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INDUCED FERMION NUMBER, PHASE-SHIFT FLIP, AND THE AXIAL ANOMALY IN THE AHARONOV-BOHM POTENTIAL

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ABSTRACT

The spectral properties of the Dirac and the Klein-Gordon Hamiltonians in the the Aharonov-Bohm potential are discussed. By using the Krein-Friedel formula, the density of states (DOS) for different self-adjoint extensions is calculated. As in the nonrelativistic case, whenever a bound state is present in the spectrum it is always accompanied by a (anti)resonance at the energy proportional to the absolute value of the binding energy. The presence of the bound state manifests itself by an asymmetric differential scattering cross section and gives rise to the Hall effect. The Aharonov-Casher and the index theorems must be corrected for singular field configurations. There are no zero (threshold) modes in the Aharonov-Bohm potential. For our choice of the 2d Dirac Hamiltonian, the phase-shift flip is shown to occur at only positive energies. This flip gives rise to a net surplus of η states at the lower threshold coming entirely from the continuous part of the spectrum. The results are applied to several physical quantities: the total energy, induced fermion-number, and the axial anomaly. Stability of the system is discussed. The predictions of a persistent current in the presence of a cosmic string and a gravitational vortex are made.

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1 Introduction

In this paper the change $\triangle\rho_\alpha(E)$ over all space of the density of states (DOS) induced by the Aharonov-Bohm (AB) potential $\mathbf{A}(r)$ [1],

$$A_r = 0, \qquad A_\varphi = \frac{\Phi}{2\pi r} = \frac{\alpha}{2\pi r}\,\Phi_0, \tag{1}$$

in the radial gauge, is calculated for the two-dimensional (2d) Dirac and the Klein-Gordon Hamiltonians. We shall talk about the AB potential in a more general sense here. Usually $\Phi = \alpha\,\Phi_0$ is the total flux through the flux tube and Φ_0 is the flux quantum, $\Phi_0 = hc/|e|$. However, the same potential (1) is induced by a cosmic string provided the identification $\alpha = e/Q_{Higgs}$ is made, where e and Q_{Higgs} are respectively the charge of a test particle and the charge of the Higgs particle [2, 3]. In what follows, n and η will be respectively the integer and the fractional part of $\alpha = n + \eta$.

In Sec. 2 basic properties of the Dirac Hamiltonian are summarized. If one neglects anomalous magnetic moment of the electron, the Dirac Hamiltonian in the AB potential (and in any magnetic field) considered as an operator in L^2 is a formal operator with supersymmetry [4]. In Sec. 2.2 the point spectrum [5] of the Dirac Hamiltonian is discussed. A contradiction between periodicity of the spectrum under the substitution $\alpha \to \alpha \pm 1$ (as long as α does not change its sign) and the Aharonov-Casher theorem [6] is resolved. One finds that there are no threshold (zero) modes in the AB potential for any α (see Fig. 1). As will be argued throughout the paper at many occasions, the Aharonov-Casher and the index theorems [4, 6] must be corrected for singular field configurations in the sense that for such configurations these theorems give rather an upper bound on the number of threshold or zero modes in a given finite-flux background than their actual number. Recent calculation of the Witten index in the AB potential confirms this conclusion [7].

In Secs. 2.3 and 3 phase shifts in the AB potential in the supesymmetric case and for a general self-adjoint extension are reviewed. Phase shifts are shown to be invariant under the change of the sign of energy, $E \to -E$, except for the single channel, $l = -n - 1$. A nontrivial self-adjoint extension is characterized by the presence of a bound state, and, as we shall show later, by supersymmetry breaking. In Sec. 3.3 it will be shown that a nontrivial self-adjoint extension manifests itself in an asymmetric differential scattering cross section (cf. [8, 9]) which gives rise to the Hall effect.

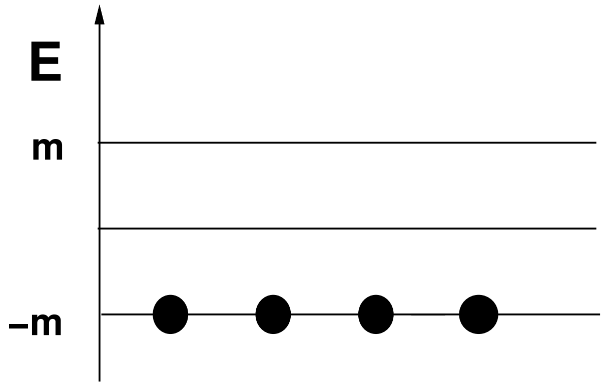

Figure 1: In the case of a 2d Dirac Hamiltonian in a generic finite-flux magnetic field the threshold states occur at one and only one threshold. In the case of the 2d Euclidean massless Dirac operator they are actually zero modes. There are, however, neither threshold nor zero modes in the AB potential. In the latter case, the spectral asymmetry is entirely accounted for by the continuous part of the spectrum. The phase-shift flip which occurs at positive energies in the $l = -n - 1$ channel gives rise to the net surplus of η states at the lower threshold.

Similarly as in the nonrelativistic case [8, 9], our calculation of the contribution of continuous spectrum to the integrated DOS (IDOS) will be based on the analysis of phase shifts and use of the Krein-Friedel formula [10, 11]. This is discussed in Sec. 4. The Krein-Friedel formula [10, 11] gives the change of the IDOS as the sum over all phase-shifts. This formula is the basic tool for calculating the change of the IDOS in solid state physics induced by potentials of a finite range. Recently we have shown in the discussion of the nonrelativistic AB scattering that, when regularized by the zeta function, it gives the correct answer even for the long-ranged AB potential [8, 9]. First, we shall consider a generic self-adjoint extension, when the supersymmetry of the Dirac Hamiltonian is lost and a bound state with energy E_b occurs in the spectrum in the channel $l = -n - 1$. We shall calculate the change of the IDOS and show that the bound state is always accompanied by a (anti)resonance [12]. The shape of the (anti)resonance is not of the Breit-Wigner type. A physical scenario under which a nontrivial self-adjoint extension describes physics in the AB potential is then discussed in Sec. 5.

In the supersymmetric case, after taking the limits $E_b \to \pm m$ in the channel $l = -n-1$, it is shown that the compatibility of our results requires the phase-shift flip [13] to occurs at positive energies but not at negative energies.

The DOS $\rho_\alpha(E)$ provides an important link between different physical quantities. It enables one to calculate the total energy and effective action, and to discuss the stability properties of matter against the spontaneous creation of a magnetic field. This question is of principal importance in understanding the nature of the ground state both in field theories and in condensed matter physics. The knowledge of $\triangle\rho_\alpha(E)$ determines the spectral asymmetry $\sigma_\alpha(\mathcal{E})$,

$$\sigma_\alpha(\mathcal{E}) = \triangle\rho_\alpha(\mathcal{E}) - \triangle\rho_\alpha(-\mathcal{E}), \tag{2}$$

where $\mathcal{E} = |E| > 0$. In contrast to three dimensions, the spectrum of the 2d Dirac Hamiltonian is in general *asymmetric* [4], and, if the supersymmetry holds, the only asymmetry can occur at the mass thresholds [4]. Roughly speaking, nonvanishing spectral asymmetry means that there is a surplus of states at positive energies with respect to the states at negative energies, or vice versa. Because in the supersymmetric case the point spectrum is empty, spectral asymmetry comes entirely from the continuous part of the spectrum. As we shall show in Sec. 6, the phase-shift flip which occurs in the $l = -n-1$ channel at positive energies gives rise to the net surplus of η "states" at the *lower* threshold, i. e., exactly at the threshold where zero modes would occur in the case of a regular flux tube with nonzero radius (see Fig. 1).

