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An algebraic approach to the Kolmogorov-Sinai entropy

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Abstract

We revisit the notion of Kolmogorov-Sinai entropy for classical dynamical systems in terms of an algebraic formalism. This is the starting point for defining the entropy for general non-commutative systems. Hereby typical quantum tools are introduced in the statistical description of classical dynamical systems. We illustrate the power of these techniques by providing a simple, self-contained proof of the entropy formula for general automorphisms of n-dimensional tori.

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1. Introduction

The aim of this paper is to investigate the compatibility of the construction of the dynamical entropy for general non-commutative dynamical systems, as proposed in [AF1], with the well-established notion of entropy for classical dynamical systems [AA, CFS, P, Wal]. We follow the common strategy that a classical notion is formulated on the level of functions on phase space and then quantized by extending the construction to general non-Abelian algebras. This approach is quite similar to that of introducing quantum groups, quantum stochastic processes or non-commutative differential geometry. As classical systems arise as classical limits of quantum systems, one should logically reverse the approach.

In the statistical description of discrete time classical dynamical systems one produces a coarse-grained model of the system by partitioning the phase space X into a finite number of disjoint measurable subsets $(C_0, \ldots C_{k-1})$. The measure $\mu(C_i)$ of a set C_i gives the probability that the set C_i is occupied and defines, putting $\lambda_i = \mu(C_i)$, a discrete probability space $(\mathbf{Z}_k, \lambda)$ which is the coarse-grained statistical model of the system. In the course of time, the orbit of a point $x \in \mathsf{X}$ visits the sets $C_{i_0}, C_{i_1}, \ldots$ The probability of the points in phase space that will visit at time j the set C_{i_j}, $j = 0, 1, \ldots m-1$ is given by

$$\mu\big(C_{i_0} \cap \theta^{-1}(C_{i_1}) \cap \cdots \cap \theta^{-m+1}(C_{i_{m-1}})\big), \tag{1.1}$$

where $x \mapsto \theta(x)$ is the single time step evolution. The mapping of the orbit of a point to the "configuration" $(i_0, i_1, \ldots)$ defines a symbolic dynamics. A single step in the evolution translates into a left-shift of the configuration. Hence, a partition of the phase space leads to a classical spin model on a half-chain with a shift invariant measure determined by (1.1). The dynamical entropy or Kolmogorov-Sinai invariant is the usual statistical entropy density of the half-chain or, more precisely, the maximal one, varying over the partitions of the phase space. Alternatively, the half-chain model can be interpreted as a stationary stochastic process with values in the index set $\{0, \ldots k-1\}$.

The natural way to obtain a discrete statistical picture of a non-commutative system is to consider density matrices ρ as states on the algebra $\mathcal{M}_k$ of $k \times k$ matrices. Such a picture can easily be produced in terms of operational partitions of the unity [Lin], which are collections $\mathcal{X} = (x_0, \ldots x_{k-1})$ of observables satisfying $\sum_i x_i^* \, x_i = \mathbb{1}$. The analogue of the measure λ of above is the density matrix ρ with matrix elements $\omega(x_j^* \, x_i)$. A construction similar to that of above can now be performed. It yields a state on a quantum spin half-chain which can be interpreted either as a symbolic dynamics or as a quantum stochastic process in the sense of [AFL]. Unlike the classical case however, the stochastic process is generally not stationary, due to the non-commutativity of the quantum observables. The dynamical entropy can still be defined in terms of the mean entropy of the state on the half-chain. A number of technical problems limits the computations in specific quantum models to operational partitions of a special nature [AF1, AF2, AN, AFTA]. The problem of computing the entropy for general partitions in the von Neumann algebra $\mathfrak{M}$ associated with the dynamical system is still open but is crucial in order to obtain a proper invariant.

One of the main motivations of this paper is to obtain a full understanding of the algebraic construction of the dynamical entropy for classical systems. Despite the commutativity of the observables, the statistical models of above still produce truly non-commutative systems. It was unclear in [AF1] whether allowing operational partitions constructed with general measurable functions instead of the limited class of finite linear combinations of characteristic functions, yields the standard Kolmogorov-Sinai invariant. A second motivation is to test how effective it is to describe a classical system not in terms of the statistical properties of the time evolution of phase space partitions but rather in terms of multi-time correlation matrices and their von Neumann entropies.

