \{\psi \in \mathcal{D}_w | \int |\psi|^2(y)\,dy \,<\, \infty\}, \tag{2.5}$$

equipped with the obvious scalar product $\langle \psi_1, \psi_2 \rangle_w \;:= \int \bar{\psi_1}\psi_2(y)\,dy$. The choice of the Lebesgue measure in (2.5) is dictated by the requirement that $U(a,b,\phi)$ be unitary on $\mathcal{H}_w$.

Let $w = (w_1, w_2)$ and $z = (z_1, z_2)$ be linearly independent and consider the two corresponding Hilbert spaces $\mathcal{H}_w$ and $\mathcal{H}_z$. We denote by $v = (v_1, v_2)$ and $u = (u_1, u_2)$ two fixed vectors such that $\omega(w,v) = \omega(z,u) = 1$. Consider $\psi \in \mathcal{H}_w$ and $\varphi \in \mathcal{H}_z$. It is then easy to see that $\bar{\varphi}\psi$ belongs to $L^1(\mathbb{R}^2, dq\,dp)$. The following proposition then follows from a straightforward calculation that we omit [GuSt].

Proposition 2.2. *Let $w, z \in \mathbb{R}^2$ be linearly independent. Let $\Delta = \omega(w, z)$. Then there exists a unique continuous linear map $P_{zw} : \mathcal{H}_w \to \mathcal{H}_z$ such that, $\forall \psi \in \mathcal{H}_w$ and $\forall \varphi \in \mathcal{H}_z$*

$$\langle \varphi, P_{zw}\psi \rangle_z = \int \bar{\varphi}\psi \, \frac{dq\, dp}{2\pi\hbar} \,. \tag{2.6}$$

Moreover, if $D_\hbar \in \mathbb{C}$, with $|D_\hbar| = \sqrt{2\pi\hbar\,|\Delta|}$, then $D_\hbar P_{zw}$ is unitary.

The proof of the proposition provides an explicit expression for P_{zw}:

$$(P_{zw}\,\psi)(x) = \frac{1}{2\pi\hbar\Delta} \exp\,[\,-\frac{i}{2\hbar}\,h_z(x)h_u(x)\,] \int f(y) \,\exp\,[\,-\frac{i}{\hbar}S_{zw}(y, h_z(x))\,]\, dy\,,$$

where S_{zw} is the quadratic form

$$S_{zw}(y_1, y_2) = \frac{1}{2\Delta}\left[\,(y_1, y_2)\begin{pmatrix} \omega(v,z) & 1 \\ 1 & \omega(u,w) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]. \tag{2.7}$$

Note that P_{zw} extends to $\mathcal{D}_w$ (see [Fo]).

The previous result allows to associate to any linear map $A \in SL(2,\mathbb{R})$ and to any given $z \in \mathbb{R}^2$ a well defined unitary operator, unique up to a phase, in the following manner. Given $A \in SL(2,\mathbb{R})$ and $z \in \mathbb{R}^2$, it follows immediately that $\forall \psi \in \mathcal{H}_z$ of the form (2.4), we have

$$(U(A)\,\psi)(x) = f(h_{\,{}^TA^{-1}z}(x))\,\exp\,[\,-\frac{i}{2\hbar}\,h_{\,{}^TA^{-1}z}(x)\,h_{\,{}^TA^{-1}u}(x)\,],$$

where $U(A)$ is the previously defined prequantum action. Hence $U(A)\mathcal{H}_z = \mathcal{H}_{\,{}^TA^{-1}z}$ and we can define $V(A) : \mathcal{H}_z \to \mathcal{H}_z$ by

$$V(A) = D_\hbar P_{z,{}^TA^{-1}z} \circ U(A). \tag{2.8}$$

$V(A)$ is an unitary integral operator representing the quantum propagator associated to the classical symplectic transformation A. Indeed, to see that it agrees with Schrödinger quantum mechanics (up the the choice of a phase), note that in the case where $z = (1,0)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $b \neq 0$, we recover the well known formula ($u = (0,1)$, $w = (d,-b)$, $v = (-c,a)$, $\Delta = b$) for the integral kernel of $V(A)$, *i.e.*

$$V(A)(y_1, y_2) = \frac{1}{\sqrt{2\pi\hbar b}} \exp\,[\frac{i}{2\hbar b}(a y_1^2 - 2y_1y_2 + d y_2^2)\,]\,.$$

The correct phase for $V(A)$ is not obtained by the very simple approach we have presented. This can be done with a considerable amount of additional work [Fo, GuSt]: this problem is however of no concern in the present framework, since a global phase does not change the quantum dynamics of a single transformation.

3. Quantization of toral automorphisms.

We shall now apply our previous construction to quantization on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with canonical symplectic structure $\omega_{\mathbb{T}^2}$, such that $d\pi^*\,\omega_{\mathbb{T}^2} = dq \wedge dp$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the usual covering map. In the first place, we need to identify the quantum Hilbert space. The periodicity of the system in q and in p will be taken into account along the same lines well known in solid state physics, namely by considering distributions on $\mathbb{R}^2$ with quasiperiodic boundary conditions both in q and p. This approach is calculationally convenient and we shall show its equivalence with the geometric quantization procedure. It has the advantage of being readily extendable to the geometrically non-natural situations of Section 4.

Let us introduce $u_1 = U(1,0,0)$ and $u_2 = U(0,1,0)$ as in (2.2). Given $\hbar \in \mathbb{R}^+$ and $\theta \in \mathbb{T}^2$, we denote by $\mathcal{S}'_\hbar(\theta)$ the space of all the tempered distributions ψ on the plane satisfying the following conditions:

$$u_1\,\psi(q,p) = \exp[\,2\pi i\,\theta_1\,]\,\psi(q,p)\,, \qquad u_2\,\psi(q,p) = \exp[\,2\pi i\,\theta_2\,]\,\psi(q,p)\,. \tag{3.1}$$

Computing $(u_1u_2 - u_2u_1)\psi$ using (2.2) and (3.1) one can easily see that this space is non trivial if and only if $2\pi\hbar N = 1$ for some $N \in \mathbb{N}$. We shall refer to this as the *prequantum condition* and, from now on, we shall assume it to be satisfied. In this case, $\forall n = (n_1,n_2) \in \mathbb{Z}^2$ and $\psi \in \mathcal{S}'_\hbar(\theta)$,

$$\psi(x+n) = \exp\left[\,-2\pi i\,(\theta_1 n_1 + \theta_2 n_2)\,\right]\,\exp\left[\,\frac{i}{2\hbar}(qn_2 - pn_1)\,\right]\exp\left[\,-\frac{i}{2\hbar}n_1n_2\,\right]\psi(x)\,, \tag{3.2}$$

where, as in Section 2, $x = (q,p)$. Given now $\psi(q,p) \in \mathcal{S}'_\hbar(\theta)$ and $w \in \mathbb{R}^2$, one checks readily that $\nabla_w\psi \in \mathcal{S}'_\hbar(\theta)$. We then define, in analogy with (2.3), the corresponding space of linearly polarized sections $\mathcal{D}_w(\theta,N) = \mathcal{S}'_\hbar(\theta) \cap \mathcal{D}_w = \{\psi \in \mathcal{S}'_\hbar(\theta) |\, \nabla_w\psi = 0\}$. We will only consider polarizations of the torus for which $w_2/w_1 \in \mathbb{Q}$. This is equivalent to requiring that the flow lines of X_w are circles. In this case, up to rescaling w by a constant multiple, we can assume $w = (w_1,w_2) \in \mathbb{Z}^2$ with $g.c.d.\,(w_2,w_1) = 1$.

Theorem 3.1. *Let $w = (w_1,w_2) \in \mathbb{Z}^2$ as above, where $g.c.d.(w_1,w_2) = 1$. Then $\mathcal{D}_w(\theta,N)$ is a complex vector space of dimension N. Choosing $v \in \mathbb{Z}^2$ with $\omega(w,v) = 1$, any $\psi \in \mathcal{D}_w(\theta,N)$ can be written uniquely as*

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k\,\exp\left[\,-i\pi N\,h_w(x)h_v(x)\,\right]\delta(h_w(x) - q_k(w,\theta))\,, \tag{3.3}$$

where

$$q_k(w,\theta) = k/N - (1/2)\, w_1 w_2 + (1/N)\, \omega(w,\theta)\,, \tag{3.4}$$

and $\forall k \in \mathbb{Z}$, *the* $c_k \in \mathbb{C}$ *satisfy*

$$c_{k+N} = \exp\left[\, 2\pi i\, \alpha_\theta(N,w)\, \right] c_k\,, \tag{3.5}$$

with

$$\alpha_\theta(N,w) = (N/2)\, v_1 v_2 + \omega(v,\theta)\,. \tag{3.6}$$

Conversely, any $\psi \in \mathcal{S}'(\mathbb{R}^2)$ *of the form* (3.3) *with the* c_k*'s satisfying* (3.5 − 6) *belongs to* $\mathcal{D}_w(\theta,N)$.

