Almost Everything about the Almost Mathieu Operator I*

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Abstract: We review some aspects of the spectral theory of the Almost Mathieu operator (acting on $\ell^2(Z)$):

$$(H_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n) \ ,$$

where $\alpha, \lambda, \theta \in R$. We concentrate on the spectrum as a set, and describe the partial results obtained so far on the conjectures that, for irrational α, it is a Cantor set and has Lebesgue measure $|4 - 2|\lambda||$.

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1. Introduction

The Almost Mathieu operator is the discrete one dimensional Schrödinger operator (acting on $\ell^2(Z)$) given by:

$$(H_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n) \ , \tag{1.1}$$

where $\alpha, \lambda, \theta \in R$. Its name comes from the similarity to the Mathieu equation:

$$-y''(x) + \lambda\cos(x)\, y(x) = Ey(x) \ . \tag{1.2}$$

It is often also called Harper's operator, although many think that Harper's name should be reserved for the special case of $H_{\alpha,\lambda,\theta}$ with $\lambda = 2$.

$H_{\alpha,\lambda,\theta}$ is a tight binding model for the Hamiltonian of an electron in a one dimensional lattice, subject to a commensurate (if α is rational) or incommensurate (if α is irrational) potential. It is also related to the Hamiltonian of an electron in a two dimensional lattice, subject to a perpendicular (uniform) magnetic field. There are two different ways (or limits) in which this relation can be obtained. The first, going back to Harper [11], is to start with a tight binding model of a two dimensional rectangular lattice (which only takes into account nearest neighbors interaction), and then to consider a Landau gauge for the magnetic field. Namely, the vector potential is taken to be in one direction, parallel to one of the directions of the lattice and perpendicular to the other. This makes the Hamiltonian separable, such that the eigenfunctions are plain waves in the direction which is perpendicular to the vector potential, and one obtains $H_{\alpha,\lambda,\theta}$ for the direction of the vector potential. α is the magnetic flux per unit cell (in quantum flux units), θ is the wave-number of the plane waves in the transversal direction, and $\lambda/2$ is the ratio between the length of a unit cell in the direction of the vector potential and its length in the transversal direction. In particular, $\lambda = 2$ corresponds to a square lattice. This approach is closely related to the standard Landau gauge solution for free electrons (in the plane) in a uniform magnetic field, where one gets plane waves in one direction and the harmonic oscillator (which gives rise to Landau levels) in the other direction. In this sense, $H_{\alpha,\lambda,\theta}$ appears as a tight binding analog of the harmonic oscillator. The second way, going back at least to Rauh [17], is to start with free electrons (in the plane) in a uniform magnetic field, and to consider the perturbation of a single Landau level arising from a weak periodic (sinusoidal) potential. The magnetic flux per unit cell is $1/\alpha$ in this case. In both of these ways, the relevant energy spectrum is the union over θ of the individual energy spectra of $H_{\alpha,\lambda,\theta}$. Namely, it is the set $S(\alpha,\lambda)$ defined by:

$$S(\alpha,\lambda) \ \equiv \ \bigcup_{\theta} \sigma(\alpha,\lambda,\theta) \ \ , \tag{1.3}$$

where

$$\sigma(\alpha,\lambda,\theta) \;\equiv\; \mathrm{Spec}\,(H_{\alpha,\lambda,\theta}) \;\; . \tag{1.4}$$

Other than its relations with some fundamental physical problems, $H_{\alpha,\lambda,\theta}$ is fascinating also because of the incredible spectral richness obtained by varying the parameters α, λ, θ, along with the fact that, to a large extent, this richness can be rigorously analyzed. It serves as a primary example for many spectral phenomena.

This is the first of two papers, in these proceedings, which are intended to survey the spectral theory of the Almost Mathieu operator, and it will concentrate on the spectrum as set. The other paper, by S. Jitomirskaya, will concentrate on aspects of the spectral decomposition, and the remarkable metal-insulator transition occurring (for irrational α) at $|\lambda| = 2$. In Section 2, we review some elementary spectral properties of $H_{\alpha,\lambda,\theta}$. In Section 3, we discuss the "Ten Martini Problem," namely, we describe the partial results obtained so far on the conjecture that the spectrum (for irrational α) is a Cantor set. In Section 4, we discuss the Lebesgue measure of the spectrum.