The spectral asymmetry measures how the vacuum is distorted under the presence of an external field, and therefore it is natural that spectral asymmetry $\sigma_\alpha(\mathcal{E})$ determines the one-loop contribution E^1_{eff} to the effective energy [14], the induced fermion number Q, and, in the case of the massless Euclidean Dirac operator, the axial anomaly $\mathcal{A}$. One has

$$E^1_{eff} = \int_0^\infty \mathcal{E}\,\sigma_\alpha(\mathcal{E})\,d\mathcal{E}, \tag{3}$$

$$Q = -\frac{1}{2}\int_0^\infty \sigma_\alpha(\mathcal{E})\,d\mathcal{E}, \qquad \mathcal{A} = \int_0^\infty \sigma_\alpha(\mathcal{E})\,d\mathcal{E}, \tag{4}$$

and $Q = -\mathcal{A}/2$ [15, 16]. By using the above relations one can check for the consistency of the result for either of these quantities. Moreover, thanks to the recent advances in the fabrication of microstructures and in mesoscopic physics, one can realize the AB potential and study its influence in many physical systems (see [17] for a recent review). The flux tube in the above experiments can be either penetrable or impenetrable [18]. By the latter we mean the standard Aharonov-Bohm experiment when the interior of the flux tube was

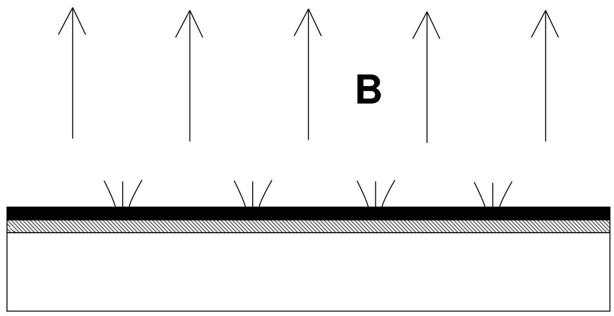

Figure 2: Black layer is a superconductor of type II put on top of the heterostructure containing the two-dimensional electron gas (shaded region). When this sample is put in a homogeneous magnetic field, magnetic field penetrates the superconductor in Abrikosov vortices. Therefore, electrons from the heterostructure do not move in the homogeneous magnetic field but in the field of a (penetrable) flux tube.

isolated from the physical system under consideration [1]. This is also the case of electrons moving on a ring or an annulus threaded by the flux tube. A realistic physical realization of the penetrable flux tube is that suggested originally by Rammer and Shelankov [18] and later realized experimentally by Bending, Klitzing, and Ploog [19], i. e., to put a type II superconducting gate on top of the heterostructure containing the two-dimensional electron gas (2DEG) (see Fig. 2). When a magnetic field is switched on the conventional superconductor is penetrated by vortices of flux with $\alpha = 1/2$. Therefore, electrons from the heterostructure do not move in the homogeneous magnetic field but in the field of a penetrable flux tube.

In Sec. 8 we shall check our method for the calculation of the DOS and the spectral asymmetry for the case of a gravitational vortex. In Sec. 9, the change of the IDOS is calculated for the Klein-Gordon equation. Eventually, in Sec. 10, our results are summarized and their experimental consequences are considered.

2 Basic properties of the Dirac Hamiltonian

Let us collect the basic properties of two (2d) and three (3d) dimensional Dirac Hamiltonians that will be used in the paper (see [4] for a recent review). The Dirac Hamiltonian $H(A)$ in the *standard* representation in a magnetic field (in units with $\hbar = c = 1$) is

$$H(A) = \begin{pmatrix} m & D^\dagger \\ D & -m \end{pmatrix}, \tag{5}$$

where

$$D = \begin{cases} \sum_{j=1}^{3} \sigma_j (p_j + A_j) & n = 3, \\ \sum_{j=1}^{2} (p_1 + A_1) + i(p_2 + A_2) & n = 2, \end{cases} \tag{6}$$

and $p_j = i\partial_j$. We have taken the standard representation of γ matrices in 2d,

$$\gamma^o = \beta = \sigma_3, \qquad \gamma^1 = i\sigma_2, \qquad \gamma^2 = -i\sigma_1, \tag{7}$$

where σ_j are the Pauli matrices. Let us lift the coordinate z in the 3d Dirac Hamiltonian and restrict ourselves to the plane perpendicular to the flux tube. Then

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \sigma_1 p_1 + \sigma_2 p_2 = i \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix}. \tag{8}$$

This term is nothing but the massless 2d Euclidean Dirac operator. After some algebra one finds that the components (χ^1, χ^4) and (χ^3, χ^2) of the four-spinor $\chi = (\chi_1, \chi_2, \chi_3, \chi_4)$ combine to give the components of two spinors that satisfy 2d Dirac equations with *opposite* signs of the mass term. In two space dimensions, mass breaks the parity: one has two irreducible representation of the γ matrices since, in contrast to 3d, $\{\gamma_0, \gamma_1, \gamma_2\}$ and $\{-\gamma_0, -\gamma_1, -\gamma_2\}$ are not unitary equivalent. The above decomposition just tells us why the 3d Dirac Hamiltonian with non-zero mass does not break parity despite the fact that the 2d Dirac Hamiltonian with non-zero mass does.

In $3d$ one has $D^\dagger = D$ and the spectrum is symmetric in energy, while $D^\dagger \neq D$ in $2d$ and the spectrum is asymmetric. In the latter case, if supersymmetry holds, one can show that respective restrictions of D on $(\mathrm{Ker}\, D)^\perp$ and $D^\dagger$ on $(\mathrm{Ker}\, D^\dagger)^\perp$, where Ker denotes zero-energy subspace, are unitary equivalent (see [4], p. 144) and, as we have stated, the only asymmetry occurs at mass thresholds.

2.1 Dirac Hamiltonian in the AB potential

Let us fix a 2d Dirac Hamiltonian $H(A)$ in the AB potential by fixing the sign of mass m and the flux $\alpha \geq 0$ to be positive, and $e = -|e|$. After separation of variables in polar coordinates, $H(A)$ reduces to the direct sum, $H(A) = \oplus_l h_{m,l}$, of channel operators $h_{m,l}$ in $L^2[(0,\infty), rdr]$,

$$h_{m,l} = \begin{bmatrix} m & -i\left(\partial_r + \frac{\nu+1}{r}\right) \\ -i\left(\partial_r - \frac{\nu}{r}\right) & -m \end{bmatrix}, \tag{9}$$

where $\nu = l + \alpha$ [3]. One finds immediately that

$$h^*_{m,l}|_{m\to -m} = -h_{m,l}, \tag{10}$$

where '$*$' means the complex conjugation. Therefore, for $E = -\mathcal{E} < -m$, the scattering states are given by

$$\Psi_{-\mathcal{E};l}(t,r,\varphi) = \Psi^*_{\mathcal{E};l}(t,r,\varphi)|_{m\to -m}. \tag{11}$$

We recall that the parameter $\mathcal{E}$ stands for $|E|$.

2.2 Point spectrum and the Aharonov-Casher theorem

Formally, the set (9) of radial Hamiltonians is invariant under the substitution $\alpha \to \alpha \pm 1$. As long as α does not change its sign, the spectrum of $H(A)$ must be periodic in α (see discussion below). This is, however, seemingly in contradiction with the Aharonov-Casher theorem [6] which states that the point spectrum is a function of α. In a general finite-flux magnetic field $B(\mathbf{r})$,

$$\int_\Omega B(\mathbf{r})\, d^2\mathbf{r} = \Phi = \text{const}, \tag{12}$$

the Aharonov-Casher theorem tells us that the 2d Dirac Hamiltonian $H(A)$ has exactly either $\alpha - 1 = n - 1$ or n threshold states at one and only one of the thresholds $E = \pm m$ (in the present case at $E = -m$) depending whether the flux α is an integer or not. In the case of the 2d massless Euclidean Dirac operator, the threshold states are actually zero modes. The reason is that the component of the threshold mode that would be multiplied by mass m is zero and so the threshold mode is the eigenmode of $\boldsymbol{\sigma} \cdot \mathbf{p}$, too [4, 6].