Proof. Let $\psi \in \mathcal{D}_w(\theta,N)$, then Lemma 2.1. implies that it is of the form

$$\psi(x) = f(h_w(x))\, \exp\left[\, -i\pi N\, h_w(x) h_v(x)\, \right], \tag{3.7}$$

where $v \in \mathbb{R}^2$ is chosen such that $\omega(w,v) = 1$. It will be convenient to take $v \in \mathbb{Z}^2$. Note that, since $g.c.d.\,(w_1,w_2) = 1$, such v always exists. Using (3.2) and making the simple observation that

$$h_w(a)h_v(b) - h_v(a)h_w(b) = \omega(a,b) \quad \forall a,b \in \mathbb{R}^2\,, \tag{3.8}$$

one obtains, for any $n \in \mathbb{Z}^2$ and $t \in \mathbb{R}$, that f must satisfy

$$\begin{aligned} f(t + h_w(n)) = &\exp\left[\, i\pi N\, (2t h_v(n) + h_w(n) h_v(n))\, \right] \\ &\exp\left[\, -i\pi N\, n_1 n_2\, \right]\, \exp\left[\, -2\pi i\, h_n(\theta)\, \right] f(t)\,. \end{aligned} \tag{3.9}$$

Choosing $n = m$, where $m = (-w_2, w_1)$, and noting that $h_w(m) = 0$, $h_v(m) = \omega(w,v) = 1$, $h_m(\theta) = \omega(w,\theta)$, one concludes that f is of the form ($c_k \in \mathbb{C}$)

$$f(t) = \sum_{k\in\mathbb{Z}} c_k\, \delta(t - q_k(w,\theta))\,, \tag{3.10}$$

where the $q_k(w,\theta)$ are given in (3.4). Therefore (3.9) yields

$$\sum_{k\in\mathbb{Z}} c_k\, \delta(t - q_k + h_w(n)) = \sum_{k\in\mathbb{Z}} c_k\, \exp\left[\, 2\pi i\, \beta_\theta(k,n,N)\, \right] \delta(t - q_k)\,,$$

where

$$\beta_\theta(k,n,N) = N q_k h_v(n) + \frac{N}{2}\, h_w(n) h_v(n) - \frac{N}{2}\, n_1 n_2 - h_n(\theta)\,. \tag{3.11}$$

Note that $\beta_\theta(k,n,N)$ depends on k only through a term $kh_v(n) = 0 \mod 1$. Clearly we can drop this term and replace $\beta_\theta(k,n,N)$ by $\beta_\theta(n,N)$ defined by

$$\beta_\theta(n,N) = -\frac{N}{2}w_1w_2h_v(n)+\omega(w,\theta)h_v(n)+\frac{N}{2}\big(h_w(n)h_v(n)-n_1n_2\big)-h_n(\theta). \quad (3.12)$$

Observing that $q_k - h_w(n) = q_{k-Nh_w(n)}$ (see (3.4)), we find the following condition on the c_k, $\forall n \in \mathbb{Z}^2$ and $\forall k \in \mathbb{Z}$:

$$c_{k+Nh_w(n)} = \exp\left[\,2\pi i\,\beta_\theta(n,N)\,\right]c_k. \quad (3.13)$$

We will show that the solution space of (3.13) is exactly N-dimensional.

First note that, for (3.13) to have any non-trivial solution at all, it is necessary that

$$\beta_\theta(n,N) = \beta_\theta(\widetilde{n},N) \mod 1\,, \quad (3.14)$$

whenever $h_w(n) = h_w(\widetilde{n})$, *i.e.* whenever $\exists r \in \mathbb{Z}$ so that $\widetilde{n} = n + rm$, $(m = (-w_2, w_1))$. To prove (3.14), remark first that $\forall n, n' \in \mathbb{Z}^2$

$$\beta_\theta(n+n',N) = \beta_\theta(n,N) + \beta_\theta(n',N) \mod 1\,. \quad (3.15)$$

This follows immediately from (3.11) upon using (3.8). Moreover, one has

$$\beta_\theta(m,N) = N\left[-\frac{1}{2}\,w_1w_2 + \frac{1}{N}\,\omega(w,\theta)\,\right] + \frac{N}{2}\,w_1w_2 - \omega(w,\theta) = 0 \mod 1\,.$$

This, together with (3.15) implies (3.14). We can then choose $c_0, c_1, \cdots, c_{N-1}$ freely and define c_k for all other k using (3.13). To assure that the resulting solutions satisfy (3.13) $\forall k \in \mathbb{Z}$, and not only for $k = 0, 1, \cdots, N-1$, condition (3.15) is necessary and sufficient.

Finally, to compute $\alpha_\theta(N)$, note that $\alpha_\theta(N) = \beta_\theta(n,N)$ for any $n \in \mathbb{Z}^2$ such that $h_w(n) = 1$. If we take $n = (v_2, -v_1)$, then $h_v(n) = 0$, and (3.12) yields (3.6). $\square$

Remark. In particular, if $w = (1,0)$, it easy to see that the corresponding space of polarized sections contains all distributions of the form $f(q) = \sum_k c_k\,\delta(q - k/N - \theta_2/N)$, where $c_{k+N} = \exp\left[\,-2\pi i\,\theta_1\,\right]c_k$.

Given now $w, v \in \mathbb{Z}^2$ and $\theta \in \mathbb{T}^2$ as before, the previous proposition allows us to identify the space of sections $\mathcal{D}_w(\theta,N)$ with $\mathbb{C}^N$, as follows:

$$(c_0, \cdots, c_{N-1}) \in \mathbb{C}^N \mapsto \psi(q,p) \in \mathcal{D}_w(\theta,N), \quad (3.16a)$$

where

$$\psi(q,p) = \sum_{k\in\mathbb{Z}} c_k \, \exp\left[\, -i\pi N \, h_w(x) h_v(x) \,\right] \delta(h_w(x) - q_k(w,\theta)). \tag{3.16b}$$

Here, for $k \notin \{0, \cdots, N-1\}$, the c_k's are defined by (3.5).

In analogy with the results of Section 2, we give $\mathcal{D}_w(\theta, N)$ a Hilbert space structure. Here also, the choice of the inner product will be dictated by the requirement that the Heisenberg group acts unitarily. We shall denote by $\mathcal{H}_w(\theta, N)$ the quantum Hilbert space thus obtained.

Setting $m = (-w_2, w_1)$ and $\widetilde{m} = (v_2, -v_1)$ it is easy to see that m and $\widetilde{m}$ form a basis of $\mathbb{R}^2$ and, in addition, that $\forall n \in \mathbb{Z}^2$, there exist unique $\alpha, \beta \in \mathbb{Z}$ such that $n = \alpha m + \beta \widetilde{m}$. Moreover, by using (2.2), one computes, for all $\psi \in \mathcal{D}_w(\theta, N)$ and for any $\alpha, \beta \in \mathbb{R}$ as in (3.5),

$$\begin{aligned}
\big(U(\alpha m)\psi\big)(x) &= \sum_{k\in\mathbb{Z}} [U(\alpha m)c]_k \, \exp\left[\, -i\pi N \, h_w(x) h_v(x) \,\right] \, \delta(h_w(x) - q_k)\,, \\
\big(U(\beta \widetilde{m})\psi\big)(x) &= \sum_{k\in\mathbb{Z}} c_k \, \exp\left[\, -i\pi N \, h_w(x) h_v(x) \,\right] \, \delta(h_w(x) - (q_k + \beta))\,,
\end{aligned}$$

where

$$\big(U(\alpha m)c\big)_k = \exp\left[\, 2\pi i N \, q_k \alpha \,\right] c_k\,.$$

¿From these results and Theorem 3.1, we see that $U(a,b)\mathcal{D}_w(\theta,N) \subset \mathcal{D}_w(\theta,N)$ if and only if $N(a,b) \in \mathbb{Z}^2$ and then

$$\big(U(\frac{\ell}{N}\, m)\, c\big)_k = \exp[\, 2\pi i \, q_k \ell \,]\, c_k\,, \qquad \big(U(\frac{\ell}{N}\, \widetilde{m})\, c\big)_k = c_{k-\ell}\,. \tag{3.17}$$

The natural Hilbert structure making $U(m)$ and $U(\widetilde{m})$ unitary is given by

$$\langle \psi_2, \psi_1 \rangle_w = \sum_{k=0}^{N-1} \bar{d}_k c_k. \tag{3.18}$$

where $\psi_1 \cong (c_0, \cdots c_{N-1})$, $\psi_2 \cong (d_0, \cdots d_{N-1})$.