2. Some Elementary Spectral Properties

If α is a rational number, $\alpha = p/q$, where p and q are relatively prime, $H_{\alpha,\lambda,\theta}$ is a periodic Jacobi matrix, and by Bloch-Floquet theory, the structure of the spectrum $\sigma(\alpha,\lambda,\theta)$ is well understood. It consists of q bands (closed intervals), which are usually separated by gaps. As θ is varied, these bands move, and their length may change, but this happens in such a way that different bands never overlap (other than in band edges). Namely, an energy which is inside a given band (suppose that we always label the bands by their order of occurrence on the real line) for some θ, will never be in a different band for any other θ (see, e.g., [3]). Thus, the set $S(\alpha,\lambda)$ is similar to $\sigma(\alpha,\lambda,\theta)$ for any individual θ, and also consists of q bands. If α is irrational, it follows from general principles [7] that the spectrum is independent of θ. Thus, we have in such case $S(\alpha,\lambda) = \sigma(\alpha,\lambda,\theta)$ for any θ.

An important property of $H_{\alpha,\lambda,\theta}$ is the Aubry duality [1], which allows to relate eigenfunctions and spectra of the Almost Mathieu operator with some given λ to those of the Almost Mathieu operator with λ replaced by $4/\lambda$. This duality can be understood from a physical viewpoint: If we consider the magnetic field problem described above, and change the gauge of the vector potential into a Landau gauge in the transversal direction, we obtain again the Almost Mathieu operator, but with λ replaced by $4/\lambda$, and the whole operator rescaled by a factor of $\lambda/2$. Since the energy spectrum of the magnetic field Hamiltonian must be gauge independent, we should expect:

$$S(\alpha,\lambda) \;=\; \frac{\lambda}{2} S(\alpha, 4/\lambda) \; . \tag{2.1}$$

Indeed, (2.1) was rigorously proven (to hold for any real α) by Avron and Simon [2]. ¿From (2.1), we see, in particular, that it is sufficient to study $S(\alpha,\lambda)$ for $0 \le \lambda \le 2$, since for $\lambda > 2$, $S(\alpha,\lambda)$ is obtained immediately from the $\lambda < 2$ case.

In the rational case $\alpha = p/q$, $S(\alpha,\lambda)$ (which consists of q bands) can have at most $q-1$ gaps. It turns out that these gaps are always open, other than the middle gap for even q. This fact was recently proven by van Mouche [25], and also, independently, by Choi, Elliot, and Yui [6]. In fact, Choi, Elliot, and Yui have even obtained an explicit lower bound on the size of each open gap, such that the theorem is:

Theorem 2.1. *If $Q \ni \alpha = p/q$, and $|\lambda| \le 2$, then all gaps (except the middle gap, for q even) of $S(\alpha,\lambda)$ are open, and have width larger than $(|\lambda|/16)^q$.*

Since $H_{\alpha,\lambda,\theta}$ is strongly continuous in α, $\sigma(\alpha,\lambda,\theta)$ is also continuous, in the sense that if $E \in \sigma(\alpha,\lambda,\theta)$ and $\alpha_n \to \alpha$, then there are points $E_n \in \sigma(\alpha_n,\lambda,\theta)$ such that $E_n \to E$. The set $S(\alpha,\lambda)$ has even better continuity properties, and various results on this continuity have been obtained. The strongest (at least as far as the implications we discuss below are concerned) is due to Avron, van Mouche, and Simon, who have shown uniform (in both α and the energy) Hölder continuity of order $1/2$. Namely,

Theorem 2.2. *For fixed λ and $|\alpha - \alpha'| < C(\lambda)$, each $E \in S(\alpha,\lambda)$ has $E' \in S(\alpha',\lambda)$ with: $|E - E'| < 6|\lambda(\alpha-\alpha')|^{1/2}$.*

In particular, Theorem 2.2 implies that for every gap in $S(\alpha,\lambda)$ with measure $|g|$ larger than $12|\lambda(\alpha-\alpha')|^{1/2}$, there is a corresponding gap in $S(\alpha',\lambda)$ with measure larger than $|g| - 12|\lambda(\alpha-\alpha')|^{1/2}$.