To resolve this contradiction, one must check for validity of the Aharonov-Casher theorem in the case of a singular flux tube with zero radius. In the latter case, one must check for square integrability not only at infinity but at the position of singularities of the field, too. It is here where the theorems fail. If one takes proof of the Aharonov-Casher theorem as presented, for example in Ref. [4], p. 198, and substitutes for $B(\mathbf{r})$,

$$B(\mathbf{r}) = 2\pi\alpha\, \delta^{(2)}(\mathbf{r}) = \frac{\alpha}{r}\, \delta(r), \tag{13}$$

one finds immediately that threshold (zero) modes are

$$e^{-\phi(r)},\ e^{-\phi(r)}(x_1 - ix_2),\ \ldots,\ e^{-\phi(r)}(x_1 - ix_2)^{n-1}, \tag{14}$$

where

$$\phi(\mathbf{r}) = \frac{e}{2\pi} \int_{R^2} \ln |\mathbf{r} - \mathbf{r}'| \, B(\mathbf{r}') \, d^2\mathbf{r}' = \alpha \ln |\mathbf{r}|. \tag{15}$$

Threshold (zero) modes are obviously singular at the origin and are not elements of $L^2[(0,\infty), rdr]$. One can show this directly by using $h_{m,l}$ as well. For a threshold state at $E = m$ to exist, the upper component χ_1 of the Dirac spinor has to obey

$$\left(\partial_r - \frac{\nu}{r}\right) \chi_1(r) = 0. \tag{16}$$

At the threshold $E = -m$ one then obtains

$$\left(\partial_r + \frac{\nu + 1}{r}\right) \chi_2(r) = 0. \tag{17}$$

These two equations can be easily integrated. The solutions are

$$\chi_1(r) = r^{\nu} \qquad \text{and} \qquad \chi_2(r) = r^{-(1+\nu)}, \tag{18}$$

and they are obviously not in $L^2[(0,\infty), rdr]$ for any l. If they are square integrable at the infinity they are not so at the origin and *vice versa*.

2.3 Continuous spectrum

In the case of a continuous spectrum, the respective radial equation for the up (down) component of the spinor reduces to the Bessel-like equation with a δ-function potential,

$$H_l^{\pm} \chi(r) = k^2 \chi(r) \tag{19}$$

where

$$H_l^{\pm} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\nu_{\pm}^2}{r^2} \pm g_m \frac{\alpha}{r} \delta(r). \tag{20}$$

Here $g_m = 1$, $(\nu_+ = l + \alpha / \nu_- = l + 1 + \alpha)$ for the (up/down) component [13]. The dispersion relation is $k = \sqrt{E^2 - m^2}$. The spectrum of the Dirac Hamiltonian for $E > m$ is given in terms of the Bessel functions,

$$\Psi_{\mathcal{E},l} = \chi(r) e^{il\varphi} e^{-i\mathcal{E}t/\hbar}, \tag{21}$$

where

$$\chi(r) = \frac{1}{N} \begin{pmatrix} \sqrt{\mathcal{E} + m} \, (\varepsilon_l)^l J_{\varepsilon_l \nu}(kr) \\ i\sqrt{\mathcal{E} - m} \, (\varepsilon_l)^{l+1} J_{\varepsilon_l(\nu+1)}(kr) e^{i\varphi} \end{pmatrix}, \tag{22}$$

N is a normalization factor, and $\varepsilon_l = \pm 1$. Then square integrability at the origin fixes the sign of ε_l except for the channel $l = -n-1$ [3]. The two-component solutions of the massive Dirac equation have only *one degree of freedom* that is reflected in the equality of up and down phase shifts [20]. Except for the channel $l = -n-1$, phase shifts are invariant under the change of the sign of the energy, $E \to -E$, and are given by

$$\delta_l^u = \delta_l^d = \begin{cases} -\pi\alpha, & l > -n-1, \\ \pi\alpha, & l < -n-1. \end{cases} \tag{23}$$

The AB potential is *long-ranged* and the phase shifts δ_l are *singular*: they do not depend on the energy and do not decay to zero for $E \to \infty$. In the channel $l = -n-1$ *two different solutions with opposite phase shifts* are possible,

$$\chi^-(r) = \frac{1}{N}\begin{pmatrix} \sqrt{\mathcal{E}+m}\, J_{\eta-1}(kr) \\ i\sqrt{\mathcal{E}-m}\, J_\eta(kr)e^{i\varphi} \end{pmatrix}, \tag{24}$$

and

$$\chi^+(r) = \frac{1}{N}\begin{pmatrix} \sqrt{\mathcal{E}+m}\, J_{1-\eta}(kr) \\ -i\sqrt{\mathcal{E}-m}\, J_{-\eta}(kr)e^{i\varphi} \end{pmatrix}. \tag{25}$$

By looking at the behaviour as $r \to \infty$ of these two solutions, one finds, for example that for $n = 0$, their phase shifts are

$$\delta^- = -\frac{1}{2}\pi\eta, \qquad\qquad \delta^+ = \frac{1}{2}\pi\eta. \tag{26}$$

Note that state χ^+ (χ^-) is *regular* (singular) in the limit $\eta \to 0_+$ and *singular* (regular) in the limit $\eta \to 1_-$. If one takes solution χ^+ for $l = -n-1$, then phase shifts for all l are given by

$$\delta_l^u = \delta_l^d = \frac{1}{2}\pi(|l| - |l+\alpha|). \tag{27}$$

This ambiguity in the channel $l = -n-1$ is the consequence of the fact that the AB potential is in the so-called *limit circle case* at the origin (see Ref. [21], p.152) where the boundary conditions have to be specified. A given combination of χ^+ and χ^- then characterizes a particular self-adjoint extension of $H(A)$ [3]. Provided that $\alpha = -n-\eta < 0$, the critical channel is $l = n$. Therefore, despite the fact that the set (9) of radial Hamiltonians is invariant under the substitution $\alpha \to \alpha \pm 1$, the spectrum is not. The spectrum is periodic only as long as α does not change its sign.

3 Self-adjoint extensions

Distinguished feature of a generic self-adjoint extension is that the point spectrum is not empty and a bound state can occur with energy inside the mass gap $(-m, m)$ [3, 22]. Both supersymmetry and the scale invariance are broken in this case.

3.1 Point spectrum

The bound state of energy E_b in the channel $l = -n - 1$ is given in terms of the modified Bessel functions [3, 23],

$$B_{E_b;-n-1}(t, r, \varphi) = \frac{1}{N} \begin{pmatrix} \sqrt{m + E_b}\, K_{1-\eta}(\kappa r) \\ i\sqrt{m - E_b}\, K_\eta(\kappa r) e^{i\varphi} \end{pmatrix} e^{-i(n+1)\varphi} e^{-iE_b t/\hbar}. \tag{28}$$

Here $K_\nu(z)$ is the modified Bessel function [24] and $\kappa = \sqrt{m^2 - E_b^2}$. The wave function is in $L^2[(0, \infty), r dr]$ as it decays exponentially outside the flux tube and is square integrable at the origin. However, in the limit $E_b \to \pm m$ the square integrability is lost, in accord with our statement about the nonexistence of threshold states in the AB potential.

3.2 Continuous spectrum

In the presence of a bound state of energy E_b scattering states have to be modified. The reason is that the Dirac Hamiltonian $h_{m,l}$ has to be a symmetric operator when defined on a dense subset of $L^2[(0, \infty), r dr]$, such as the set of absolutely continuous functions regular at the origin [3]. The condition implies that any two states $\chi_1(\mathbf{r})$ and $\chi_2(\mathbf{r})$ in the Hilbert space $L^2[(0, \infty), r dr]$ have to satisfy

$$r\, \phi^\dagger \sigma_1 \xi = r\, \left[\chi_2^* \chi^1 + \chi_1^* \chi^2\right] \to 0 \qquad (r \to 0), \tag{29}$$

where σ_1 is the Pauli matrix. Eq. (29) is nothing but the boundary condition at the origin. To satisfy (29) scattering states with energy $E > m$ in the channel where the bound state is present have to take a general form

$$\Psi_{\mathcal{E};-n-1}(t, r, \varphi) = \left[\chi^+(r)\, \sin\mu + \chi^-(r)\, \cos\mu\right] e^{-i(n+1)\varphi} e^{-i\mathcal{E}t/\hbar}, \tag{30}$$

with $\chi^+(r)$ and $\chi^-(r)$ given respectively by (24) and (25) [25]. In contrast to [3] we shall not use the factor $(-1)^{n+1}$ for $\chi^-(r)$. Similarly, as in the non-relativistic case [8, 9], in the presence of a bound state of energy E_b the condition (29) determines the parameter μ of the self-adjoint extension to be a function of energy E,

$$\tan\mu = \frac{m-E_b}{(m^2-E_b^2)^\eta}\frac{(\mathcal{E}^2-m^2)^\eta}{\mathcal{E}-m}\,. \tag{31}$$

The formula is only valid for $E>m$. Scattering states at energies $E<-m$ are obtained directly from positive energy states by using relation (11).