The paper is organized as follows: in Section 2 we describe the algebraic construction of the dynamical entropy for general non-commutative dynamical systems. In Section 3 we restrict our attention to classical systems and prove that the algebraic construction produces the Kolmogorov-Sinai invariant. Section 4 provides a condition, H-density, on a subalgebra $\mathfrak{B}$ of functions which guarantees that the Kolmogorov-Sinai invariant will be reached when restricting attention to operational partitions in $\mathfrak{B}$. In particular continuous and smooth functions on phase space form sufficiently large algebras. Finally, we consider in Section 5 automorphisms of n-dimensional tori. We prove that the H-density condition is satisfied for periodic functions with finite Fourier series and use this explicit algebra to produce a self-contained computation of the entropy.

2. Preliminaries

We briefly remind the construction of the dynamical entropy for a discrete dynamical system [AF1]. Let $\mathfrak{M}$ be a von Neumann algebra of operators acting on a Hilbert space $\mathfrak{H}$ and let Ω be a normalized vector in $\mathfrak{H}$ which is cyclic for $\mathfrak{M}$ i.e. $\mathfrak{M}\,\Omega$ is dense in $\mathfrak{H}$. Furthermore we are given a unitary operator U on $\mathfrak{H}$ such that $U\,\Omega = \Omega$ and $U\,\mathfrak{M}\,U^* = \mathfrak{M}$. The single time-step evolution is given by the $*$-automorphism $x \mapsto U\,x\,U^*$ of $\mathfrak{M}$. Mostly, a dynamical system is given in terms of a C*-algebra $\mathfrak{A}$, an automorphism Θ of $\mathfrak{A}$ and a Θ-invariant state ω of $\mathfrak{A}$. By the GNS construction, there is a unique representation π of such an abstract dynamical system onto a Hilbert space $\mathfrak{H}$ in such a way that $\omega(x) = \langle \Omega, \pi(x)\,\Omega\rangle$ and $\pi(\Theta(x)) = U\,\pi(x)\,U^*$. The von Neumann algebra $\mathfrak{M}$ of above is then the weak closure of $\pi(\mathfrak{A})$. In the sequel, we will identify $\pi(x)$ with x, $x \in \mathfrak{A}$ and use the notation Θ for the automorphism $x \mapsto U\,x\,U^*$ on $\mathfrak{M}$.

An operational partition of unity of size $k = 1, 2, \ldots$ is a k-tuple $\mathcal{X}$ of elements x_i of $\mathfrak{M}$ satisfying:

$$\sum_{i=0}^{k-1} x_i^* x_i = \mathbb{1}.$$

Two partitions $\mathcal{X} = \bigl(x_0, x_1, \ldots, x_{k-1}\bigr)$ and $\mathcal{Y} = \bigl(y_0, y_1, \ldots, y_{\ell-1}\bigr)$ can be composed to yield a new partition $\mathcal{X} \circ \mathcal{Y} = \bigl(x_0\,y_0, x_0\,y_1, \ldots, x_{k-1}\,y_{\ell-1}\bigr)$ of size $k\,\ell$. Also, if $\mathcal{X}$ is a partition, then $\Theta(\mathcal{X}) = \bigl(\Theta(x_0), \Theta(x_1), \ldots, \Theta(x_{k-1})\bigr)$ is again a partition.

To any size k partition $\mathcal{X}$ we associate a $k \times k$ density matrix $\rho[\mathcal{X}]$ with (i,j) matrix-element $\langle x_j\,\Omega, x_i\,\Omega\rangle$. A partition $\mathcal{X}$ introduces therefore a coarse-grained description of the system in terms of the $k \times k$ matrices $\mathcal{M}_k$ and of a density matrix $\rho[\mathcal{X}]$. More explicitly, consider $\mathbf{C}^k \otimes \mathfrak{H}$ and the algebra $\mathcal{M}_k \otimes \mathfrak{M}$ acting on it. Let $(e_0, \ldots, e_{k-1})$ be a fixed orthonormal basis in $\mathbf{C}^k$, then the vector $\Psi_{\mathcal{X}} = \sum_i e_i \otimes x_i\,\Omega$ is normalized and cyclic for $\mathcal{M}_k \otimes \mathfrak{M}$. The restriction of the vector state $\langle \Psi_{\mathcal{X}}, \cdot\, \Psi_{\mathcal{X}}\rangle$ to $\mathcal{M}_k$ is precisely $\rho[\mathcal{X}]$. Indeed,

$$\langle \Psi_{\mathcal{X}}, A \otimes \mathbb{1}\,\Psi_{\mathcal{X}}\rangle = \sum_{i,j=0}^{k-1} A_{i,j}\,\langle \Omega, x_i^*\,x_j\Omega\rangle = \mathrm{Tr}\,\Big(\rho[\mathcal{X}]\,A\Big).$$