3. The Ten Martini Problem

It is a long standing conjecture that for all irrational α's and all λ's, $S(\alpha,\lambda)$ is a Cantor set; namely, a closed, nowhere dense set, which has no isolated points. The absence of isolated points for $S(\alpha,\lambda)$ is known from general principles about ergodic Jacobi matrices [7]. Thus, the remaining question is whether or not the spectrum is nowhere dense.

The Cantor structure was conjectured in a 1964 paper by Azbel [4]. In 1976, Hofstadter [13] found clear numerical evidence that for $|\lambda| = 2$, $S(\alpha,\lambda)$ is a Cantor set of zero Lebesgue measure. In 1981, during a talk at the AMS annual meeting, Mark Kac offered ten Martinis to anyone who shows that $S(\alpha,\lambda)$ "has all its gaps there." This led

Simon [19] to name it "The Ten Martini Problem." In fact, there are two versions of the problem: The regular form is to prove that for all λ's and irrational α's, $S(\alpha,\lambda)$ is a Cantor set. The strong (or dry) form is to prove that all the gaps which are allowed by the Gap Labeling Theorem (see [19] and references therein) are really open. For simplicity, we shall only relate to the (weaker) regular form.

Several partial results on the Ten Martini Problem have been obtained so far. They establish the Cantor structure either in certain limits or for special values of α and/or λ. In general, however, the problem is still open.

The first rigorous result on the Cantor structure of $S(\alpha,\lambda)$ has been given in 1982 by Bellissard and Simon [5]. They have shown that for a dense G_δ of pairs $(\lambda,\alpha)\in R^2$ (and thus "generically," in the commonly used topological sense), $S(\alpha,\lambda)$ is a Cantor set. The α's which are relevant to their proof need to be very well approximated by rationals, but are not really specified. Nor are the λ's.

Sinai, in a 1987 paper [20], has proven that for (Lebesgue) a.e. α, and sufficiently large (or small) $|\lambda|$, $S(\alpha,\lambda)$ is a Cantor set. Sinai's proof is perturbative and quite complicated. The relevant α's are those with typical diophantine properties (namely, are not too well approximated by rationals), such that they are explicitly given. The λ's are not explicitly given, and the required largeness (or smallness) of $|\lambda|$ may depend on α. An alternative perturbative proof of Sinai's result was recently given by Goldstein [8].

In 1989, Helffer and Sjöstrand [12] have proven that $S(\alpha,\lambda)$ is a Cantor set, for the special case $|\lambda| = 2$, and for a special class of irrationals characterized through their continued fraction expansions as follows: Let

$$\alpha = [n_1, n_2, n_3, \dots] = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}} , \tag{3.1}$$

then there exists C such that if $n_i > C \;\; \forall i$ then $S(\alpha,2)$ is a Cantor set (of zero Lebesgue measure). Their analysis is based on a unique spectral renormalization procedure, which exploits elaborate semi-classical analysis, and is based on ideas (and a heuristic analysis) of Wilkinson [26]. The class of relevant α's is nowhere dense and of zero Lebesgue measure.

In their 1990 paper, Choi, Elliot, and Yui [6] have proven that $S(\alpha,\lambda)$ is a Cantor set for all λ's, and for (Liouville) α's obeying the diophantine condition: $|\alpha - p/q| < C^{-q}$ for infinitely many rationals p/q. C is an explicitly given constant. This set of relevant α's is of zero Lebesgue measure, but it contains a dense G_δ. In particular, their result strengthens the Bellissard-Simon result. The Choi-Elliot-Yui result is a simple consequence of their lower bound on gap sizes for rational α's (Theorem 2.1), along with the continuity properties of $S(\alpha,\lambda)$ (Theorem 2.2).