If necessary, $\mu_\pm$ will be written to distinguish between positive and negative energies. A close inspection of (31) reveals that under $m\to -m$ both the sign and the value of $\tan\mu_+$ change,

$$\tan\mu_+ \to \tan\mu_- = -\frac{m+E_b}{(m^2-E_b^2)^\eta}\frac{(\mathcal{E}^2-m^2)^\eta}{\mathcal{E}+m}\,. \tag{32}$$

The change amounts to the following two changes in (31),

$$E_b\to -E_b, \qquad \mathcal{E}\to-\mathcal{E}. \tag{33}$$

One cannot simply replace $\mathcal{E}$ by $-\mathcal{E}$ in (31) to obtain the value of μ_- for the negative energy states.

The spectrum [see (30) and (31)] is different for different bound state energies E_b. Therefore, in physical terms, it is the bound state energy E_b that parametrizes different self-adjoint extensions. Now, by comparing the asymptotic behaviour of (30) for $r\to\infty$ with the behaviour when $\alpha=0$ one finds the channel S-matrix $S_{-n-1}=e^{2i\delta_{-n-1}}$ with the phase shift given by

$$\delta_{-n-1}(E) = \frac{1}{2}\pi\alpha + \triangle_{-n-1}(E), \tag{34}$$

where

$$\triangle_{-n-1} = -\arctan\frac{\sin(\eta\pi)}{\cos(\eta\pi)-\tan\mu}\,. \tag{35}$$

The analysis presented here again shows consistency of our result about the absence of threshold (zero) modes in the AB potential. As we have seen, the $l=-n-1$ channel is the only channel where the sign of factor ε_l in (22) is not fixed by the square integrability condition at the origin and where, in principle, a bound state can appear (see Sec. 3.1). On the other hand, the Aharonov-Casher theorem allows for zero modes only in the channels $l=0,-1,\ldots,-n+1$ [cf. (14)], none of which is the critical channel $l=-n-1$.

3.3 Differential scattering cross sections and the Hall effect

Once the phase shifts are known, one can calculate the S matrix, $s_\alpha(\varphi)$, according to

$$s_\alpha(\varphi) := \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{2i\delta_l + il\varphi}, \tag{36}$$

where φ is the scattering angle. One can show that the result is given by [9]

$$s_\alpha = s_\alpha^0 + \triangle s_\alpha^0 \tag{37}$$

where

$$s_\alpha^0(\varphi) = \delta(\varphi)\cos\pi\alpha + i\,\frac{\sin\pi\alpha}{\pi}\, e^{-in\varphi}\, \frac{P}{e^{i\varphi}-1}, \tag{38}$$

is the S matrix in the norelativistic case [26], and

$$\triangle s_\alpha^0(\varphi) = \left[e^{i\pi\alpha + 2i\triangle_{-n-1}} - e^{i\pi\alpha}\right] e^{-i(n+1)\varphi} = 2ie^{i\pi\alpha + i\triangle_{-n-1} - i(n+1)\varphi} \sin\triangle_{-n-1}. \tag{39}$$

P in (38) denotes an analog of the principal value [26]. Formally,

$$P\,\frac{1}{e^{i\varphi}-1} = \lim_{\epsilon\downarrow 0} \frac{1}{2}\left[\frac{1}{e^{i\varphi}-(1-\epsilon)} + \frac{1}{e^{i\varphi}-(1+\epsilon)}\right]. \tag{40}$$

Similarly as in the nonrelativistic case [9], one can verify that although the S matrix is not symmetric with respect to $\varphi \to -\varphi$, the S matrix still satisfies relation

$$s_{-|\alpha|}(\varphi) = s_{|\alpha|}(-\varphi) \tag{41}$$

under the sign reversal of the flux α.

The differential scattering cross section $d\sigma/d\varphi$ is then extracted from the S matrix $s_\alpha(\varphi)$. One finds for $\varphi \neq 0$ that

$$\begin{aligned}\left(\frac{d\sigma}{d\varphi}\right)(k,\varphi) &= \left(\frac{d\sigma^0}{d\varphi}\right)(k,\varphi) + \\ &\frac{8\pi}{k}\sin^2\triangle_{-n-1} + \frac{4}{k}\frac{\sin(\pi\alpha)}{\sin(\varphi/2)}\sin\triangle_{-n-1}\cos\left(\triangle_{-n-1} + \pi\alpha - \varphi/2\right),\end{aligned} \tag{42}$$

where

$$\left(\frac{d\sigma^0}{d\varphi}\right)(k,\varphi) = \frac{1}{2\pi k}\frac{\sin^2(\pi\alpha)}{\sin^2(\varphi/2)}. \tag{43}$$

is the standard nonrelativistic AB differential scattering cross section [26]. Now, it can be verified explicitly from (42) that differential scattering cross section $d\sigma/d\varphi$ is periodic

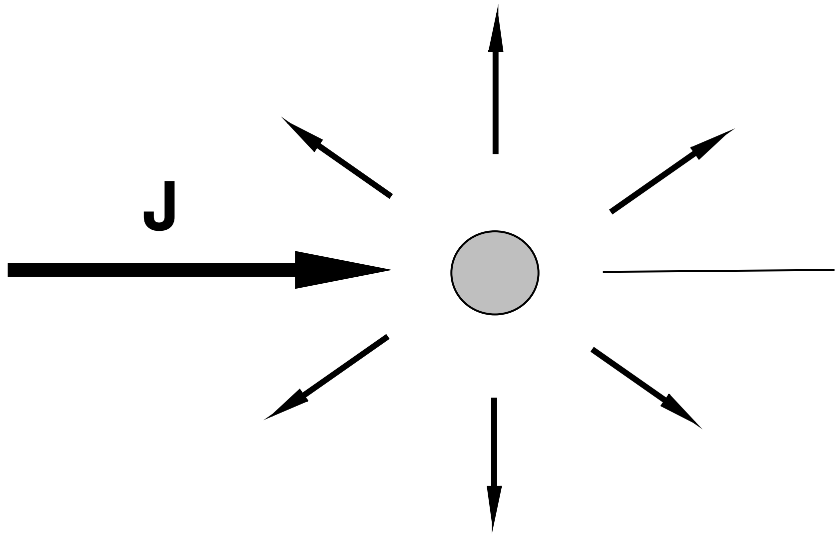

Figure 3: Incident current **J** scatters off the flux tube. The fraction of incident current **J** which is scattered to the angle φ is proportional to the differential scattering cross section $d\sigma(k,\varphi)$. In the case of an asymmetric differential scattering cross section there is generically a surplus of particles flowing in one of the transverse directions.

under the substitution $\alpha \to \alpha \pm 1$ (as long as α does not change its sign) and *asymmetric* with regard to $\varphi \to -\varphi$. The asymmetry is both expected and easy to understand as one has only one bound state (28) which, for $\alpha \geq 0$, occurs for $l < 0$. We shall confirm this by explicite calculation.