As $\langle \Psi_{\mathcal{X}}, \cdot\, \Psi_{\mathcal{X}}\rangle$ is a vector state, both $\rho[\mathcal{X}]$ and the restriction $\sum_i |x_i\,\Omega\rangle\langle x_i\,\Omega|$ of the vector state to $\mathfrak{B}(\mathfrak{H})$ have, up to multiplicities of zero, the same eigenvalues [AL]. In particular both density matrices have equal von Neumann entropy

$$\mathsf{H}[\mathcal{X}] = \mathsf{S}(\rho[\mathcal{X}]) = \mathsf{S}\Big(\sum_{i=0}^{k-1} |x_i\,\Omega\rangle\langle x_i\,\Omega|\Big). \tag{2.1}$$

The von Neumann entropy $\mathsf{S}(\rho)$ of a density matrix is computed as $\mathrm{Tr}\,\eta(\rho)$ where η is the standard entropy function

$$\eta(x) = \begin{cases} -x\,\log x & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0. \end{cases} \tag{2.2}$$

The entropy $\mathsf{H}[\Theta^{m-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]$ will typically grow linearly with m. We define the dynamical entropy $\mathsf{h}[\mathcal{X}]$ of the partition $\mathcal{X}$ as:

$$\mathsf{h}[\mathcal{X}] = \limsup_m \frac{1}{m}\,\mathsf{H}[\Theta^{m-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}].$$

Consider a unital $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{M}$ which is globally invariant under the dynamics. The dynamical entropy $\mathsf{h}_{\mathfrak{B}}$ is obtained by taking the supremum of $\mathsf{h}[\mathcal{X}]$ over all finite partitions $\mathcal{X}$ in $\mathfrak{B}$:

$$\mathsf{h}_{\mathfrak{B}} = \sup_{\mathcal{X} \subset \mathfrak{B}} \mathsf{h}[\mathcal{X}].$$

Finally, we will put $\mathsf{h} = \mathsf{h}_{\mathfrak{M}}$.

3. Classical systems

¿From now on, we will restrict our attention to classical systems. Consider a measurable space X with σ-algebra $\mathfrak{S}$ and probability measure μ and assume that $\mathfrak{S}$ is complete i.e. any subset B of a null-set A automatically belongs to $\mathfrak{S}$ and has zero measure. The dynamics is given in terms of an automorphism of X i.e. a measure preserving transformation θ of X with measure preserving inverse:

$$\mu(A) = \mu(\theta(A)) = \mu(\theta^{-1}(A)), \quad A \in \mathfrak{S}.$$

The triple $(\mathsf{X}, \theta, \mu)$ is usually called a classical dynamical system, often the space X carries more structure such as that of a manifold.

The Kolmogorov-Sinai invariant [AA, CFS, P, Wal] h_{KS} of $(\mathsf{X}, \theta, \mu)$ is constructed as follows: if $\mathcal{C} = (C_0, C_1, \ldots, C_{k-1})$ is a partition of X into disjoint, measurable subsets, the entropy $\mathsf{H}[\mathcal{C}]$ of $\mathcal{C}$ is computed as

$$\mathsf{H}[\mathcal{C}] = \sum_{i=0}^{k-1} \eta(\mu(C_i)),$$

η defined as in (2.2). If $\mathcal{C}$ and $\mathcal{D}$ are two partitions of X, then $\mathcal{C} \vee \mathcal{D}$ is the partition of X generated by $\mathcal{C}$ and $\mathcal{D}$. It consists of the non-empty intersections of sets C_i and D_j. Choosing a fixed partition $\mathcal{C}$, it can be shown that the following limit exists:

$$\mathsf{h}[\mathcal{C}] = \lim_{m \to \infty} \frac{1}{m}\, \mathsf{H}\left[\theta^{-m+1}(\mathcal{C}) \vee \cdots \theta(\mathcal{C}) \vee \mathcal{C}\right].$$