More recently, Last [15] has shown that for $|\lambda|=2$, and (Lebesgue) a.e. α, $S(\alpha,\lambda)$ is a (zero measure) Cantor set. The Cantor structure, in this case, follows immediately from showing that $S(\alpha,\lambda)$ has zero measure. The relevant α's are explicitly given in terms of a diophantine condition or, equivalently, as having unbounded continued fraction expansions (see more details below). While this class of α's is intuitively larger than the class of α's discussed by Helffer and Sjostrand, it should be pointed out that their result covers some α's that are excluded from Last's result.

For an additional review of results on the Ten Martini Problem and many references, we refer the reader to the recent paper by Shubin [18].

4. Lebesgue Measure of the Spectrum

In a 1980 paper [1], Aubry and Andre described numerical evidence that for p,q relatively prime:

$$\lim_{q\to\infty} |S(p/q,\lambda)| \;=\; |4-2|\lambda||\,, \tag{4.1}$$

where $|\cdot|$ denotes Lebesgue measure. This led to the conjecture that for every irrational α:

$$|S(\alpha,\lambda)| = |4-2|\lambda||\,, \tag{4.2}$$

which, in particular, agrees with Hofstadter's 1976 numerical findings [13] that $|S(\alpha,2)| = 0$ (α irrational). It should be pointed out that the set $S(\alpha,\lambda)$ and its structure do depend strongly on α. Clear evidence for this can be found in Guillement, Helffer, and Treton's 1989 paper [9], which contains many numerically computed illustrations of Almost Mathieu spectra. The universality of the measure, predicted by (4.2), is, therefore, a rather remarkable property.

The first rigorous work on this issue was done by Thouless [21] in 1983. He has shown that for every rational p/q: $|S(p/q,\lambda)| \;\geq\; |4-2|\lambda||$, which, by the continuity of $S(\alpha,\lambda)$, implies:

$$|S(\alpha,\lambda)| \;\geq\; |4-2|\lambda|| \tag{4.3}$$

for any real α.

In their 1990 paper [3], Avron, van Mouche, and Simon, have proven that for $|\lambda|\leq 2$ and p,q relatively prime:

$$4-2|\lambda| \;\leq\; |S(p/q,\lambda)| \;\leq\; 4-2|\lambda| + 4\pi\left(\frac{|\lambda|}{2}\right)^{q/2}\,, \tag{4.4}$$

and, moreover,

$$|S_-(p/q,\lambda)| \;=\; 4-2|\lambda| \;, \tag{4.5}$$

where $S_-(\alpha,\lambda) \equiv \bigcap_\theta \sigma(\alpha,\lambda,\theta)$. (4.4) proves the numerical observation (4.1) of Aubry and Andre, for any $|\lambda| \neq 2$. A 1993 observation of Last [14] has shown that (4.4) combined with Theorem 2.2 implies the conjecture (4.2) for $|\lambda| \neq 2$ and (Lebesgue) a.e. α. For $|\lambda| = 2$, however, (4.4) implies nothing, and the vanishing of the measure in this case remained open.

The $|\lambda| = 2$ case was recently tackled by Last and Wilkinson [16] and Last [15], who have shown that

$$\frac{2(\sqrt{5}+1)}{q} \;<\; |S(p/q,2)| \;<\; \frac{8e}{q} \;, \tag{4.6}$$

where $e \equiv \exp(1) = 2.71\ldots$. It is obtained as a $\lambda \to 2$ limit of the remarkable exact equality (4.5) of Avron, van Mouche, and Simon. (4.6) is strongly related to a conjecture of Thouless [21,22,23,24] that $\lim_{q\to\infty} q|S(p/q,2)| = \text{Const} = 9.32\ldots$, and, in particular, it allows to complete the proof of the following theorem:

Theorem 4.1. *If α is an irrational, for which there is a sequence of rationals $\{p_n/q_n\}$ obeying:*

$$\lim_{n\to\infty} {q_n}^2 \left|\alpha - \frac{p_n}{q_n}\right| \;=\; 0 \;,$$

then for every $\lambda \in R$:

$$|S(\alpha,\lambda)| = |4-2|\lambda|| \;.$$

Remark. The set of irrationals characterized in Theorem 4.1 is precisely the set of irrationals having unbounded continued fraction expansions. This set is known to have full Lebesgue measure [10].

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