The asymmetry has an important experimental consequences since it give rise to the Hall effect [9]. The differential scattering cross section gives the fraction of particles from the incident beam that are scattered off to the angle φ (see Fig. 3). One can easily find the net surplus of particles flowing in one of the transverse direction and find the Hall resistivity, ρ_{xy},

$$\rho_{xy} = \rho_H^0 \frac{k_F}{\alpha} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \sin\varphi \, \frac{d\sigma}{d\varphi}(k_F, \varphi), \tag{44}$$

$\rho_H^0 = Bc/n_e e$ is the Hall resistivity in a uniform magnetic field B, n_e is the density of electrons, e is the electronic charge, and k_F is the charactersitic momentum (Fermi energy in the nonrelativistic case) [9]. The uniform magnetic field B is obtained by averaging over the field produced by vortices with the density n_v,

$$B = n_v \alpha \Phi_0. \tag{45}$$

When the differential scattering cross section (42) is inserted in Eq. (44) one finds that only the last term in (42) contributes and

$$\rho_{xy} = \frac{4n_v}{n_e}\frac{hc^2}{e^2}\sin(\pi\alpha)\sin\triangle_{-n-1}\cos(\triangle_{-n-1}+\pi\alpha). \tag{46}$$

As a self-consistency check, the Hall resistivity ρ_{xy} vanishes whenever α is an integer.

4 The density of states and the Krein-Friedel formula

As has been established earlier, the point spectrum consists at best of one point that lies in the interval $-m < E < m$. Let us denote by $N_V(E)$ and $N_0(E)$ respectively the integrated density of states (IDOS) induced by the presence of a scatterer V and the free IDOS. The contribution of scattering states to the change $\triangle N_V(E)$ of the IDOS induced by the presence of a scatterer with a finite range of interaction is known to be determined by the Krein-Friedel formula [10] directly as the sum over all phase shifts,

$$\triangle N_V(E) = N_V(E) - N_0(E) = \frac{1}{\pi}\sum_l \delta_l(E) = (2\pi i)^{-1}\ln\det \mathrm{S}, \tag{47}$$

S being the total on-shell S matrix. As usually, the IDOS $N_V(E)$ here is defined as

$$N_V(E) = \int_{-\infty}^{E} \rho_V(E')\,dE', \tag{48}$$

where

$$\rho_V(E) = -\frac{1}{\pi}\mathrm{Im}\,\mathrm{Tr}\,G_V(\mathbf{x},\mathbf{x},E+i\epsilon), \tag{49}$$

and $G_V(\mathbf{x},\mathbf{x},E+i\epsilon)$ is the resolvent of $H = H_0 + V$ in the presence of a scatterer V. Similarly for $N_0(E)$. Formula (47) is the consequence of a more general Krein's formula,

$$\mathrm{Tr}\,[G_V(z) - G_0(z)] = \int_{-\infty}^{\infty}\frac{\xi(\lambda)}{(\lambda-z)^2}\,d\lambda, \tag{50}$$

where

$$\xi(\lambda) = \frac{i}{2\pi}\ln\,\det\mathbf{S}(\lambda). \tag{51}$$

$\xi(\lambda)$ is a bounded function, called the Krein *spectral displacement operator* [10, 28]. The formula (50) is valide whenever $H = H_0 + V$ and H_0 are trace comparable [28], which means essentially that the operation on the left-hand side in Eq. (50) has a meaning.

In a recent papers [8, 9] we have shown that the Krein-Friedel formula (47) can have a wider range of applicability. In particular, we have established that the Krein-Friedel formula can also be used for the long-ranged AB potential, provided that the sum over phase shifts is regularized by the ζ function. The fact that phase shifts can be rather easily calculated without any care of the proper normalization of wave functions greatly facilitates the calculation. Moreover, by means of the Krein-Friedel formula it is rather easy to calculate the change of the IDOS for all possible self-adjoint extensions of (20). In the presence of the bound state, the phase shift is only changed from its value given by Eq. (23) in the channel $l = -n - 1$. It the latter channel, the phase shift is given by formulae (34) and (35). Therefore, by using the ζ-function regularization [8, 9] one finds in each energy continuum that

$$\begin{aligned}
\ln \det \mathrm{S} &= \sum_{l=-\infty}^{\infty} 2i\delta_l = 2i\triangle_{-n-1} + i\pi \sum_{l=-\infty}^{\infty} (|l| - |l+\alpha|) \\
&= 2i\triangle_{-n-1} + i\pi \left[2\sum_{l=1}^{\infty} l^{-s} - \sum_{l=0}^{\infty} (l+\eta)^{-s} - \sum_{l=1}^{\infty} (l-\eta)^{-s} \right]\Bigg|_{s=-1} \\
&= 2i\triangle_{-n-1} + i\pi \left[2\zeta_R(s) - \zeta_H(s,\eta) - \zeta_H(s,1-\eta) \right]\big|_{s=-1} \\
&= 2i\triangle_{-n-1} - i\pi\eta(1-\eta),
\end{aligned} \tag{52}$$

where ζ_R and ζ_H are the Riemann and the Hurwitz ζ-functions. Therefore, according to the Krein-Friedel formula (47),

$$\triangle N_\alpha(E) = -\frac{1}{2}\eta(1-\eta) - \frac{1}{\pi} \arctan \frac{\sin(\eta\pi)}{\cos(\eta\pi) - \tan\mu} \cdot \tag{53}$$

One can verify that the presence of a bound state does not spoil periodicity of $\triangle N_\alpha(E)$ as long as α does not change its sign. One can repeat the calculation for $\alpha \leq 0$ to check that

$$\triangle N_{-|\alpha|}(-E) = \triangle N_{|\alpha|}(E). \tag{54}$$

For $E > m$ one finds that $\tan\mu_+$ is positive [see (31)] and hence there is a typical *antiresonance* in the relativistic case for $0 < \eta < 1/2$ at the energy $\mathcal{E}_{res}$ which satisfies relation

$$\frac{\mathcal{E}_{res} - m}{(\mathcal{E}_{res}^2 - m^2)^\eta} = \frac{1}{\cos\eta\pi} \frac{m - E_b}{(m^2 - E_b^2)^\eta}. \tag{55}$$

The phase shift $\delta_{-n-1}(E)$ (34) changes by $-\pi$ in the direction of increasing energy and the IDOS (53) has a sharp *decrease* by one.

For $1/2 < \eta < 1$ the cosine in (53) changes its sign and the antiresonance disappears at positive energies [27]. Instead, a resonance occurs at the negative energy continuum at the energy $E_{res} = -\mathcal{E}_{res}$ which is determined by the equation

$$\frac{\mathcal{E}_{res} + m}{(\mathcal{E}_{res}^2 - m^2)^\eta} = \frac{1}{|\cos\eta\pi|}\frac{m + E_b}{(m^2 - E_b^2)^\eta}. \tag{56}$$

$\eta = 1/2$ is a special point since the (anti)resonance is at infinity. Obviously, the shape of the (anti)resonance is not of the Breit-Wigner type.

4.1 Supersymmetric case and the phase-shift flip

One knows that in the supersymmetric case there is neither bound state nor the (anti)resonance. Therefore, to get rid of the bound state we shall consider limits $E_b \to \pm m$. If E_b changes, so do E_{res} and $\tan\mu_\pm$. Limiting values of E_{res} and $\tan\mu_\pm$ are presented in Tab. I. Formally, both cases $E_b \uparrow m$ and $E_b \downarrow -m$ seem to be plausible. In each case, scattering states in one of the energy continuums are described by state χ^+ and in the other continuum by state χ^-. As a result, one has the *phase-shift flip* in one energy continuum,

$$\delta^+ \to \delta^- = -\frac{1}{2}\pi\eta,$$

and the conventional phase shift $\delta^+ = \pi\eta/2$ in the second energy continuum.

TABLE I. Limiting values of E_{res} and $\tan\mu_\pm$ when $E_b \to \pm m$.

	$0<\eta<1/2$	$1/2<\eta<1$	$0<\eta<1$	$0<\eta<1$
$E_b \uparrow m$	$E_{res} \downarrow m$	$E_{res} \uparrow -m$	$\tan\mu_+ \to 0$	$\tan\mu_- \to -\infty$
$E_b \downarrow -m$	$E_{res} \uparrow \infty$	$E_{res} \downarrow -\infty$	$\tan\mu_+ \to \infty$	$\tan\mu_- \to 0$

However, these two limits are different. In the first case, the limiting values of E_{res} are at thresholds, while in the second case the limiting values of E_{res} are at infinity. If E_b approaches the upper threshold and $0 < \eta < 1/2$, one finds that the antiresonance also approaches the upper threshold,

$$\mathcal{E}_{res} \sim m + (\cos\eta\pi)^{1-\eta}(m - E_b), \tag{57}$$

and in the limit the antiresonance “annihilates” with the bound state.