As in section 2, we can construct a natural identification (or pairing) between $\mathcal{H}_w(\theta, N)$ and $\mathcal{H}_z(\theta, N)$ when w and z are linearly independent. We first introduce the equivalent of the right hand side of (2.6). If $\psi_1 \in \mathcal{H}_w(\theta, N)$ and $\psi_2 \in \mathcal{H}_z(\theta, N)$, then $\bar{\psi}_2 \psi_1$ can be interpreted as a distribution on the plane. Indeed, although the product of distributions is not defined in general, it makes sense in this case because of the particular form of ψ_1 and ψ_2: $\delta(h_w(x) - q_l(w,\theta))$ and $\delta(h_z(x) - q_k(z,\theta))$ are supported on transversal lines, so that we have no trouble defining their product. Clearly, $\bar{\psi_2}\psi_1$ is $\mathbb{Z}^2$-periodic and, as such, passes to a distribution on $\mathbb{T}^2$. Hence $\int_{\mathbb{T}^2} \bar{\psi_2}\psi_1 \frac{dq\,dp}{2\pi\hbar}$ makes sense as the value of the distribution $\bar{\psi_2}\psi_1$ on the function $(2\pi\hbar)^{-1}$ on $\mathbb{T}^2$. We then have, in analogy with Proposition 2.2:

Proposition 3.1. *Given $w, z \in \mathbb{Z}^2$ as above with $\Delta = \omega(w,z) > 0$ and $\theta \in \mathbb{T}^2$, there exist a unique vector space homomorphism $P_{zw}(\theta, N) : \mathcal{H}_w(\theta, N) \to \mathcal{H}_z(\theta, N)$, such that $\forall \psi \in \mathcal{H}_w(\theta, N), \forall \varphi \in \mathcal{H}_z(\theta, N)$*

$$\langle \varphi, P_{zw}(\theta, N)\psi\rangle_{\mathcal{H}_z(\theta,N)} = \int_{\mathbb{T}^2} \bar{\varphi}\psi \,\frac{dq\,dp}{2\pi\hbar}. \tag{3.19}$$

Moreover, using the identifications defined in (3.16)*, the matrix representation of $P_{zw}(\theta, N)$ is*

$$P_{zw}(\theta, N)_{kr} = \frac{N}{\Delta}\sum_{p=0}^{\Delta-1} \exp\left[2\pi i\,\alpha_\theta(N,w)p\right]\,\exp\left[-2\pi i N\,S_{zw}(q_r(w,\theta)+p, q_k(z,\theta))\right]. \tag{3.20}$$

Proof. That P_{zw} is defined as a vector space homomorphism by (3.19) is clear. To prove the rest of the proposition, we compute the right hand side of (3.19). Recall that this can be done by "integrating" $\bar{\varphi}\psi$ over *any* fundamental domain of the torus. We start by describing a suitable choice. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and define $g_1 = (1/\Delta)\,Jz, \quad g_2 = -(1/\Delta)\,Jw$. Then g_1, g_2 is a basis of $\mathbb{R}^2$ dual to w, z since $h_w(g_1) = h_z(g_2) = 1\,,\ h_w(g_2) = h_z(g_1) = 0$. The unit cell of the dual lattice has volume Δ^{-1}. Taking $L = (L_1, L_2) \in \mathbb{R}^2$, define

$$T(L) = \{x \in \mathbb{R}^2\,|x = \alpha g_1 + \beta g_2, L_1 \le \alpha < L_1 + \Delta, L_2 \le \beta < L_2 + 1\}, \tag{3.21}$$

which is the union of Δ dual unit cells. It is easy to see that $T(L)$ is a fundamental domain for the torus. Indeed, suppose that $x = \alpha g_1 + \beta g_2$ and $x' = \alpha' g_1 + \beta' g_2$ belong to $T(L)$ and that $\exists n \in \mathbb{Z}^2$ so that $x' = x + n$. Then, (3.21) implies that ${}^T(\alpha' - \alpha, \beta' - \beta) = A(t)\,{}^T(h_w(n), h_z(n))$. But $-1 < \beta' - \beta < 1$ and $h_z(n) \in \mathbb{Z}$, so $h_z(n) = 0$, which implies that $\exists \gamma \in \mathbb{R}$ so that $n = \gamma(z_2, -z_1)$. Since *g.c.d.* $(z_1, z_2) = 1$, it follows that $\gamma \in \mathbb{Z}$. Finally, this implies that $\alpha' - \alpha = h_w(n) = \gamma\Delta$ and, since $-\Delta < \alpha' - \alpha < \Delta$, $\gamma = 0$, so $n = 0$. We will use $T(L)$ with a suitable choice of L to compute $\int_{\mathbb{T}^2} \bar{\varphi}\psi \frac{dq\,dp}{2\pi\hbar}$.

For that purpose, recall that $\psi \in \mathcal{H}_w(\theta, N)$ is supported on the lines $h_w(x) = q_\ell(w,\theta)$, $\ell \in \mathbb{Z}$ and $\varphi \in \mathcal{H}_z(\theta, N)$ on the lines $h_z(x) = q_k(z,\theta)$, $k \in \mathbb{Z}$, which intersect in the points $\{x_{k\ell} \,|k, \ell \in \mathbb{Z}\}$ defined uniquely by $h_w(x_{k\ell}) = q_\ell(w,\theta), h_z(x_{k\ell}) = q_k(z,\theta)$. It is then clear that $x_{k\ell} = q_\ell(w,\theta)g_1 + q_k(z,\theta)g_2$, so that, $\forall r, s \in \mathbb{Z}$, $x_{k+rN,\ell} = x_{k\ell} + rg_2$ and $x_{k,\ell+sN} = x_{k\ell} + sg_1$. As a result, for a suitable choice of L the points $x_{k\ell}$ belonging to $T(L)$ are $\{x_{k\ell}\,|\, 0 \le k < N-1,\ 0 \le \ell < \Delta N - 1\}$. Taking $\psi \cong (c_0, \cdots, c_{N-1}) \in \mathbb{C}^N \cong \mathcal{H}_w(\theta, N)$ and $\varphi \cong (d_0, \cdots, d_{N-1}) \in \mathbb{C}^N \cong \mathcal{H}_z(\theta, N)$,(see (3.16)) we then readily obtain that

$$\int_{\mathbb{T}^2} \bar{\varphi}\psi \,\frac{dq\,dp}{2\pi\hbar} = \frac{N}{\Delta} \sum_{k=0}^{N-1} \sum_{\ell=0}^{\Delta N-1} \overline{d}_k c_\ell \exp\left[-2i\pi N S_{zw}(q_\ell(w,\theta), q_k(z,\theta))\right]$$

$$= \frac{N}{\Delta} \sum_{k=0}^{N-1} \overline{d}_k \sum_{r=0}^{N-1} c_r \sum_{p=0}^{\Delta-1} \exp\left[2i\pi\alpha_\theta(N,w)p\right] \exp\left[-2i\pi N S_{zw}(q_r(w,\theta)+p, q_k(z,\theta))\right]$$

$$= \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} \overline{d}_k P_{zw}(\theta, N)_{kr} c_r,$$

where we wrote $\ell = pN + r$ and used $c_{pN+r} = c_r \,\exp\left[2i\pi\alpha_\theta(N,w)p\right]$, (see (3.5)). In conclusion, the matrix representation of $P_{zw}(\theta)$ is given in (3.20). $\square$

The above definition of the pairing $P_{zw}(\theta, N)$ is a special case of a very general definition in the context of geometric quantization [Sn]. It should be remarked however that the general theory does not guarantee that the pairing is unitary: this has to be checked in each case separately. We now turn to this task. Note that the explicit expression of the matrix of $P_{zw}(\theta, N)$ is sufficiently complicated to make a direct computation of $P_{zw}^*(\theta, N)\, P_{zw}(\theta, N)$ difficult (except in the case when $\Delta = 1$, in which case it is trivial). We therefore develop a different argument which uses the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$ and the known unitarity of the pairing there (Proposition 2.2). This yields a proof for all $P_{zw}(\theta, N)$ at once. It would be nice to have a direct geometric proof for each fixed θ.

