EXISTENCE OF INVARIANT TORI FOR NON HAMILTONIAN PERTURBATIONS OF INTEGRABLE SYSTEMS

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Abstract We extend known results on the existence of invariant tori for non hamiltonian perturbations of integrable systems, to the case where the unperturbed system is anisocronus. The result holds for perturbations admitting a definite finite number of derivatives, and belonging to a generic class.

We address here the problem of existence of invariant tori for systems of the form

$$\begin{aligned} \dot{I} &= \varepsilon \ f(I, \varphi, \varepsilon) \\ \dot{\varphi} &= \omega(I) + \varepsilon \ g(I, \varphi, \varepsilon) \end{aligned} \tag{1}$$

where $I \in \mathcal{G} \subset \Re^n$ (with $\mathcal{G}$ open) are the slow variables, $\varphi \in \mathbf{T}^m$ are the fast angular variables, and $\varepsilon \in [0, \eta]$ is a (small) real parameter ($\eta > 0$). All functions are assumed to be at least C^2. We recall that this kind of systems arise naturally for example as non hamiltonian perturbations of integrable hamiltonian systems. Notice also that if the unperturbed system is resonant, then there are also some angles that play the role of slow variables.

We remark that for $m = 1$ the problem is of a different nature, as on the one side we are dealing with periodic solutions, and on the other we have no problems due to resonances. Thus, in the following we will assume we have to deal with a proper torus T^m, i.e. we will assume $m > 1$ (for the case $m = 1$ see e.g. [1]).

The search of invariant tori for system (1) is usually performed in three steps: first one performs a first order averaging, i.e. a coordinate transformation $(I, \varphi) = \mathcal{T}(J, \psi, \varepsilon)$ taking the system (1) into the system

$$\begin{aligned} \dot{J} &= \varepsilon \ F(J) + \varepsilon^{1+p} \ \mathcal{R}(J, \psi, \varepsilon) \ , \\ \dot{\psi} &= \omega(J) + \varepsilon \tilde{g}(J, \psi, \varepsilon) \ . \end{aligned} \tag{2}$$

where

$$F(I) := \frac{1}{(2\pi)^m} \int_{\mathbf{T}^m} f(I, \varphi, 0)\ d\varphi_1 ... d\varphi_m\ , \tag{3}$$

and p is a suitable positive number. Then one considers the truncated system

$$\begin{aligned} \dot{J} &= \varepsilon\ F(J) \\ \dot{\varphi} &= \omega(J) \end{aligned} \tag{4}$$

and looks for a zero J_0 of F; obviously the manifold $T_0 := J_0 \times \mathbf{T}^m$ is an invariant torus for the averaged systems (4). Finally one has to prove that such a torus is only deformed by a small perturbation, so that there exists an invariant torus also for system (2) and therefore also for the original system (1).

The technical difficulties one meets in making the above procedure rigorous rest essentially in justifying the averaging procedure transforming (1) into (2), and in proving that the invariant tori of (4) can be continued to invariant tori of (2).

It is particularly simple to solve the above thechnical problems when the frequency vector is diophantine and constant, i.e. independent of the slow variables I. Indeed, in such a case the averaging transformation which maps (1) in (2) is globally defined (on $\mathcal{G}$), and the problem of continuing a torus can be solved in different ways. Bogolyubov et al. (see [2]) solved it by using a KAM type technique, obtaining that there exists a large set of $\Re^m$ such that if the frequency vector belongs to such a set, and if F has a zero in $\mathcal{G}$, then the original system has an invariant torus. A different solution was given by Yagasaki[3] who realized that, since the zero of F is generically hyperbolic (i.e. all the eigenvalues of the linearization of F at such a point have non vanishing real part), the invariant torus of (4) is a normally hyperbolic invariant manifold, and therefore it persists under small perturbation[4,5].

On the other side, we were not able to find in the literature any result on the general case where $\omega = \omega(I)$ is non constant. In the present note we prove under generic conditions that an invariant torus of the averaged system can be continued to an invariant torus of the original system (1) even when the frequency is actually dependent on I.

Here the question is more delicate than in the case of constant frequency. Indeed, the averaging transformation is defined only on the set of slow variables on which the frequency vector is sufficiently non resonant; therefore the system (1) is equivalent to (2) only on such a set. It follows that an invariant torus of (4) is also an invariant torus of (1) only if it is completely contained in such a nonresonant set. Moreover, the nonresonant set depends on ε, and when $\varepsilon = 0$ it has a dense complement; therefore the question of existence of invariant tori for (1) requires some care.

The rest of the paper is devoted to the proof of existence of an invariant torus for the general case.

In order to obtain such a proof we have to look more in detail at the averaging procedure. Indeed it is well known that the averaging method holds as long as the frequencies are nonresonant enough. The whole problem consists in specifying quantitatively what does it means "enough".