Compatibility of our results with previous calculations of the axial anomaly [16] requires that the limit $E_b \uparrow m$ be taken. In the latter case, irrespective of the value of η, one finds for any fixed $E > m$ that $\tan\mu_+ \to 0$. Therefore, $\mu_+ = 0$ for positive energies [see (31)], and scattering states are

$$\Psi_{\mathcal{E};-n-1}(t,r,\varphi) = \chi^-(r)\, e^{-i(n+1)\varphi} e^{-i\mathcal{E}t/\hbar}. \tag{58}$$

In the same limit, for a fixed negative energy then [see (32)]

$$\tan\mu_- \to -\infty, \tag{59}$$

and hence $\mu_- = -\pi/2$. Therefore, at negative energies, $E = -\mathcal{E} < -m$ scattering states are

$$\Psi_{-\mathcal{E};-n-1}(t,r,\varphi) = \chi^+(r)\, e^{-i(n+1)\varphi} e^{i\mathcal{E}t/\hbar}. \tag{60}$$

One can rephrase this calculation by saying that the phase-shift flip occurs only either at positive energies or only at negative energies depending on whether the bound state (28) “annihilates” with the (anti)resonance into the continuous spectrum either at the upper or at the lower threshold.

Therefore, in the *supersymmetric* case, the contribution of scattering states with energy $E \geq m$ to the change of the DOS is

$$\triangle\rho_\alpha(E) = \rho_\alpha(E) - \rho_0(E) = -\frac{1}{2}\eta(3-\eta)\,\delta(E-m). \tag{61}$$

while the contribution of scattering states with $E \leq -m$ to the change of the DOS is

$$\triangle\rho_\alpha(E) = -\frac{1}{2}\eta(1-\eta)\,\delta(E+m). \tag{62}$$

As expected, the change of the DOS is concentrated at the thresholds where it is proportional to delta functions.

5 Regularization, $R \to 0$ limit, and the interpretation of selfadjoint extensions

To identify the physics that corresponds to different self-adjoint extensions one starts with a flux tube of finite radius R and a magnetic field B satisfying (12), and considers the limit $R \to 0$ [3, 22]. The limit is curious in the following sense. Since the flux tube is not singular any more the Aharonov-Casher [4, 6] (or index [29]) theorem applies and one has, in general, $n-1$ or n threshold (zero) modes (at the lower thershold) depending on whether the flux is an integer or not. Then, in the limit $R \to 0$ these modes merge with the continuous spectrum. To obtain the bound state in the spectrum for $R \neq 0$, the resulting interaction inside the flux tube for both up and down components of the Dirac spinor has to be an attractive potential $V(r)$ of strength at least equal to

$$V(r)|_{r\leq R} = -\frac{\hbar^2}{2m}\frac{\alpha}{R^2}c(R), \tag{63}$$

where $V(r) = 0$ for $r > R$. Here, $c(R) = 2[1+\varepsilon(R)]$ and $\varepsilon(R) > 0$ [22, 30]. Note that in the limit $R \to 0$

$$V(r)|_{r\leq R} \to -\left[1+\varepsilon(0)\right]\frac{\hbar^2}{m}\frac{\alpha}{r}\delta(r). \tag{64}$$

If one starts with a homogeneous field regularization [$B(x,y) = const$ inside the flux tube] the magnetic moment coupling induces an *attractive* potential $V_m(r)$ for the *down* component,

$$V_m(r)|_{r\leq R} = -g_m\frac{\alpha}{R^2}, \tag{65}$$

and a *repulsive* potential $-V_m(r)$ for the *up* component inside the flux tube. Therefore, in order to obtain a bound state in the spectrum the potential $W_c(r)$,

$$W_c(r) = V(r) + V_m(r), \tag{66}$$

has to be put in by hand. Here $V_m(r)$ cancels the repulsive potential $-V_m(R)$ for the up component. An arbitrary weak attractive potential *cannot* lead to bound states (cf. [13]). For a bound state to survive as $R \to 0$ the coupling constant has to be renormalized, $\varepsilon(R) \to 0$ as $R \to 0$ [31]. The potential $W_c(r)$ is then the *critical* potential. For either weaker or for stronger potentials and without the renormalization, bound states do not survive the limit $R \to 0$ [9]. If $\varepsilon(R)$ in (63) is small but kept constant as $R \to 0$ then, in the nonrelativistic case, $E_b \to -\infty$ [8, 9, 32]. Nevertheless, in the limit $R \to 0$ bound states decouple. This is seen in the phase shifts, since $\triangle_l(E) \to 0$ when $E_b \to -\infty$ [8, 9]. In the relativistic case then the bound states moves through the gap $(-m, m)$ from one continuum into another and no trace of them remains as $R \to 0$, except when $W_c(R)$ is present. The same situation persists with the cylindrical shell regularization but the number of bound states for a given α can be *different* [8, 9].

6 Spectral asymmetry, effective energy, induced fermion number, and the axial anomaly

In the absence of the bound state, when supersymmetry holds, the spectral asymmetry is entirely determined by the continuous part of the spectrum. One finds from (61) and (62) that the contribution is actually determined by the phase-shift flip,

$$\sigma_\alpha(\mathcal{E}) = -\,\eta\,\delta(\mathcal{E}-m). \tag{67}$$

The phase-shift flip which occurs at positive energies in the $l=-n-1$ channel gives rise to the net surplus of η states at the lower threshold with respect to the upper energy threshold (see Fig. 1). Correspondingly, according to (3),

$$E^1_{eff} = -\,m\,\eta, \tag{68}$$

and consequently, according to (4),

$$Q = \frac{1}{2}\,\eta, \qquad\qquad \mathcal{A} = -\,\eta. \tag{69}$$

When α reaches 1, then, in the limit $\alpha \to 1_-$, there is seemingly a surplus of one state at the lower threshold with respect to the upper threshold. However, for $\alpha = 1$ all quantities of interest, including basic set (9) of radial Hamiltonians and wave functions (22) are identical to those in the case for $\alpha = 0$. By using the fact that phase shifts are only defined modulo π, one finds that phase shifts depend periodically (although discontinuously) on α [cf. (23)]. Therefore, we argue that spectral asymmetry σ_α, and consequently E^1_{eff}, Q, and $\mathcal{A}$ change *discontinuously* to their original zero values for $\alpha = 1$. This can already be seen from our results (61-62) for the change of the DOS. Otherwise, periodicity of the spectrum under the substitution $\alpha \to \alpha \pm 1$ (as long as α does not change its sign) would be lost. Actually, we have already encountered a discontinuous behaviour in the discussion of the phase-shift flip. Despite that χ^- is the singular state, nonintegrable at the limit $\eta \to 0_+$, it is this state which occurs in the spectrum for $E \geq m$, and not state χ^+, square integrable at the origin for $\eta = 0$.