Proposition 3.2.

(1) *For any* $w \in \mathbb{Z}^2$ *with* $g.c.d.(w_2, w_1) = 1$, $\mathcal{H}_w \cong N^2 \int d^2\theta\, \mathcal{H}_w(\theta, N)$.

(2) $U(a,b)P_{zw} = P_{zw}U(a,b)$ *for any* $w, z \in \mathbb{R}^2$ *and* $\forall (a,b) \in \mathbb{R}^2$.

(3) $P_{zw} = N \int d^2\theta\, P_{zw}(\theta, N)$. *Given* $D_\hbar \in \mathbb{C}$, $\mid D_\hbar \mid = (\Delta/N^3)^{1/2}$, $D_\hbar P_{zw}(\theta, N)$ *is unitary.*

Remark. Note that if $w = (1, 0)$, $z = (0, 1)$ and $\theta = (0, 0)$ then $N^{-3/2}\, P_{zw}(\theta, N) = \mathcal{F}_N$. Here $\mathcal{F}_N$ denotes the usual finite Fourier transform namely,

$$f_\ell = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\,[\,-\frac{2\pi i}{N}\,\ell k\,]\, c_k\,.$$

Before proving this proposition, note that the quantum map associated to any $A \in SL(2, \mathbb{Z})$ is now constructed exactly as in Section 2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and take $\psi \in \mathcal{S}'_\hbar(\theta)$. It is easy to see that $U(A)\psi = \psi \circ A^{-1}$ defines a map from $\mathcal{S}'_\hbar(\theta)$ to $\mathcal{S}'_\hbar(\theta')$, where $\theta' = \beta(\theta) = {}^T\!A\,\theta + \frac{1}{2} N\,{}^T(ac, db) \mod 1$. Moreover, for any $z \in \mathbb{Z}^2$ we have a natural map $U(A) : \mathcal{H}_z(\theta, N) \to \mathcal{H}_{\,{}^T\!A^{-1}z}(\theta', N)$, as we can check by an easy calculation. Its unitarity can be checked either by a direct computation or by remarking that $U(A)$ is unitary on $\mathcal{H}_z = N^2 \cdot \int_0^1 \int_0^1 d^2\theta\, \mathcal{H}_z(\theta, N)$ and hence is also unitary on $\mathcal{H}_z(\theta, N)$. If θ has the property that $\beta(\theta) = \theta$, we can again define the quantum propagator (up to a normalization factor) $V(A) : \mathcal{H}_z(\theta, N) \to \mathcal{H}_z(\theta, N)$ by the formula $V(A, \theta, N) = D_\hbar P_{z,\,{}^T\!A^{-1}z}(\theta, N) \circ U(A)$, which is the restriction of $V(A)$ in (2.8) to $\mathcal{H}_z(\theta, N)$. This yields exactly the same propagators as in [DE]. A particularly simple expression for $V(A, \theta, N)$ is obtained when A is of the special form considered in the following Corollary (see also [HB]).

Corollary 3.1. *If $z = (1, 0)$, $\theta = (0, 0)$ and $A = \begin{pmatrix} 2g & 1 \\ 4g^2 - 1 & 2g \end{pmatrix} \in SL(2, \mathbb{Z})$, $(g > 1$, $g \in \mathbb{Z})$ then:*

$$V(A, \theta, N)_{\ell,k} = \frac{1}{\sqrt{N}} \exp\,[\frac{2\pi i}{N}\,(g\ell^2 - \ell k + gk^2)\,]\,.$$

Proof of Proposition 3.2.

1) Let $v \in \mathbb{Z}^2$ with $\omega(w, v) = 1$ and set $m = -Jw$, $\widetilde{m} = Jv$. We define, for $\tilde\theta \in [0, 1[\times [0, 1[$,

$$S(\tilde\theta) = \sum_{\alpha,\beta \in \mathbb{Z}} (-1)^{N\alpha\beta} \exp\,[\,-2\pi i(\alpha\tilde\theta_1 + \beta\tilde\theta_2)\,]\, U(\alpha m + \beta\widetilde{m})\,.$$

It is then easy to see that $S(\tilde\theta)$ is a continuous operator from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$ which extends uniquely to a map from $\mathcal{S}'(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$. Moreover, a calculation shows

$$\begin{aligned} S(\tilde\theta) \;&=\; \Big(\sum_{\alpha\in\mathbb{Z}} \exp[\,-2\pi i\tilde\theta_1\alpha\,]\, U(\alpha m)\Big)\Big(\sum_{\beta\in\mathbb{Z}} \exp[\,-2\pi i\tilde\theta_2\beta\,]\, U(\beta\widetilde{m})\Big) \\ &= \Big(\sum_{\beta\in\mathbb{Z}} \exp[\,-2\pi i\tilde\theta_2\beta\,]\, U(\beta\widetilde{m})\Big)\Big(\sum_{\alpha\in\mathbb{Z}} \exp[\,-2\pi i\tilde\theta_1\alpha\,]\, U(\alpha m)\Big) \end{aligned}$$

and

$$U(\alpha' m + \beta' \widetilde{m})\, S(\tilde{\theta}) \;=\; (-1)^{N\alpha'\beta'}\; \exp[\, 2\pi i\, (\tilde{\theta}_1 \alpha' + \tilde{\theta}_2 \beta')\,]\, S(\tilde{\theta})\,.$$

Since

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

for $\alpha' m + \beta' \widetilde{m} = n$, we have

$$u_1\, S(\tilde{\theta}) \;=\; (-1)^{N v_1 w_1}\; \exp[\, 2\pi i\, (v_1 \tilde{\theta}_1 + w_1 \tilde{\theta}_2)\,]\, S(\tilde{\theta})\,, \tag{3.22}$$

$$u_2\, S(\tilde{\theta}) \;=\; (-1)^{N v_2 w_2}\; \exp[\, 2\pi i\, (v_2 \tilde{\theta}_1 + w_2 \tilde{\theta}_2)\,]\, S(\tilde{\theta})\,.$$

As a result $S(\tilde{\theta})\, \mathcal{S}'(\mathbb{R}^2) \subset \mathcal{S}'_\hbar(\theta)$, with

$$\theta_1 = (N/2) v_1 w_1 + [v_1 \tilde{\theta}_1 + w_1 \tilde{\theta}_2] \qquad \theta_2 = (N/2) v_2 w_2 + [v_2 \tilde{\theta}_1 + w_2 \tilde{\theta}_2] \tag{3.23}$$

For $\psi \in \mathcal{D}_w$ as in (2.4), a simple computation using the Poisson formula yields

$$[\, S(\tilde{\theta})\psi\,](x) = \exp[\, -i\pi N\, h_w(x) h_v(x)\,] \sum_{k\in\mathbb{Z}} d_k(\theta)\, \delta[\, h_w(x) - (k + \tilde{\theta}_1)/N\,] \tag{3.24}$$

where

$$d_k(\theta) = \frac{1}{N} \sum_{\beta\in\mathbb{Z}} \exp[\, -2\pi i\, \beta \tilde{\theta}_2\,]\, f((k+\tilde{\theta}_1)/N - \beta)\,. \tag{3.25a}$$

