Viscosity of Quantum Hall Fluids

J. E. Avron

Department of Physics, Technion, 32000 Haifa, Israel

R. Seiler and P. G. Zograf *

Fachbereich Mathematik, Technische Universität Berlin, 10623 Berlin

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Abstract

The viscosity of quantum fluids with an energy gap at zero temperature is non-dissipative and is related to the adiabatic curvature on the space of flat background metrics (which plays the role of the parameter space). For a quantum Hall fluid on two dimensional tori this viscosity is computed. In this case the average viscosity is quantized and is proportional to the total magnetic flux through the torus.

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* On leave from Steklov Mathematical Institute, St. Petersburg, Russia

Classically, the elastic modulus λ and viscosity η are tensors of rank 4, which relate the stress tensor T to the strain tensor e and strain-rate tensor $\dot{e}$ [1]. In the limit of small strain-rates

$$T_{\alpha\beta}(e,\dot{e}) = T_{\alpha\beta}(e,0) + \sum_{\gamma,\delta} \eta_{\alpha\beta\gamma\delta}\ \dot{e}_{\gamma\delta}, \quad \alpha,\beta,\gamma,\delta = 1,\ldots,d, \tag{1}$$

where d is the dimension of configuration space. The elastic modulus tensor $\lambda_{\alpha\beta\gamma\delta} = -\frac{1}{2}\ \frac{\partial T_{\alpha\beta}(e,0)}{\partial e_{\gamma\delta}}$ is symmetric in all its indices. Classical Newtonian fluids [2] offer no resistance to shear; so, all shear modes are soft and λ has rank one.

The term $\sum \eta_{\alpha\beta\gamma\delta}\ \dot{e}_{\gamma\delta}$ in Eq. (1) is *the viscosity stress tensor* [2]. For Newtonian fluids, T and $\dot{e}$ are symmetric tensors, and consequently, $\eta_{\alpha\beta\gamma\delta} = \eta_{\beta\alpha\gamma\delta} = \eta_{\alpha\beta\delta\gamma}$ [1, 3]. With respect to substitution of indices $(\alpha\beta\gamma\delta) \mapsto (\gamma\delta\alpha\beta)$, the viscosity tensor splits into symmetric and anti-symmetric parts, $\eta = \eta^S + \eta^A$, where

$$\eta^S_{\alpha\beta\gamma\delta} = \eta^S_{\gamma\delta\alpha\beta}, \quad \eta^A_{\alpha\beta\gamma\delta} = -\eta^A_{\gamma\delta\alpha\beta}. \tag{2}$$

The symmetric part, being associated with dissipation, is a definite quadratic form on the space of 2-tensors (strain-rates). If a fluid is isotropic, then

$$\eta^S_{\alpha\beta\gamma\delta} \ = \eta\ \delta_{\alpha\gamma}\ \delta_{\beta\delta} + \xi\delta_{\alpha\beta}\ \delta_{\gamma\delta}, \tag{3}$$

where ξ and η are the two *viscosity constants* [2]. If the fluid is also incompressible, ξ may be put to zero with no loss. This is the normal situation in Newtonian fluid mechanics. The anti-symmetric part of the viscosity tensor describes non-dissipative response. One normally assumes that it is absent because of no compelling evidence to think otherwise.

Quantum fluids can have a ground state which is separated by a finite gap from the rest of the spectrum. At zero temperature such a fluid will have a non-dissipative response, with $\eta^S = 0$. The anti-symmetric part η^A may or may not vanish at zero temperature. For instance, η^A vanishes for systems with time reversal symmetry (and this may be viewed as a consequence of Onsager relation [4]).

Quantum fluids with energy gap and broken time reversal symmetry will, in general, have $\eta^A \neq 0$ at zero temperature. A quantum Hall fluid with a full Landau level gives such an example. The viscosity tensor at low temperature could then be dominated by the non-dissipative part η^A. The study of the non-dissipative viscosity bears analogies with the Quantum Hall effect [5], the Magnus force in superconductors [6], and even with gravity [7]. Like the Hall conductance, and the Magnus force, the (non-dissipative) viscosity is related to the adiabatic curvature [8] and to topological invariants [9]. The connection with gravity comes about because the adiabatic curvature relevant for viscosity is a 2-form on the space parameterizing flat background metrics. In other words, the viscosity stress tensor can be viewed as describing the response of the system to certain adiabatic deformations of the metric.