Another justification for our conclusion for the discontinuous behaviour of the spectral asymmetry comes from the nonrelativistic case. In the latter case one knows that there

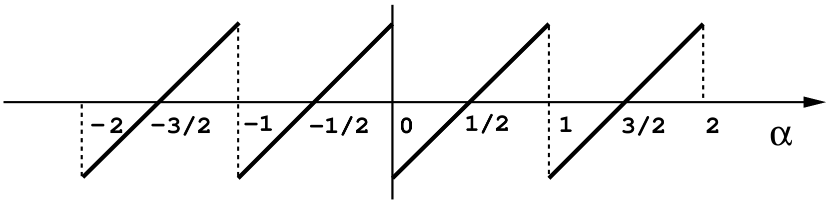

Figure 4: Persistent currents depend discontinuously on the flux α. The figure shows the persistent current of spinless nonrelativistic fermions as a function of α.

exists a physical quantity closely associated with the density of states [33], and hence with the spectral asymmetry, which changes discontinuously with the flux α (cf. Fig. 4). This quantity is the so-called *persistent current* induced by the AB potential [34]. A persistent current is essentially due to the momentum that electrons acquire in the AB potential. The persistent current is defined with respect to a point. It is given by the total current through a line that extends from that point to infinity, in the absence of currents through the external leads [35]. According to the formulae in [9, 33, 35], the scattering-state contribution to the persistent current of free electrons in the plane, pierced by the flux tube, is directly related to the change $\triangle N_\alpha(E)$ of the IDOS,

$$dI(E,\alpha) = \frac{e}{2\pi}\partial_\alpha[\triangle N_\alpha(E)]dE, \tag{70}$$

where $dI(E,\alpha)$ is the differential contribution to the persistent current at energy E. Because formula (70) involves only the IDOS, in an ideal situation measurements of a persistent current can test the results for the IDOS. For example, in the nonrelativistic case of spinless fermions [9] one has

$$\triangle N_\alpha = -\eta(1-\eta)/2. \tag{71}$$

When this results are inserted in Akkermans et el. formula (70), one obtains a discontinuous persistent current which was observed experimentally [36].

In the case for a generic self-adjoint extension when the bound state (28) is present, the supersymmetry is lost, because spectral asymmetry σ_α is no longer concentrated at the mass thresholds, and respective restrictions of D on $(\operatorname{Ker} D)^\perp$ and $D^\dagger$ on $(\operatorname{Ker} D^\dagger)^\perp$ are no longer unitary equivalent. By using results (53) for the IDOS one finds

$$\sigma_\alpha(\mathcal{E}) = \operatorname{sign}(E_b)\,\delta(\mathcal{E}-|E_b|) + \frac{1}{\pi}\frac{d}{d\mathcal{E}}\left\{\left[\triangle^+_{-n-1}(\mathcal{E}) - \triangle^-_{-n-1}(\mathcal{E})\right]\Theta(\mathcal{E}-m)\right\}, \tag{72}$$

where Θ is the Heaviside step function, and $\triangle^{\pm}_{-n-1}(\mathcal{E})$ is the change of the conventional phase shift (35) respectively for positive and negative energy states in the $l = -n-1$ channel. The superscripts '$\pm$' means that the value of $\tan\mu_{\pm}$ is taken in (35). The values of E^1_{eff}, Q, and $\mathcal{A}$ are then obtained by substituting (72) in (3) and (4). It should be stressed out that $\sigma_\alpha(\mathcal{E})$ can have both signs depending on the sign of the mass in a 2d Dirac Hamiltonian (6), (9). In the case of the 3d Dirac Hamiltonian, according to expectations

$$\sigma_\alpha(\mathcal{E}) = E^1_{eff} = Q = 0. \tag{73}$$

7 Energy calculations

As has been shown, in the case of the Dirac Hamiltonian (in contrast to the Schrödinger Hamiltonian) the bound state does not occur when the flux tube is regularized with a given distribution of magnetic field satisfying the finite flux condition (12), in the absence of a sufficiently strong attractive potential inside the flux tube. In this case, the one-loop contribution E^1_{eff} to the effective energy in 2+1 dimensions is negative but stays finite [see Eq. (68)]. However, E^1_{eff} only reflects a distortion of the fermionic sector and to discuss the total energy one must also consider energy E_B of the magnetic field

$$E_B = \frac{1}{2}\int_\Omega B^2(\mathbf{r})\, d^2\mathbf{r}. \tag{74}$$

The classical energy of the magnetic field (74) tends to infinity when the radius R of the flux tube shrinks to zero and the total flux Φ is kept fixed. For example, in the case of a homogeneous field regularization,

$$E_B = \frac{\pi}{2}B^2R^2 = \frac{\Phi^2}{2\pi R^2} \to \infty \qquad (R \to 0). \tag{75}$$

By summing the two contributions, E^1_{eff} and E_B, one finds that the total energy changes by an infinite positive amount. From the 3d point of view, acoording to Eq. (73), the one-loop contribution E^1_{eff} to the effective energy is zero and the total energy again changes by an infinite positive amount. Therefore, up to one-loop the matter is *stable* against a spontaneous field formation. The latter statement is in accord with the so-called *diamagnetic inequality* [37].

In the nonrelativistic case in $2+1$ dimensions we have shown, in an idealistic situation when magnetic moment $g_m > 2$ is kept constant and dynamics is ignored that a quantum-mechanical instability against a magnetic field formation arises. The reason is the formation of bound states that decouple from the Hilbert space by taking away negative energy [8, 9]. One finds that whenever the ratio of the energy at rest to the electromagnetic energy satisfies an inequality

$$\frac{mc^2}{e^2} < X(\alpha, g_m), \tag{76}$$

the total energy of the field and matter together goes to $-\infty$ as $R \to 0$, where R is the radius of the flux tube. The flux tube has been regularized by a homogeneous magnetic field $B(\mathbf{r})$. Note that the homogeneous magnetic field optimizes the energy functional (74) subject to the constraint (12). The function $X(\alpha, g_m)$ is determined by the solutions x_l of the matching equation,

$$X(\alpha, g_m) = \frac{1}{4\pi\alpha^2} \sum_l x_l^2(\alpha, g_m) \geq 0. \tag{77}$$

The relativistic treatment shows that the instability is in fact a nonrelativistic artefact.

8 Gravitational vortex

It is illustrative to check our method for the calculation of the DOS and the spectral asymmetry in the AB potential for the case of a gravitational vortex [38]. A localized massive particle of mass M in a 2+1 dimensional space-time induces a conical geometry with a deficit angle $\delta\varphi = 2\pi(1-\gamma) \equiv 8\pi GM$, where G is Newton's constant. Solutions of the Dirac equation in this case are given essentially by (21-22) provided $\varepsilon_l \nu = \varepsilon_l (l+\alpha)$ is replaced by $\nu = |l|/\gamma$ [38]. Since

$$0 < \gamma \leq 1, \tag{78}$$

it is impossible to find a nontrivial self-adjoint extension because one of the components of the Dirac spinor is always given by a Bessel function of order $\nu \geq 1$. Hence, the phase shift for all l has the conventional form,

$$\delta_l^u = \delta_l^d = \frac{1}{2}\pi |l| \left(1 - \gamma^{-1}\right) \tag{79}$$

(see (23) or [38]) and no phase-shift flip occurs. By using the Krein-Friedel formula, the change of the DOS is given by

$$\triangle\rho_\gamma(E) = -\frac{1}{12}\left(1-\gamma^{-1}\right)\left[\delta(E-m)+\delta(E+m)\right], \tag{80}$$

and the spectral asymmetry $\sigma_\gamma(E) \equiv 0$. Hence, there is no induced charge in this case in accord with [39].

9 The Klein-Gordon equation

Let us consider the Klein-Gordon Hamiltonian with the minimal and an additional *non-minimal* Pauli coupling,

$$J^\mu\left[eA_\mu - (1/2)g\lambda^{-2}\varepsilon_{\mu\nu\sigma}F^{\nu\sigma}\right], \tag{81}$$

with λ an arbitrary scale parameter. It is peculiar to $2+1$ dimensions that the Pauli coupling exits without any reference to the spin [40]. The first term couples the current to the gauge potential while the second, nonminimal terms, couples the current directly to the magnetic field and as such it is an analogue of the magnetic moment coupling in the Pauli Hamiltonian [40]. By using separation of variables in polar coordinates one then arrives exactly at H_l^+ [see (20)]. The latter is nothing but the radial part of the Pauli Hamiltonian [8, 9, 13] with only one change: the dispersion has the relativistic form, $k = \sqrt{E^2-m^2}$. In contrast to the Dirac equation the wave function of a scattering state has a single component which is given in terms of Bessel functions of order $|l+\alpha|$. Therefore, the equivalent results for the Klein-Gordon equation are obtained by substituting the relativistic dispersion into nonrelativistic results given in [8, 9]. Bound states can occur in two channels, $l=-n$ and $l=-n-1$. The parameter μ_l in these channel is then determined by [8, 9, 25]

$$\begin{aligned}\cot\mu_{-n} &= -A_{-n} = -(k/\kappa_{-n})^{2\eta},\\ \cot\mu_{-n-1} &= -A_{-n-1} = -(k/\kappa_{-n-1})^{2(1-\eta)},\end{aligned} \tag{82}$$

where $\kappa_l = \sqrt{m^2-E_l^2}$ is as above, with E_l, $0\le E_l\le m$, being the binding energy either in the $l=-n$ or in the $l=-n-1$ channel. The corresponding change of the conventional phase shift (23) (the latter is the same as in the case of the Schrödinger and the Dirac equations) is then

$$\triangle_l = \arctan\left(\frac{\sin(|l+\alpha|\pi)}{\cos(|l+\alpha|\pi) - A_l^{-1}}\right). \tag{83}$$