Note that

$$d_{k+N}(\theta) = \exp[\, -2\pi i\, \tilde{\theta}_2\,]\, d_k\,. \tag{3.25b}$$

Using (3.23), one establishes

$$(k + \tilde{\theta}_1)/N = q_k(w, \theta) - (1/2)\, w_1 w_2\, [\, v_2 - v_1 - 1\,], \tag{3.26}$$

with $q_k(w, \theta)$ as in (3.4), and

$$\tilde{\theta}_2 = -\alpha_\theta(N, w) - (N/2)\, v_1 v_2\, [\, w_1 - w_2 - 1\,], \tag{3.27}$$

with $\alpha_\theta(N, w)$ defined in (3.6). Note that the relation $w_1 v_2 - w_2 v_1 = 1$ implies that the last term in (3.26) and in (3.27) is an integer. Hence (3.24) becomes

$$[\, S(\tilde{\theta})\psi\,](x) = \exp[\, -i\pi N\, h_w(x) h_v(x)\,] \sum_{k\in\mathbb{Z}} c_k(\theta)\, \delta[h_w(x) - q_k(w, \theta)] \tag{3.28}$$

with $c_k(\theta) = d_{k+\frac{N}{2}w_1w_2[v_2-v_1-1]}(\theta)$ satisfying (3.5), thanks to (3.25) and (3.27). Hence (3.28) is written in the form (3.3) which shows $S(\tilde{\theta})\psi \in \mathcal{D}_w(\theta, N)$. Recall now from (3.16) and (3.18) that $\mathcal{H}_w(\theta, N) \cong \mathbb{C}^N$. As a result

$$\begin{aligned} N^2 \int_0^1 \int_0^1 d^2\theta\, \mathcal{H}_w(\theta, N) &\cong N^2 \int_0^1 \int_0^1 \mathbb{C}^N\, d^2\theta \\ &\cong L^2(\,[0,1[\,\times[0,1[\,; \mathbb{C}^N,\, N^2 d^2\theta). \end{aligned}$$

On the other hand, if $f \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, dy)$, then, using (3.25), and performing a change of variables in the integral, using (3.23), yields

$$\begin{aligned} & N^2 \int_0^1 d\theta_1 \int_0^1 d\theta_2 \sum_{r=0}^{N-1} |c_r(\theta)|^2 = \\ & \qquad N^2 \int_0^1 d\tilde{\theta}_1 \int_0^1 d\tilde{\theta}_2 \sum_{r=0}^{N-1} |c_r(\theta)|^2 = \int_{\mathbb{R}} |f(y)|^2\, dy. \end{aligned}$$

Hence the map

$$f \mapsto (c_0(\theta), \cdots, c_{N-1}(\theta)) \in L^2(\,[0,1[\,\times[0,1[\,; \mathbb{C}^N,\, N^2 d^2\theta)$$

extends to a natural isometry on all of $L^2(\mathbb{R}, dy)$. It is easily seen to be onto and hence unitary. Since $L^2(\mathbb{R}, dy) \cong \mathcal{H}_w$, this proves (1).
2) $\forall \psi_1 \in \mathcal{H}_w$, $\psi_2 \in \mathcal{H}_z$, we have

$$\begin{aligned} \langle \psi_2, P_{zw} U(a,b)\psi_1 \rangle_z &= \int \frac{dq\,dp}{2\pi\hbar}\, \overline{\psi_2}\, U(a,b)\psi_1 \\ &= \int \frac{dq\,dp}{2\pi\hbar}\, \overline{U(a,b)^*\psi_2}\, \psi_1 = \langle U(a,b)^*\psi_2, P_{zw}\psi_1 \rangle_z. \end{aligned}$$

3) We know from the remark after Proposition 2.2 that P_{zw} extends from $\mathcal{H}_w$ to a map from $\mathcal{D}_w$ to $\mathcal{D}_z$. Moreover, in view of (2), $P_{zw}\mathcal{D}_w(\theta, N) \subset \mathcal{D}_z(\theta, N)$. Hence, defining $\tilde{P}_{zw}(\theta, N)$ to be the restriction of P_{zw} to $\mathcal{D}_w(\theta, N)$, it follows that $\tilde{P}_{zw}(\theta, N) : \mathcal{H}_w(\theta, N) \to \mathcal{H}_z(\theta, N)$ and consequently that $P_{zw} = N^2 \int d^2\theta\, \tilde{P}_{zw}(\theta, N)$. By computing the explicit formula for $\tilde{P}_{zw}(\theta, N)$, we will show $N\,\tilde{P}_{zw}(\theta, N) = P_{zw}(\theta, N)$, establishing the proposition. Taking $\psi \in \mathcal{D}_w(\theta, N)$ in the form (3.3), and using the definition of P_{zw} in Proposition 2.2, we have

$$\begin{aligned} & [\,\tilde{P}_{zw}(\theta, N)\,\psi\,](x_2) = \\ & (1/2\pi\hbar\Delta)\, \exp[\,-i\pi N\, h_z(x_2) h_u(x_2)\,] \sum_{k\in\mathbb{Z}} c_k\, \exp[\,-2\pi i N\, S_{zw}(q_k(w,\theta), h_z(x_2))\,], \end{aligned} \tag{3.29}$$

where S_{zw} is given in (2.7). Letting $k = \ell\Delta N + r$, $\ell \in \mathbb{Z}$ and $r \in \{0, \cdots, \Delta N - 1\}$, the *r.h.s.* of (3.29) reads

$$\exp[\,-i\pi N\, h_z(x_2)h_u(x_2)\,]\cdot \\ \frac{N}{\Delta}\sum_{\ell\in\mathbb{Z}}\sum_{r=0}^{N\Delta-1} c_r\, \exp[\,2\pi i\,\ell\Delta\alpha_\theta(N,w)\,]\, \exp[\,-2\pi i N\, S_{zw}(q_r+\ell\Delta, h_z(x_2))\,]\,.$$

Since $(1/2)\,\omega(v,z)\,\ell^2\Delta = (1/2)\,\omega(v,z)\,\ell\Delta \mod 1$, we have

$$S_{zw}(q_r+\ell\Delta, h_z(x_2)) = S_{zw}(q_r, h_z(x_2)) + \ell\,[q_r\omega(v,z)+h_z(x_2)] + (1/2)\omega(v,z)\ell^2\Delta\,,$$

so that

$$\exp[\,-2\pi i N\, S_{zw}(q_r+\ell\Delta, h_z(x_2))\,] = \\ \exp[\,-2\pi i N\, S_{zw}(q_r, h_z(x_2))\,]\, \exp[\,-2\pi i N\,(q_r\omega(v,z)+h_z(x_2)+\omega(v,z)\Delta/2)\ell\,]\,.$$

This yields:

$$[\,\tilde{P}_{zw}(\theta,N)\psi\,](x_2) = \exp[\,-i\pi N\, h_z(x_2)h_u(x_2)\,]\cdot \\ \Big(\frac{N}{\Delta}\sum_{r=0}^{N\Delta-1} c_r\, \exp[\,-2\pi i N\, S_{zw}(q_r, h_z(x_2))\,]\Big)\Big(\sum_{\ell\in\mathbb{Z}}\exp[\,-2\pi i N\,(h_z(x_2)-A)\ell\,]\Big)\,,$$

where

$$A = -q_r(w,\theta)\,\omega(v,z) - (1/2)\,\omega(v,z)\Delta + (\Delta/N)\,\alpha_\theta(N,w),$$

and where $q_r(w,\theta)$ and $\alpha_\theta(N,w)$ are given by (3.4) and (3.6) respectively. Note that we can replace A by anything else mod 1, a freedom we will use in order to get a convenient form for $\tilde{P}_{zw}(\theta,N)$. In particular one has

$$\begin{aligned} A &= -(r/N)\,\omega(v,z) + (1/2)\,z_1z_2 - (1/N)\,\omega(\theta,z) \qquad \mod 1 \\ &= q_{[-r\omega(v,z)]}(z,\theta) \qquad \mod 1\,, \end{aligned}$$

so that

$$\begin{aligned} &[\tilde{P}_{zw}(\theta,N)\,\psi](x_2) = \exp[\,-i\pi N\, h_z(x_2)h_u(x_2)\,]\cdot \\ &\quad\Big(\frac{N}{\Delta}\sum_{r=0}^{N\Delta-1} c_r\, \exp[\,-2\pi i N\, S_{zw}(q_r, h_z(x_2))\,]\Big)\Big((1/N)\sum_{k\in\mathbb{Z}}\delta[y_2 - q_{k-r\omega(v,z)}]\Big) \\ &= \exp[\,-i\pi N\, h_z(x_2)h_u(x_2)\,] \\ &\frac{1}{\Delta}\sum_{r=0}^{N\Delta-1} c_r \sum_{k\in\mathbb{Z}}\exp[\,-2\pi i N\, S_{zw}(q_r(w,\theta), q_{k+r\omega(z,v)}(z,\theta))\,]\,\delta[\,y_2 - q_{k+r\omega(z,v)}(z,\theta)\,] \\ &= \exp[\,-i\pi N\, h_z(x_2)h_u(x_2)\,] \\ &\frac{1}{\Delta}\sum_{r=0}^{N\Delta-1} c_r \sum_{\ell\in\mathbb{Z}}\exp[\,-2\pi i N\, S_{zw}(q_r(w,\theta), q_\ell(z,\theta))\,]\,\delta[\,h_z(x_2) - q_\ell(z,\theta)\,]\,. \end{aligned}$$

Using (3.5) and comparing to (3.20), one sees that $\tilde{P}_{zw}(\theta, N) = (1/N)\, P_{zw}(\theta, N)$. Hence $P_{zw} = N \int d^2\theta\, P_{zw}(\theta, N)$. $\square$

To summarize, by applying the ideas of geometric quantization in their simplest form, one can easily quantize linear transformations on $\mathbb{R}^2$ as well as on $\mathbb{T}^2$. We stress again that the construction is simple and calculationally very convenient. Indeed, although the proofs of Propositions 3.1 and 3.2 are somewhat involved in the general case, they reduce to trivialities when $\Delta = 1$, as in Corollary 3.1 and in the following sections. In that case (3.20) does not involve a sum and the unitarity of P_{zw} is then immediate. We shall now show that the reformulation of geometric quantization we have just presented allows for an immediate generalization to a class of piecewise linear or affine linear transformations of the torus.