The Kolmogorov-Sinai entropy h_{KS} is defined as:

$$\mathsf{h}_{\mathrm{KS}} = \sup_{\mathcal{C}}\ \mathsf{h}[\mathcal{C}].$$

In an algebraic description of a classical system, the algebra of classical observables is taken as $\mathcal{L}^\infty(\mathsf{X}, \mu)$ acting on $\mathcal{L}^2(\mathsf{X}, \mu)$ as multiplication operators: for $f \in \mathcal{L}^\infty(\mathsf{X}, \mu)$, M_f is the operator $\phi \mapsto (M_f\, \phi)(x) = f(x)\, \phi(x)$ on $\mathcal{L}^2(\mathsf{X}, \mu)$. The constant function $\mathbf{1}$ is a normalized vector in $\mathcal{L}^2(\mathsf{X}, \mu)$, which is cyclic for $\mathcal{L}^\infty(\mathsf{X}, \mu)$. Because of the θ-invariance of μ

$$\phi \in \mathcal{L}^2(\mathsf{X}, \mu) \mapsto (U\, \phi)(x) = \phi(\theta(x)), \quad x \in \mathsf{X}$$

is a unitary operator and it implements the dynamics on $\mathcal{L}^\infty(\mathsf{X}, \mu)$:

$$U\, M_f\, U^* = M_{\Theta(f)}, \quad \text{with} \quad \Theta(f) = f \circ \theta.$$

A partition $\mathcal{F}$ of size k is a k-tuple $(f_0, \ldots, f_{k-1})$ of μ-measurable functions such that $\sum_i |f_i|^2 = \mathbf{1}$. The size of a partition $\mathcal{F}$ will be denoted by $|\mathcal{F}|$. The density matrix $\rho[\mathcal{F}]$ has matrix elements

$$\rho[\mathcal{F}]_{ij} = \mu(\overline{f_j}\, f_i) = \int_{\mathsf{X}} d\mu\, \overline{f_j}\, f_i.$$

We will need the following useful representation of $\rho[\mathcal{F}]$: let $(e_0, \ldots, e_{k-1})$ be an orthonormal basis for $\mathbf{C}^k$. $\Psi_{\mathcal{F}} = \sum_i f_i\, e_i$ is a measurable function from X to the unit sphere in $\mathbf{C}^k$ and defines a measurable function

$$\mathsf{P}[\mathcal{F}] = |\Psi_{\mathcal{F}}\rangle\langle\Psi_{\mathcal{F}}| \tag{3.1}$$

to the orthogonal projections of dimension 1 in the $k \times k$ matrices $\mathcal{M}_k$. The density matrix $\rho[\mathcal{F}]$ can be written as

$$\rho[\mathcal{F}] = \int_{\mathsf{X}} d\mu\, \mathsf{P}[\mathcal{F}]. \tag{3.2}$$

For n partitions $\mathcal{F}_i$ it follows that

$$\rho[\mathcal{F}_1 \circ \cdots \circ \mathcal{F}_n] = \int_{\mathsf{X}} d\mu\, \mathsf{P}[\mathcal{F}_1] \otimes \cdots \otimes \mathsf{P}[\mathcal{F}_n]. \tag{3.3}$$

Remark 3.1 The density matrix $\rho[\mathcal{F}_1 \circ \mathcal{F}_2]$ is unitarily equivalent to $\rho[\mathcal{F}_2 \circ \mathcal{F}_1]$, furthermore, the partial trace over the middle space of $\rho[\mathcal{F}_1 \circ \mathcal{F}_2 \circ \mathcal{F}_3]$ produces the density matrix $\rho[\mathcal{F}_1 \circ \mathcal{F}_3]$. Both these properties hold because of the commutativity of the algebra $\mathcal{L}^\infty(\mathsf{X}, \mu)$ and they generally fail for truly quantum mechanical dynamical systems.

We list some basic properties of the entropy H of a partition.

Proposition 3.2 *For any three partitions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ the following properties hold:*

i) $0 \le \mathsf{H}[\mathcal{F}_1] \le \log |\mathcal{F}_1|$ (3.4)

ii) $\mathsf{H}[\mathcal{F}_1 \circ \mathcal{F}_2 \circ \mathcal{F}_3] = \mathsf{H}[\mathcal{F}_2 \circ \mathcal{F}_1 \circ \mathcal{F}_3]$ (3.5)

iii) $\mathsf{H}[\mathcal{F}_1] \le \mathsf{H}[\mathcal{F}_1 \circ \mathcal{F}_2]$ (3.6)

iv) $\mathsf{H}[\mathcal{F}_1 \circ \mathcal{F}_2] \le \mathsf{H}[\mathcal{F}_1] + \mathsf{H}[\mathcal{F}_2]$ (3.7)