Let us first recall a general fact from the theory of adiabatic response [10]. Consider a family of Hamiltonians $H(X)$ which depend smoothly on a set of parameters $X = \{X_1, \ldots, X_n\}$ (X denotes a point in parameter space while x denotes a point in configuration space). Let $|\psi(X)\rangle$ be a (normalized) non-degenerate state of $H(X)$, with energy $E(X)$. Let $X(t)$ be a path in parameter space which is traversed adiabatically; $\dot{X}$ is the velocity along the path. We assume throughout that the state stays non-degenerate along the path.

By the principle of virtual work, $-\frac{\delta H}{\delta X}$ is the observable corresponding to the generalized force related to δX [11]. Adiabatic response theory says that, in the adiabatic limit,

$$\langle \frac{\partial H}{\partial X_j} \rangle = \frac{\partial E}{\partial X_j} + \sum_{k=1}^{n} \Omega_{jk} \, \dot{X}_k, \tag{4}$$

where Ω_{ij} is the (anti-symmetric) adiabatic curvature [8]:

$$\Omega_{ij} \equiv \; Im \; \langle \partial_i \psi | \partial_j \psi \rangle, \quad \Omega_{ij} = -\Omega_{ji}, \quad \partial_j = \frac{\partial}{\partial X_j}. \tag{5}$$

If H is time reversal invariant, then $\Omega_{ij} = 0$ [8]. The anti-symmetry of Ω implies no dissipation: there is no change in energy if the system is taken along a closed loop in parameter space.

In the special case where $|\psi\rangle$ is a (normalized) multiparticle state corresponding to N non-interacting fermions in the single particle (normalized) states $|\varphi_\ell\rangle$ with energies $E_\ell(X)$ (all depending smoothly on parameter X) one has

$$\Omega_{jk} = \; Im \; \sum_{\ell=1}^{N} \langle \partial_j \varphi_\ell | \partial_k \varphi_\ell \rangle, \quad E(X) = \sum_{\ell=1}^{N} \; E_\ell(X). \tag{6}$$

To apply all this to viscosity one needs to identify appropriate parameters so that $\frac{\partial H}{\partial X_j}$ is related to the energy-momentum tensor and $\dot{X}$ to the rate of strain. We shall argue that this is done by flat metrics (metrics with constant coefficients).

Consider a fluid confined to a given domain in d dimensional Euclidean space. The planar parallelogram shown in Fig. 1 is an example for $d = 2$. A uniform deformation is represented by the (symmetric) tensor of constant strain [1]

$$e_{\alpha\beta} = \frac{1}{2} \left(-\delta_{\alpha\beta} + \sum_{\gamma} \frac{\partial x'_\gamma}{\partial x_\alpha} \frac{\partial x'_\gamma}{\partial x_\beta} \right), \tag{7}$$

where x' is a linear transformation of x. In two dimensions the deformation space of flat metrics is three dimensional. A two dimensional subspace is associated with transformations that preserve the volume, and the transverse direction may be associated with scaling.

One can view deformations as changing the domain while keeping the underlying Euclidean metric fixed. An equivalent point of view is to consider a deformation of the metric, while keeping the domain fixed with $g = I + 2e$ [1]. The deformed metric is not Euclidean in general but always has constant coefficients and therefore flat. Convenient coordinates on the corresponding parameter space will be introduced later.

Let $H(g)$ denote the Hamiltonian on a domain D in d-dimensional space associated with metric tensor g with constant coefficients $g_{\alpha\beta}$ viewed as external parameters. By the principle of virtual work [11], $-\frac{\delta H}{\delta e} = -2\,\frac{\delta H}{\delta g}$ is the observable associated with *total* stress tensor, $\int_D T_{\alpha\beta}(x)\ d\,vol(x)$. (The total stress has dimension of energy, while the stress has the dimension of pressure). For example, for a free particle of mass m in D the kinetic energy is

$$H(g) = \frac{m}{2}\,\sum g^{\alpha\beta}\,v_\alpha v_\beta, \tag{8}$$

and the total stress is the observable

$$-\frac{\partial H}{\partial e_{\alpha\beta}}\Big|_{e=0} = m\ v_\alpha v_\beta. \tag{9}$$

This is reminiscent of general relativity where the (local) energy-momentum tensor is related to the variation of the action due to (local) deformation of the metric.