4. Quantization of piecewise linear and affine transformations.

(A) *Translations and skew translations.*

The simplest transformations on the torus are undoubtedly the translations $x = (q, p) \mapsto (q + a, p + b) \mod 1$. If $a = r_1/s_1$ and $b = r_2/s_2$ (with *g.c.d.* $(r_i, s_i) = 1$, $i = 1, 2$), then we can write $(a, b) = (r/s)\,(-w_2, w_1)$ for integer r, s with *g.c.d.* $(r, s) = 1$, $w \in \mathbb{Z}^2$, and *g.c.d.* $(w_1, w_2) = 1$. Here s is the least common multiple of s_1 and s_2, which is also the common period of all orbits under this translation.

Taking $k \in \mathbb{N}^*$, $N = sk$, we saw in Section 3 (see (3.17)) how to quantize this translation. The expression of the quantum translation $U(a, b)$ (i.e. (3.17) with $\ell = rk$) shows that its eigenfunctions are concentrated on the circles

$$\omega(x, (a, b)) = (r/s)\, q_i \qquad i = 0, \ldots, ks - 1$$

and that they are k-fold degenerate. The quantum propagator is easily seen to have the same period as the classical translation since

$$U^s(a, b) = \exp[\, 2\pi i\, (-\frac{w_1 w_2}{2}\, \ell s + r\omega(w, \theta))\,]\; id_{\mathcal{H}_w(\theta, N)}.$$

It follows that, as in the multidimensional harmonic oscillator with commensurate frequencies [DBIH], these degeneracies can be used to construct eigenfunctions of $U(a, b)$ that, in the classical limit ($k \to \infty$), concentrate on any given classical orbit.

The approach of Section 3 does not a priori permit the quantization of translations of the form $(a,b)=\alpha\,(r_1/s_1, r_2/s_2)$, $\alpha\notin\mathbb{Q}$, much less of ergodic translations, for which $a/b\notin\mathbb{Q}$. The reason is that the corresponding prequantized translations do not preserve the spaces $\mathcal{H}_w(\theta,N)$.

Since the ergodic translations are undoubtedly the simplest ergodic dynamical systems, it would be interesting to circumvent this difficulty and to nevertheless construct a quantum analog for them. We will see that this can be done very naturally within the framework of Section 3. The situation is actually very similar to the one encountered when quantizing linear flows. Indeed, there we saw that $U(A)\mathcal{H}_w(\theta,N)=\mathcal{H}_{\,{}^TA^{-1}w}(\theta,N)$ for a suitable choice of θ and then we used the natural pairing between $\mathcal{H}_{\,{}^TA^{-1}w}(\theta,N)$ and $\mathcal{H}_w(\theta,N)$ to construct $V(A)$. Here we will see that $U(a,b)\mathcal{H}_w(\theta,N)=\mathcal{H}_w(\theta',N)$ with θ' given in Lemma 4.1 below. Although in this case we can never choose θ so that $\theta'=\theta$, we will construct an identification $D_\hbar P_{vw}(\theta,\theta')$ between $\mathcal{H}_w(\theta',N)$ and $\mathcal{H}_v(\theta,N)$ in analogy with (3.19). Since there is also a natural identification $D_\hbar P_{wv}(\theta)$ between $\mathcal{H}_v(\theta,N)$ and $\mathcal{H}_w(\theta,N)$ (Proposition 3.1), we define the unitary quantum translation $Q_w(a,b)$ by

$$Q_w(a,b)=D_\hbar^2\,P_{wv}(\theta,\theta)\circ P_{vw}(\theta,\theta')\circ U(a,b)\;:\;\mathcal{H}_w(\theta,N)\to\mathcal{H}_w(\theta,N)\,. \tag{4.1}$$

Note that this reduces to (3.17) when the translation has the required form, and that the $Q_w(a,b)$ depend continuously on (a,b). On the other hand, the construction is w-dependent and it is clear that the $Q_w(a,b)$ can not generate a unitary representation of the full Weyl-Heisenberg group.

Lemma 4.1.

(1) $U(a,b)\,\mathcal{S}'_\hbar(\theta)=\mathcal{S}'_\hbar(\theta')$, *with* $(\theta'_1,\theta'_2)=(\theta_1-Nb,\theta_2+Na)\quad \bmod 1\,.$
(2) $U(a,b)\nabla_w\psi=\nabla_w U(a,b)\psi$ *for any* $w\in\mathbb{R}^2$, $(a,b)\in\mathbb{R}^2$, $\psi\in\mathcal{S}'(\mathbb{R}^2)\,.$
(3) $U(a,b):\mathcal{H}_w(\theta,N)\to\mathcal{H}_w(\theta',N)$ *is unitary.*

Proof. Both (1) and (2) follow from a simple computation. That $U(a,b)$ maps $\mathcal{H}_w(\theta,N)$ isomorphically onto $\mathcal{H}_w(\theta',N)$ is an immediate consequence of (1) and (2). To check the unitarity, let $\psi\in\mathcal{H}_w(\theta,N)$ with

$$\psi(q,p)=\sum_{k\in\mathbb{Z}}c_k\,\exp[\,-i\pi N\,h_w(x)h_v(x)\,]\,\delta\big(h_w(x)-q_k(\theta,w)\big)\,. \tag{4.2}$$

For convenience, we write $\tau=(\tau_1,\tau_2)=(a,b)$. Now we introduce $\ell=(\ell_1,\ell_2)\in\mathbb{Z}^2$,

$I_{1/N} =]-\frac{1}{N}, \frac{1}{N}[$ and $\beta = (\beta_1, \beta_2) \in I_{1/N}^2$, uniquely determined by

$$\tau_i = \ell_i/N + \beta_i$$
$$\theta_i' = \theta_i + (-)^i N \beta_{3-i} \in [0,1[,$$

with $i = 1, 2$. A direct calculation shows that

$$[U(a,b)\psi](q,p) = \sum_{k \in \mathbb{Z}} d_k \exp[-i\pi N h_w(x) h_v(x)] \, \delta\big(h_w(x) - q_k(\theta', w)\big),$$

where

$$d_k = c_{k-h_w(\ell)} \exp[i\pi N \left(2q_k(\theta', w) h_v(\tau) - h_w(\tau) h_v(\tau)\right)] \tag{4.3}$$

Recalling the identification $\psi \cong (c_0, \cdots, c_{N-1})$ and $[U(a,b)\psi] \cong (d_0, \cdots, d_{N-1})$, the unitarity of $U(a,b)$ is now immediate from (4.2). $\square$

Given now $U(a,b)\psi \cong (d_0, \cdots, d_{N-1}) \in \mathcal{H}_w(\theta', N)$, we can proceed in the spirit of Proposition 3.1 to define $P_{vw}(\theta, \theta') : \mathcal{H}_w(\theta', N) \to \mathcal{H}_v(\theta, N)$ as follows:

$$\langle \psi_2, P_{vw}(\theta, \theta')\psi_1 \rangle_{\mathcal{H}_v(\theta, N)} = \int_{[0,1)\times[0,1)} \overline{\psi}_2 \psi_1 \, \frac{dq\, dp}{2\pi\hbar}. \tag{4.4}$$

A simple calculation then yields

$$[P_{vw}(\theta, \theta')\psi](q,p) = N \sum_{\ell} [P_{vw}(\theta, \theta')\psi]_\ell \, \exp\left[i\pi N h_w(x) h_v(x)\right] \delta(h_v(x) - q_\ell(v, \theta))$$

where

$$\begin{aligned}
[P_{vw}(\theta, \theta')\psi]_\ell &= N \sum_{k=0}^{N-1} d_k \, \exp\left[-2i\pi N S_{vw}(q_k(w, \theta'), q_\ell(v, \theta))\right] \\
&= N \sum_{k=0}^{N-1} d_k \, \exp\left[-2i\pi N q_k(w, \theta') q_\ell(v, \theta)\right]
\end{aligned}$$

It is easy to see that $\| D_\hbar P_{vw}(\theta, \theta') U(a,b)\psi \|_{\mathcal{H}_v(\theta,N)}^2 = \| \psi \|_{\mathcal{H}_w(\theta', N)}^2$, where $|D_\hbar| = N^{-3/2}$.