For $0 < \eta < 1/2$ the *resonance* appears in the $l = -n$th channel,

$$E_{res}^2 - m^2 = \frac{m^2 - E_{-n}^2}{[\cos(\eta\pi)]^{1/\eta}} > 0. \tag{84}$$

The phase shift $\delta_{-n}(E)$ changes by π [see (83)] in the direction of increasing energy and the change of the integrated density of states,

$$\triangle N_\alpha(E) = -\frac{1}{2}\eta(1-\eta) + \frac{1}{\pi}\arctan\left(\frac{\sin(\eta\pi)}{\cos(\eta\pi) - A_{-n}^{-1}}\right) - \frac{1}{\pi}\arctan\left(\frac{\sin(\eta\pi)}{\cos(\eta\pi) + A_{-n-1}^{-1}}\right), \tag{85}$$

has a sharp increase by one. For $1/2 < \eta < 1$ the resonance is then shifted to the $l = -(n+1)$th channel.

10 Discussion of the results

By extending the Krein-Friedel formula to the case of a singular Aharonov-Bohm potential we were able to express all physical quantities discussed here in terms of scattering phase shifts. We have

- obtained closed analytical form for the density of states $\rho_\alpha(E)$ of the 2d Dirac and the Klein-Gordon Hamiltonians induced by the Aharonov-Bohm potential and predicted a resonance to occur whenever a bound is in the spectrum;
- calculated spectral asymmetry $\sigma_\alpha(\mathcal{E})$, the one-loop contribution E_{eff}^1 to the effective energy, the induced charge Q, and the axial anomaly $\mathcal{A}$ of the Dirac operator in the Aharonov-Bohm potential for a generic self-adjoint extension;
- obtained closed analytical form for the density of states $\rho_\alpha(E)$ and the spectral asymmetry $\sigma_\alpha(\mathcal{E})$ of the 2d Dirac Hamiltonian in the field of a gravitational vortex;
- pointed out that the Aharonov-Casher and index theorems have to be corrected for singular field configurations;
- determined that the phase-shift flip only occurs for positive energies.

We have established that the DOS is a periodic function of the flux as long as the flux does not change its sign. In a particular case, when the bound state is absent and the Dirac Hamiltonian supersymmetry holds, our result for $\triangle\rho_\alpha(E)$ is consistent with an earlier calculations of the spectral asymmetry $\sigma_\alpha(\mathcal{E})$, induced charge Q, and the axial anomaly $\mathcal{A}$ [16]. For a generic self-adjoint extension, a bound state is present in the spectrum and the supersymmetry is broken. Physically, a generic self-adjoint extension corresponds to the situation when an attractive δ-function potential $W_c(r)|_{R\to 0}$ (66) is put on top of the AB potential. In the case of the "magnetic" AB potential, a repulsive interaction is induced for the spin up component of the Dirac spinor and hence, the attractive potential may be easier to realize in the AB potential of non-magnetic origin, such as in the field of a cosmic string. A nontrivial self-adjoint extension manifests itself by an asymmetric differential scattering cross section which has been calculated here. Our result (69) for induced charge Q gives another example where a transcendental charge is induced (cf. [41]).

The crucial point for deriving the closed analytical form for the axial anomaly $\mathcal{A}$ [see (4) and (72)] (cf. Ref. [42]) was to understand the relevance of the phase-shift flip. In a general finite-flux field the latter was discussed in [43]. Their treatment, however, differs from ours since they considered a complementary situation of a regular field configuration and the limit $k \to 0$.

Regarding the Aharonov-Casher and index theorems, one has to check for square integrability of solutions at the position of a singularity, too. There are neither threshold nor zero modes in the AB potential and the index of massless 2d Euclidean operator is zero. A recent calculation [7] of the resolvent regularized and the Witten indices confirms this conclusion. We stress out that if the Aharonov-Casher theorem held for the AB potential, then this would be incompatible with the periodicity of the spectrum under the substitution $\alpha \to \alpha \pm 1$ (provided that α does not change its sign). For any $R \neq 0$ the Aharonov-Casher and index theorems hold, and therefore we have in general a nonzero number of zero modes. However, in the limit $R \to 0$, zero modes merge with the continuous spectrum and index is *discontinuous* in the limit $R \to 0$.

In the AB potential, spectral asymmetry comes entirely from the continuous part of the spectrum and dependends *discontinuously* on flux α. Our analysis in term of phase shifts shows that phase shifts are invariant under $E \to -E$ except for the critical channel $l = -n-1$. The latter channel is also singlet out by the fact that only in this channel both regular and singular Bessel function are square integrable at the origin. In the channel $l = -n-1$, the phase-shift flip occurs for positive energy scattering states giving rise to the net surplus of η "states" at the *lower* threshold, i. e., at the same threshold where zero modes would occur provided that the flux tube is nonsingular and has a finite radius (see Fig. 1). The phase-shift flip cannot take place for both positive and negative energy scattering states (cf. Ref. [13]). One of the arguments is that in the latter case the spectral asymmetry, and correspondingly induced charge and the axial anomaly all would be zero.

We did not find any instability (cf. result (68) for the one-loop contribution E^1_{eff} to the effective energy) of minimally-coupled relativistic matter in 2+1 dimensions (cf. [44] when the Abelian Chern-Simons term is present). Due to the symmetry of the spectrum of the 3d Dirac Hamiltonian any question about the instability is pointless (cf. the suggestion of [45] for a 'flux spaghetti' vacuum in the spirit of [46] as a mechanism for avoiding the divergence of perturbative QED).

Our results enable us to draw the following experimental predictions:

- a persistent current will occur in the field of a cosmic string, and a similar scenario can also occur in the field of a gravitational vortex;

- resonance should occur whenever a bound state is present in the spectrum. The resonance has a profile which is not of the Breit-Wigner type.

- In the latter case when a bound state is present in the spectrum, the differential scattering cross section $d\sigma/d\vartheta$ is asymmetric and

- the Hall effect occurs.

The prediction of a persistent current follows from the formal technical similarity between scattering in the AB potential and scattering in the field of a cosmic string [2, 3] and in the field of a gravitational vortex [38].

The last three phenomena occur only in the case if a nontrivial self-adjoint extension takes place. A self-adjoint extension is actually the $R \to 0$ limit, where R is the radius of a flux tube. Experimentally, infinitely thin means nothing but that the radius of the flux tube is negligibly small when compared to any other length, such as a wavelength of particles, in the system. Therefore, this is the regime in which our results can be applied. The parameter $\triangle_{-n}$ of a particular self-adjoint extension is then determined by a bound state energy in the $l = -n$ channel. It will be interesting to consider an application of our results in the set-up (see Fig. 2) proposed by Rammer and Shelankov [18], essentially because recent measurements on YBCO-delta rings with three grain-boundary Josephson junction [47], reported the observation of vortices that curry a flux $\alpha = 1/4$ which is *smaller* than the standard flux quantum $hc/2e$ (corresponding to $\alpha = 1/2$) in the superconductor. Therefore, when the high-T_c YBCO film is used as a gate on top of the heterostructure containing 2DEG, the resonance is at some finite energy and could in principle be observed. The same experimental set-up is also promissing for detecting the Hall effect for the random distribution of vortices, at least in the nonrelativistic case.

Note that one has the unitary equivalence between a spin 1/2 charged particle in a 2d magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a 2d electric field [48] and our results apply to the this case as well.

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