To this aim, fix a positive K, and split f as $f = f^- + f^+$, where

$$f^-(I,\varphi) := \sum_{|k|\le K} f_k(I)\; e^{ik\cdot\varphi}\ , \tag{5}$$

and $f_k(I)$ is the $k-th$ Fourier coefficient of $f(I,\varphi,0)$. The key remark is that, due to the decay of the Fourier coefficients of a smooth function, taking $K = O(\varepsilon^{-b})$ with some suitable $b > 0$, one can consider f^+ as being a higher order perturbation. Therefore the construction of the averaging transformation involves only f^- which, by definition, contains only Fourier components of order less than K. This allows in turn to show that the system (2) is equivalent to (1) on an open set, of size of the order of some (negative) power of K. In order to make quantitative the above procedure let us consider the distinguished subset of points $I \in \mathcal{G}$ characterized by a diophantine frequency $\omega(I)$. Formally, we fix two positive real constants γ and τ, and consider the set $\Gamma(\gamma,\tau)$ defined as

$$\Gamma(\gamma,\tau) = \big\{ I \in \mathcal{G}\ :\ |k\cdot\omega(I)| \ge \gamma |k|^{-\tau} \text{ for all } k \in \mathbf{Z}^m \setminus \{0\} \big\}\ . \tag{6}$$

Notice that for $\tau > m-1$, the union over γ of the above sets has full mesure. For positive ρ we define the extension Γ_ρ of Γ as

$$\Gamma_\rho(\gamma,\tau) = \left(\bigcup_{I\in\Gamma(\gamma,\tau)} B_\rho(I) \right) \cap \mathcal{G}\ , \tag{7}$$

where $B_\rho(I)$ is the closed sphere of radius ρ and center I. Then, following the lines of the proof of lemma 4.3 of ref. [6] it is easy to proof the following

Lemma. *Consider the system of differential equations (1); assume that f, g and ω are C^2 on the closure of $\mathcal{G} \times \mathbf{T}^m \times [0,\eta]$. Moreover assume that there exist a positive r and a constant $\mathcal{C}$ such that, for each fixed $\varepsilon \in [0,\eta]$ the C^2 norm of $f^+(I,\varphi,\varepsilon)$ is bounded by $\mathcal{C}K^{-r}$ on $\mathcal{G}\times\mathbf{T}^m$. Then, provided ε is small enough, there exists a differentiable coordinate transformation defined on $\Gamma_\rho(\gamma,\tau)$, with*

$$\rho = O(\varepsilon^{(\tau+1)/\alpha})$$

($\alpha = 2\tau + r + 1$) such that in the new variables the system (1) takes the averaged form (2) with $p = r/\alpha$. Moreover, for each fixed $\varepsilon \in [0,\tau]$, the C^1 norms of $\mathcal{R}$ and of $\tilde{g}$ are bounded on $\Gamma_\rho(\gamma,\tau) \times \mathbf{T}^m$.

We point out that the assumption on f^+ is implied for example by the assumption that the Fourier series of the differential of order $(r+2)$ of f with respect to the angles is absolutely summable. This, in turn, is implied e.g. by the property that f admits more than $(r+m+2)$ derivatives with respect to the angles.

Assume now that there exists a $J_0 \in \Gamma_\rho$ which is an hyperbolic zero of F. Then $T_0 := J_0 \times \mathbf{T}^m$ is a normally hyperbolic invariant torus of the truncated system. So, using the theory of normally hyperbolic manifolds, one can conclude that the system (4) has a true invariant normally hyperbolic torus which is $\varepsilon^{r/\alpha}$ close to the torus T_0. However, this is true only if the deformed invariant torus is contained in Γ_ρ. In order to ensure this property uniformly in ε, we first assume that $\omega(J_0)$ is diophantine (otherwise, since $\rho \to 0$ as $\varepsilon \to 0$, there would exist ε_* such that for $\varepsilon < \varepsilon_*$ one has $J_0 \notin \Gamma_\rho$). Secondly, the size ρ of the set $\Gamma_\rho(\gamma, \tau)$ must be larger than the distance between the deformed torus and the unperturbed one T_0. Looking at the proof of the theorem on the persistence of invariant iperbolic manifolds (see [4]) it is easy to see that the distance between the torus T_0 and the invariant torus of system (2) is $O(\varepsilon^{r/\alpha})$. So, provided $r > \tau + 1$, the invariant torus of the system (2) is also an invariant torus of the system (1), and we have the following

Proposition *In the same hypotheses and notation as in the lemma above, assume also that F has a hyperbolic zero J_0, which for some positive γ, τ is contained in $\Gamma(\gamma, \tau)$, and that the constant r is larger than $\tau + 1$. Then, for ε small enough, the system (1) has an invariant torus $O(\varepsilon^{r/\alpha})$ close, in the C^1 topology, to the torus $I = J_0$, $\varphi \in \mathbf{T}^m$.*

We remark that if $\tau > m - 1$ then, generically, a zero $J_0 \in \mathcal{G}$ satifyies the hypotheses of the proposition above. Therefore our assumptions are generically satisfied, for example in the class of the functions f admitting more than $(2m + 2)$ derivatives with respect to the angles.

We would also like to point out that the regularity we need is less than the one required when using KAM techniques, where more than $3m - 1$ derivatives are required (see e.g. [7]).

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