When (a,b) is ergodic, the eigenfunctions of the $Q_w(a,b)$ can on general grounds be expected to be equidistributed on the torus in the classical limit, in sharp contrast to what happens in the periodic case.

Note that it is now easy to quantize skew translations of the form $(q,p) \mapsto (q + a, p + kq)$ which are ergodic if a is irrational and k a non-zero integer [CFS]. They are just the composition of a linear transformation and a translation.

(*B*) *Piecewise affine transformations.*

A first class of piecewise affine maps studied in [Ch] is the following. Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, apply it to $[0,1) \times [0,1)$, then cut the resulting parallelogram into strips along the direction (a,c) and shift the strips around with translations parallel to (a,c). Combining Section 3 and Section 4A, one can easily obtain a quantization for this class of transformations.

Let us now turn to another class of discontinuous maps described in [Ch,LW,V]. Consider the map $A_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{R}$ restricted to the strip $0 \le p \le 1$ and taken modulo 1 in q. This defines a map A_1 on the torus, discontinuous on the circle $\{p \in \mathbb{Z}\}$ if $b \notin \mathbb{Z}$. Similarly, construct a map A_2 on the torus by restricting $A_2 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, $b \in \mathbb{R}$ to the strip $0 \le q \le 1$ and taking p modulo 1. This map will be discontinuous on the circle $\{q \in \mathbb{Z}\}$ if $b \notin \mathbb{Z}$. The map $A = A_2A_1$, which is a discontinuous hyperbolic area preserving map on the torus, is ergodic and exponentially mixing, [Ch,Li,LW,V].

We now propose a quantization of A_i, $i = 1,2$ in the spirit of Section 3. Calling V_i the quantization of A_i, we will define the quantum propagator V of A by $V = V_2V_1$. We saw in Section 2 that $U(A_1)\mathcal{D}_w = \mathcal{D}_{(A_1^{-1})^T w}$. If, however, $a \notin \mathbb{Z}$ then $U(A_1)\mathcal{S}'_\hbar(\theta) \not\subset \mathcal{S}'_\hbar(\theta')$ for any choice of θ and θ'. This situation is similar to, but slightly more complicated than, the one of the previous paragraph, where $U(a,b)\mathcal{D}_w = \mathcal{D}_w$, but $U(a,b)\mathcal{S}'_\hbar(\theta) = \mathcal{S}'_\hbar(\theta')$. So there is again no geometrically natural definition of the quantum propagator associated to A_1. This reflects the fact that A_1 is not a continuous automorphism of the torus. The approach of Section 3 nevertheless suggests an obvious way to quantize A_1. For that purpose, note that the image of $[0,1) \times [0,1)$ under A_1 is

$$F_1 = \{(q,p) \in \mathbb{R}^2 \mid 0 \le p < 1,\ bp \le q < bp + 1\},$$

which is again a fundamental domain of the torus. Let $w = (1,0)$, $v = (0,1)$. Then, if $\psi \in \mathcal{H}_w(\theta,N)$ and $\varphi \in \mathcal{H}_v(\theta,N)$, it is immediately clear, because of the transversality of the lines $p = p_k$, $q + bp = q_\ell$, that $\overline{\varphi}U(A_1)\psi$ still defines a distribution on the plane.

As a result, there exists a unique map $PU(A_1) : \mathcal{H}_w(\theta,N) \to \mathcal{H}_v(\theta,N)$ defined by

$$\langle \varphi, PU(A_1)\psi \rangle_{\mathcal{H}_v(\theta,N)} = \int_{F_1} \overline{\varphi}U(A_1)\psi \, \frac{dq\,dp}{2\pi\hbar}. \tag{4.5}$$

Here the right-hand side of (4.5) is to understood as the value of the distribution $\overline{\varphi}U(A_1)\psi$ on a smooth characteristic function of F_1. Explicitly, a simple calculation shows that, for any $k, \ell = 0, \cdots, N-1$

$$[PU(A_1)]_{k\ell} = N \exp\left[-i\pi N b p_k^2\right] \exp\left[-2i\pi N q_\ell p_k\right]$$

where $q_\ell = \ell/N + \theta_2/N$, $p_k = k/N - \theta_1/N$.

The resulting quantum propagator on $\mathcal{H}_w(\theta, N)$ is then, using the natural identification between $\mathcal{H}_v(\theta, N)$ and $\mathcal{H}_w(\theta, N)$:

$$V_1 = N^{-3/2}\mathcal{F}_N^{-1} \circ PU(A_1).$$

Note that $N^{-3/2}PU(A_1)$ itself is the product of the finite Fourier transform with the diagonal matrix D_1 with entries $\exp\left[-i\pi N b p_k^2\right]$. So

$$V_1 = \mathcal{F}_N^{-1} \circ D_1 \circ \mathcal{F}_N. \tag{4.6}$$

Remark that for $b \in \mathbb{Z}$ and for the appropriate θ this reduces to the result obtained in Section 3, as is easily checked. Note furthermore that the map A_1 behaves as a completely integrable transformation with invariant circles $p = const.$ This is perfectly reflected in the structure of V_1. ¿From equation (4.6) one sees that its eigenfunctions are indeed concentrated on the invariant circles.

Finally, the construction of V_2 is completely analogous, with the roles of w and v interchanged. The resulting quantum propagator $V = V_2V_1$ on $\mathcal{H}_w(\theta, N)$ is readily seen to be

$$V = D_2 \circ \mathcal{F}_N^{-1} \circ D_1 \circ \mathcal{F}_N. \tag{4.7}$$

Here D_2 is the diagonal matrix with entries $\exp\left[i\pi N b q_\ell^2\right]$. The non trivial structure of V comes from the fact that it is the product of two non commuting matrices V_1, V_2.

(C) The Baker transformation.

Given the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, we consider the piecewise affine map B defined on the unit square $(0 \le q < 1, 0 \le p \le 1)$ by

$$B(q,p) = \begin{cases} Ax\,, & 0 \le q < 1/2\,, \\ T(-1, 1/2) \circ A\,, & 1/2 \le q < 1\,, \end{cases}$$

where $T(a,b)x = (q+a, p+b)$. This map is called *the Baker transformation*, and its dynamical properties have been studied in detail (see [AA,LW]). Note that it has the same structure as the piecewise affine maps described above. First one applies a linear map, then one slices the resulting rectangle and shift the parts around. There is one major difference, however, leading to some additional complications for the quantization. The linear part of the Baker transformation does not send $[0,1)\times[0,1)$ into another fundamental domain of $\mathbb{T}^2$.

Even though the Baker transformation is not continuous on the torus, the tools we developed in the previous section can again be used to associate a corresponding quantum operator to this map, as we now show. In particular, as in [BV, Sa], we take the point of view that the correct quantum Hilbert spaces for this problem are still the ones constructed in Section 3 (see below). It then suffices to mimic the approach of the previous section, with some minor changes to account for the discontinuities of the map. The resulting quantum operator is identical to the one obtained in [BV,Sa] by completely different arguments.

We shall first define a prequantized version $\widehat{B}$ of B on distributions on $\mathbb{R}^2$ with support in the left or right half of the unit square. Suppose ψ is a distribution supported on $0\le q<\frac{1}{2}\,,\;\;0\le p\le 1\,.$ Then we define

$$(\widehat{B}\psi)(q,p)=U(A)\,\psi(q,p)\,.$$

Note that the support of $\widehat{B}\psi$ is contained in $0\le q<1\,,\;\;0\le p\le\frac{1}{2}\,.$ If, on the other hand, ψ is supported in $\frac{1}{2}\le q<1\,,\;0\le p\le 1$, then

$$(\widehat{B}\psi)(q,p)=[\,U(-1,1/2)\circ U(A)\,]\psi(q,p)$$

and its support is now contained in $0\le q<1\,,\;\;\frac{1}{2}\le p\le 1$.