Proof: (3.4) is an obvious bound, (3.5) is an immediate consequence of the commutativity and (3.7) follows from subadditivity. In order to prove (3.6), we use strong subadditivity and the representations (3.2) and (3.3). Let ρ_{123} be a density matrix on a tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \mathfrak{H}_3$ of three Hilbert spaces. The partial trace of ρ_{123} over the third space will be denoted by ρ_{12}, with similar notations for the other partial traces. The following inequality is known as the strong subadditivity of the entropy [LR, Weh]

$$\mathsf{S}\big(\rho_{123}\big) + \mathsf{S}\big(\rho_2\big) \le \mathsf{S}\big(\rho_{12}\big) + \mathsf{S}\big(\rho_{23}\big). \tag{3.8}$$

First consider the case of an atomic probability measure μ supported in n points and apply strong subadditivity to the density matrix

$$\bigoplus_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j) \otimes \mathsf{P}[\mathcal{F}_2](j),$$

on $\mathbf{C}^n \otimes \mathbf{C}^{|\mathcal{F}_1|} \otimes \mathbf{C}^{|\mathcal{F}_2|}$. Then

$$\mathsf{S}\Big(\bigoplus_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j) \otimes \mathsf{P}[\mathcal{F}_2](j)\Big) + \mathsf{S}\Big(\sum_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j)\Big) \le$$
$$\mathsf{S}\Big(\bigoplus_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j)\Big) + \mathsf{S}\Big(\sum_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j) \otimes \mathsf{P}[\mathcal{F}_2](j)\Big).$$

As

$$\mathsf{S}\Big(\bigoplus_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j) \otimes \mathsf{P}[\mathcal{F}_2](j)\Big) = \mathsf{S}\Big(\bigoplus_{j=0}^{n-1} \mu(j)\, \mathsf{P}[\mathcal{F}_1](j)\Big) = \sum_{j=0}^{n-1} \eta(\mu(j)),$$

it follows that

$$\mathsf{S}\Big(\int_{\mathsf{X}} d\mu\, \mathsf{P}[\mathcal{F}_1]\Big) \le \mathsf{S}\Big(\int_{\mathsf{X}} d\mu\, \mathsf{P}[\mathcal{F}_1] \otimes \mathsf{P}[\mathcal{F}_2]\Big).$$

As the $\mathsf{P}[\mathcal{F}]$ are one-dimensional projections in a Hilbert space of fixed, finite dimension, we may approximate any measure by atomic measures and use the continuity of the entropy to get (3.6).

∎

Theorem 3.3 *The entropy* h *of a classical dynamical system* $(\mathcal{L}^\infty(\mathsf{X}, \mu), \Theta, \mu)$ *is equal to the Kolmogorov-Sinai invariant* h_{KS} *of* $(\mathsf{X}, \theta, \mu)$.

Proof: By restricting in the computation of $\mathsf{h}[\mathcal{F}]$ to partitions in characteristic functions, it follows immediately that $\mathsf{h}_{\mathrm{KS}} \le \mathsf{h}$.

It remains to be shown that h_{KS} is an upper bound for the entropy $h[\mathcal{F}]$ generated by an arbitrary size k partition $\mathcal{F}$ of the unity. Let $\mathcal{C} = (C_0, \ldots, C_{\ell-1})$ be a measurable partition of X into disjoint sets. We will denote by $\mathcal{C}^{(m)}$ and $\mathcal{F}^{(m)}$ the partitions which are generated from $\mathcal{C}$ and $\mathcal{F}$ by applying $m-1$ time steps of the dynamics:

$$\begin{aligned}\mathcal{C}^{(m)} &= \theta^{-m+1}(\mathcal{C}) \vee \cdots \vee \theta^{-1}(\mathcal{C}) \vee \mathcal{C}\\ \mathcal{F}^{(m)} &= \Theta^{m-1}(\mathcal{F}) \circ \cdots \circ \Theta(\mathcal{F}) \circ \mathcal{F}.\end{aligned}$$

We now write

$$\begin{aligned}\rho[\mathcal{F}^{(m)}] &= \int_{\mathsf{X}} d\mu\, \mathsf{P}[\mathcal{F}^{(m)}]\\ &= \sum_{C_\alpha \in \mathcal{C}^{(m)}} \int_{C_\alpha} d\mu\, \mathsf{P}[\mathcal{F}^{(m)}]\\ &= \sum_{C_\alpha \in \mathcal{C}^{(m)}} \int_{C_\alpha} d\mu\, \mathsf{P}[\Theta^{m-1}(\mathcal{F})] \otimes \cdots \otimes P[\mathcal{F}].\end{aligned}$$