Adiabatic deformations of the strain give the quantum version of Eq. (1):

$$\langle\frac{\partial H}{\partial e_{\alpha\beta}}\rangle = \frac{\partial E}{\partial e_{\alpha\beta}} + \sum_{\gamma,\delta=1}^{d} \Omega_{\alpha\beta\gamma\delta}\,\dot e_{\gamma\delta}. \tag{10}$$

The elastic modulus tensor $\lambda_{\alpha\beta\gamma\delta} = \frac{1}{2}\,\frac{\partial^2 E}{\partial e_{\alpha\beta}\,\partial e_{\gamma\delta}}$ is manifestly symmetric in all its indices. The adiabatic curvature plays here the role of non-dissipative viscosity. For homogeneous fluids the viscosity tensor and the adiabatic curvature are related by

$$\Omega = V\ \eta^A, \tag{11}$$

where V is the volume of the fluid. The adiabatic curvature Ω has units of Planck constant $\hbar$. This is analogous to the situation in the quantum Hall effect where the adiabatic curvature (=conductance) has units of $e^2/\hbar$.

Let us illustrate these ideas on a concrete example. Consider two dimensional Quantum Hall fluid on a torus, such as in Fig.1, with opposite sides identified. Explicitly, let Q be the unit square in configuration space: $(x,y) \in Q = [0,1]\times[0,1]$. We shall choose the following coordinates on the deformation space of flat metrics: $\{V, \tau_1, \tau_2 \mid V, \tau_2 > 0\}$. By $V = \sqrt{\det g}$ we denote the area and we use complex variable $\tau = \tau_1 + i\,\tau_2$ to parameterize deformations that preserve the volume.

The flat metrics $g(V,\tau)$ are:

$$g(V,\tau) = \frac{V}{\tau_2}(dx^2 + 2\tau_1 dx\,dy + |\tau|^2\,dy^2), \quad x, y \in Q. \tag{12}$$

The Landau Hamiltonian describes the kinetic energy of a charged (spinless) particle in a constant magnetic field, and Aharonov-Bohm gauge fields. It is given by

$$H(V,\tau,\phi) = \frac{1}{V\tau_2}\left(|\tau|^2 D_x^2 - \tau_1\,(D_x D_y + D_y D_x) + D_y^2\right), \tag{13}$$

where $D_x = -i\partial_x + 2\pi(By + \phi_1 + B/2)$, $D_y = -i\partial_y + 2\pi(\phi_2 + B/2)$; ϕ_1 and ϕ_2 are associated with two Aharonov-Bohm fluxes, and the integer B is the number of magnetic flux quanta through torus. (Note that in our units 1 is the unit of quantum flux, hc/e.) We impose the usual magnetic translation boundary conditions [12]:

$$\psi(x+1,y) = \psi(x,y), \quad \psi(x,y+1) = e^{-2\pi i B x}\,\psi(x,y). \tag{14}$$

The (single particle) ground state is B-fold degenerate with energy $E = 2\pi B/V$ independent of τ and ϕ. The ground state of a full Landau level has energy $E = 2\pi B^2/V$ and is separated by a gap from the rest of the spectrum. (For a non-relativistic spinning electron, described by Pauli equation, one has $E = 0$). It follows from Eq. (10) that the only non-zero component of the elastic modulus tensor is $\lambda_{VV} = 2\pi B^2/V^3$ corresponding to finite compressibility. All other components of λ vanish, as they should, for a fluid; the two shear modes are soft.

An orthonormal, smooth, family of single particle states that span the lowest Landau level is in fact given by theta functions [13]:

$$\varphi_\ell(x,y) = \frac{(2\,\tau_2\;B)^{1/4}}{V^{1/2}} \sum_{n=-\infty}^{\infty} e^{i\pi\tau B(\tilde{y}+n)^2}\, e^{-2i\pi(\phi_2+B/2)(\tilde{y}+n)}\, e^{2\pi i(nB+\ell)x}, \tag{15}$$

where $\tilde{y} = y + \ell/B + \phi_1/B + 1/2$ and $\quad \ell = 0, \ldots, B-1$.