Given $N\in\mathbb{N}$, and $w=(1,0)$, recall that $\mathcal{D}_w(\theta,N)$ is the space of distributions ψ of the form:

$$\psi(q,p)=\sum_{k\in\mathbb{Z}}c_k\,\exp[\,-i\pi N\,pq\,]\,\delta(q-q_k)\,,$$

where $q_k=k/N+\theta_2/N$ and, in addition, $c_{k+N}=e^{-2\pi i\,\theta_1}\,c_k$ for any $k\in\mathbb{Z}$. Therefore, because of the latter relations, no information is lost if we restrict ψ to the unit square, namely

$$\psi(q,p)=\sum_{k=0}^{N-1}c_k\,\exp[\,-i\pi N\,pq\,]\,\delta(q-k/N-\theta_2/N)\;\chi_{[0,1]}(p)\,,\tag{4.8}$$

where $\chi_{[0,1]}$ is the characteristic function of the unit interval. We shall write $H_1(\theta)$ for the space of distributions of the form (4.8), equipped with the inner product (3.18).

This is the quantum Hilbert space for the Baker map in the *position representation*, which is realized as a space of distributions on the phase space of the problem. Similarly, we introduce $H_2(\theta)$, which is the space of distributions $\mathcal{D}_v(\theta, N)$ with $v = (0,1)$, restricted to the unit square, i.e., $\psi \in H_2(\theta)$ iff

$$\psi = \sum_{\ell=0}^{N-1} d_l \exp\left[i\pi N pq\right] \delta(p - p_l)\, \chi_{[0,1]}(q),$$

where $p_\ell = \ell/N + \theta_1/N$. $H_2(\theta)$ is the quantum Hilbert space of the Baker transformation in the *momentum* representation. We have a natural identification between $H_1(\theta)$ and $H_2(\theta)$, given by the pairing of section 3, which in this case is just the finite Fourier transform (see the remark after Proposition 3.1).

We now observe that we have a natural decomposition $H_1(\theta) = H_L(\theta) \bigoplus H_R(\theta)$. Indeed, each $\psi \in H_1(\theta)$ can be uniquely written as $\psi = \psi_L + \psi_R$, where

$$\psi_L = \sum_{0 \le q_k < \frac{1}{2}} c_k \exp\left[\,-i\pi N\, pq\,\right] \delta(q - q_k)\ \chi_{[0,1]}(p)\,,$$

$$\psi_R = \sum_{\frac{1}{2} \le q_k < 1} c_k \exp\left[\,-i\pi N\, pq\,\right] \delta(q - q_k)\ \chi_{[0,1]}(p)\,,$$

have their respective supports in $0 \le q < \frac{1}{2}$, and $\frac{1}{2} \le q < 1$. We can now compute

$$(\widehat{B}\psi)(q,p) = (\widehat{B}\psi_L)(q,p) + (\widehat{B}\psi_R)(q,p)\,.$$

This gives

$$(\widehat{B}\psi_L)(q,p) = 2 \sum_{0 \le q_k < \frac{1}{2}} c_k\ \exp[\,-2\pi i N\, q_k p\,]\, \delta(q - 2q_k)\ \chi_{[0,1]}(2p)\,,$$

$$(\widehat{B}\psi_R)(q,p) = 2\ \exp[\,2\pi i\,(\theta_2 - N/4)\,] \sum_{\frac{1}{2} \le q_k < 1} c_k\ \exp[\,-2\pi i N\,(q_k - 1/2)\,p\,] \cdot$$

$$\delta(q + 1 - 2q_k)\ \chi_{[0,1]}(2p - 1)\,.$$

Note that the support of $\widehat{B}\psi_L$ is contained in $0 \le q < 1$, $0 \le p \le \frac{1}{2}$, whereas the support of $\widehat{B}\psi_R$ is contained in $0 \le q < 1$, $\frac{1}{2} \le p \le 1$. It is clear that $\widehat{B}\psi$ obtained in this way is not an element of $H_1(\theta)$ (for any θ), nor of any $\mathcal{D}_z(\theta, N)$. Hence, we have no hope of applying the general results on pairing of the previous section directly to define the quantum propagator. It will nevertheless turn out that we can again define, in the spirit of (3.19), a natural projection $P\hat{B}\psi$ of the distribution $\hat{B}\psi$ onto $H_2(\theta)$.

Proposition 4.1. *If N is even and $0 < \theta_1,\, \theta_2 < 1$, then there exists a unitary map $2^{-1/2}N^{-3/2}\, P\widehat{B} : H_1(\theta) \to H_2(\theta)$, uniquely defined by:*

$$\langle \psi_2, P\hat{B}\psi_1\rangle_{H_2(\theta)} = \int_{[0,1)\times[0,1)} \overline{\psi}_2 \hat{B}\psi_1 \, \frac{dq\, dp}{2\pi\hbar}. \tag{4.9}$$

Specifically,

$$P(\widehat{B}\psi)(q,p) = \sum_{\ell=0}^{N-1} \big(P(\widehat{B}\psi)\big)_\ell \, \exp[\, i\pi N\, qp\,]\, \delta(p - p_\ell)\, \chi_{[0,1]}(q),$$

with, for $\ell < N/2$,

$$\big(P(\widehat{B}\psi)\big)_\ell = 2N \sum_{k=0}^{N/2-1} c_k \, \exp[\, -4\pi i N\, q_k p_\ell\,]\,,$$

and for $\ell \geq N/2$,

$$\big(P(\widehat{B}\psi)\big)_\ell = 2N \; \exp[\, 2\pi i\,(\theta_2 - N/4)\,] \sum_{k=N/2}^{N-1} c_k \, \exp[\, -4\pi i N\,((q_k - 1/2)\, p_\ell)\,]\,.$$

Remark. We omit the proof, obtained by a simple computation. Let us point out that the conditions on θ assure that $\overline{\psi}_2\hat{B}\psi_1$ is a distribution on the plane, with support in the unit square. They guarantee in particular that ψ_2 does not have support on the line $p = 1/2$, which would lead to technical problems involving the multiplication with $P\widehat{B}\,\psi_1$. The right hand side of (4.9) is to be understood as the value of this distribution on a smooth characteristic function of the unit square, and is independent of its choice. The unitarity statement does not follow from the results of section 3, since $P\widehat{B}\,\psi_1$ is not a polarized section, as pointed out above. In this sense, the unitarity of the construction is somewhat surprising. It breaks down when N is odd, although the block diagonal structure of $P\widehat{B}$ would permit to restore it by hand.

We now define the quantum Baker transformation V_B in the spirit of section 3 as follows:

$$V_B = 2^{-1/2}N^{-3/2}\, \mathcal{F}_N^{-1} \circ P\widehat{B} : H_1(\theta) \to H_1(\theta),$$

where we used the natural pairing between $H_2(\theta)$ and $H_1(\theta)$ described above. A simple calculation now shows that if $\theta = (0,0)$ V_B is exactly the operator obtained in [BV]. If $\theta = (1/2, 1/2)$, V_B coincides with the quantum Baker map of [Sa]. Although the value $\theta = 0$ is strictly speaking excluded by the Proposition, it can be obtained in the limit. We mention that this construction can be immediately extended to a more general class of Baker like transformations [BV].

In conclusion, these examples show that the framework of section 3 permits the treatment of situations that are not geometrically natural and would therefore not be tractable within the framework of geometric quantization as such. Remark for example that, although the right hand side of equation (4.9) makes sense, it is not geometrically intrinsic, unlike the right hand side of (3.19). Similarly, the identification of the quantum Hilbert spaces with $\mathbb{C}^N$ in section 3 was merely a calculational devise, which is again no longer the case here. Nevertheless, it is clear that the phase space formulation of quantum mechanics given by geometric quantization automatically reproduces the clever intuitive arguments used to construct the quantized Baker transformation in [BV]. In particular, the prequantized map is very close to the classical map: this is clear from the general expression for $\exp[-\frac{i}{\hbar}]\hat{f}t$ in Section 2. As a result, it still has the "left to bottom", "right to top" structure of the classical map. In [BV] this feature was built into the construction of the quantized Baker transformation by assumption .

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