A little computation shows that the component of the adiabatic curvature in Eq. (6), which corresponds to the ℓ-th state, is:

$$\sum_{i<j} \Omega^{\ell}_{ij}\; dX_i \wedge dX_j = \frac{d\tau_1 \wedge d\tau_2}{4\tau_2^2} - \frac{2\pi}{B} d\phi_1 \wedge d\phi_2 \tag{16}$$

(cf. [14]). The basic equations of transport (the main result of this letter) follow from Eqs. (6,10) and (16):

$$\langle\frac{\partial H}{\partial V}\rangle = -2\pi B^2/V^2, \quad \langle\frac{\partial H}{\partial \tau_1}\rangle = \frac{B}{4\tau_2^2}\;\dot{\tau}_2, \quad \langle\frac{\partial H}{\partial \tau_2}\rangle = -\frac{B}{4\tau_2^2}\;\dot{\tau}_1. \tag{17}$$

There is a single transport coefficient related to viscosity and it is $B/(4\tau_2^2)$. The interpretation of this result is as follows. Consider two dimensional Hall fluid on a surface of a cylinder. Compressing it in the radial direction (or in the axial direction) results in a twist rate of one boundary circle relative to the other. And vice versa: a shear of the two boundary circles results in a compression rate of the radial direction and stretching rate of the axial direction.

It may be instructive to compare these transport equations, Eq. (17), with the transport equations for the Hall conductance for Landau levels on the torus. These too follow from Eqs. (4) and (16). Namely,

$$\langle\frac{\partial H}{\partial \phi_1}\rangle = 2\pi\dot{\phi}_2, \quad \langle\frac{\partial H}{\partial \phi_2}\rangle = -2\pi\dot{\phi}_1. \tag{18}$$

The generalized force in this case is the current operator $\frac{\partial H}{\partial \phi}$, and the generalized velocity $\dot{\phi}$ is the electromotive force. By Eq. (18) the conductance of a full Landau level is 2π in our units and e^2/h in ordinary units.

Comparing the conductance and viscosity one sees that the viscosity increases linearly with magnetic field, while the conductance is independent of it, and that the conductance is constant on the flux space while the viscosity is τ_2-dependent.

To appreciate the geometric significance of Eq. (17) we will look at the space of parameters (V, τ, ϕ) more thoroghly. The modular group $SL(2, Z)$ acts naturally on this parameter space as a symmetry group of the family of Landau Hamiltonians $H(V, \tau, \phi)$, Eq. (13). To describe this action explicitly let us note first that an element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$ provides an isometry between metrics $g(V, \tau)$ and $g(V, \tau')$ with $\tau' = \frac{a\tau + b}{c\tau + d}$ by the formula

$$x \mapsto ax - by, \; y \mapsto -cx + dy.$$

We can therefore consider M as a unitary operator in the Hilbert space. Then operator $MH(V, \tau, \phi)M^{-1}$ coincides up to a gauge transformation with $H(V, \tau', \phi')$, where

$$\phi_1' = d\phi_1 + b\phi_2, \; \phi_2' = c\phi_1 + a\phi_2.$$

The last formula shows, in particular, that the family of Landau Hamiltonians $H(V, \tau, 0)$ is $SL(2, Z)$-invariant and the corresponding parameter space is nothing else but the moduli space of elliptic curves. It is a 2-sphere with two conical points and one puncture. It can be conveniently represented by the fundamental domain of $SL(2, Z)$-action on the upper half plane of complex variable τ, Fig. 2 [13]. It coincides with the parameter space of flat metrics on a torus of fixed area. This is an analog of the Aharonov-Bohm flux torus in the theory of Hall conductance.

The geometric significance of the viscosity in Eq. (17) is now apparent: the first term in Eq. (16) is the invariant area form on the upper half plane of complex variable τ. Similar to the Hall conductance, which is constant in the flux space, the viscosity is proportional to the area form in the moduli space of elliptic curves.

In the Hall effect, the conductance is associated with a topological invariant - Chern number, which is an integer. This integer comes from integrating the curvature over parameter space. In the case of Eq. (18) this integer is 1. For the viscosity the situation is almost the same. Integration of the adiabatic curvature over the moduli space (or fundamental domain F of the group $SL(2,Z)$) gives

$$\frac{1}{2\pi}\int_F \frac{B}{4}\frac{d\tau_1\wedge d\tau_2}{\tau_2^2}=\frac{B}{24}, \tag{19}$$

where we have used the fact that the area of the fundamental domain is $\pi/3$. Though not an integer in general, this is still a topological invariant - Chern number (in the orbifold sense) of the ground state bundle on the parameter space. (It is not an integer because the parameter space is not a smooth compact manifold in this case).

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Figure Captions

Fig. 1. Parallelogram associated with complex parameter τ.

Fig. 2. Fundamental domain of $SL(2,Z)$.