In order to get an upper bound for $\mathsf{H}[\mathcal{F}^{(m)}]$, we will use the estimates [Weh]

$$\mathsf{S}\Big(\sum_i \lambda_i\, \rho_i\Big) \leq \sum_i \lambda_i\, \mathsf{S}(\rho_i) + \sum_i \eta(\lambda_i) \tag{3.9}$$

for density matrices ρ_i and $0 \leq \lambda_i$ with $\sum_i \lambda_i = 1$ and

$$\mathsf{S}(\rho_{12}) \leq \mathsf{S}(\rho_1) + \mathsf{S}(\rho_2),$$

where ρ_{12} is a density matrix on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and ρ_1 and ρ_2 denote the partial traces of ρ_{12} over $\mathfrak{H}_2$ and $\mathfrak{H}_1$. Then

$$\begin{aligned}
\mathsf{H}[\mathcal{F}^{(m)}] &= S\Big(\rho[\mathcal{F}^{(m)}]\Big) \\
&\leq \sum_{C_\alpha \in \mathcal{C}^{(m)}} \eta(\mu(C_\alpha)) + \sum_{C_\alpha \in \mathcal{C}^{(m)}} \mu(C_\alpha)\, \mathsf{S}\Big(\frac{1}{\mu(C_\alpha)} \int_{C_\alpha} d\mu\, \mathsf{P}[\mathcal{F}^{(m)}]\Big) \\
&\leq \sum_{C_\alpha \in \mathcal{C}^{(m)}} \eta(\mu(C_\alpha)) + \sum_{C_\alpha \in \mathcal{C}^{(m)}} \mu(C_\alpha) \sum_{j=0}^{m-1} \mathsf{S}\Big(\frac{1}{\mu(C_\alpha)} \int_{C_\alpha} d\mu\, \mathsf{P}[\Theta^j(\mathcal{F})]\Big).
\end{aligned}$$

For a suitable choice of $\mathcal{C}$, $\mathsf{P}[\Theta^j(\mathcal{F})]$ remains on each C_α sufficiently close to a one-dimensional projection in a fixed, finite-dimensional space of dimension at most k. As the entropy S is a continuous function on the $k \times k$ density matrices, which vanishes on the pure states, we can bound

$$\mathsf{S}\Big(\frac{1}{\mu(C_\alpha)} \int_{C_\alpha} d\mu\, \mathsf{P}[\Theta^j(\mathcal{F})]\Big)$$

by a small number which is independent of α and m. The proof is finished by dividing by m and taking the limit $m \to \infty$.

∎

4. H-dense subalgebras

In specific examples, one would hardly take the supremum in the definition of the entropy over all possible measurable partitions of unity but rather only over a limited class. The aim of this section is to state a condition that guarantees that the Kolmogorov-Sinai entropy will indeed be reached by considering this restricted supremum.

Let $\mathcal{C} = (C_0, \ldots, C_{k-1})$ be a measurable partition of X. We can identify such a partition with $(\chi_{C_0}, \ldots, \chi_{C_{k-1}})$, which is an operational partition of unity. For such partitions, the composition in the sense of operational partitions agrees with that in the sense of partitions of the measurable space into disjoint sets. As there is therefore no danger of confusion we will keep the same notation $\mathcal{C}$ for both.

Definition 4.1 *A unital subalgebra $\mathfrak{B}$ of $\mathcal{L}^\infty(X, \mu)$ is called* H-dense *in $\mathcal{L}^\infty(X, \mu)$ if for any finite partition $\mathcal{C} = (C_0, \ldots, C_{k-1})$ of X into measurable sets and for any $\epsilon > 0$ there exists in $\mathfrak{B}$ a partition $\mathcal{G} = (g_0, \ldots, g_{\ell-1})$ of unity such that*

$$\mathsf{H}[\mathcal{C} \circ \mathcal{G}] - \mathsf{H}[\mathcal{G}] \leq \epsilon.$$

Theorem 4.2 *If $\mathfrak{B}$ is Θ-invariant and* H*-dense, then $\mathsf{h}_{\mathfrak{B}} = \mathsf{h}_{\mathrm{KS}}$.*

For the proof of this theorem we need a few auxiliary results which we first state as a lemma.

Definition 4.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be two partitions of unity. The* conditional entropy $\mathsf{H}[\mathcal{F} \mid \mathcal{G}]$ *of $\mathcal{F}$ with respect to $\mathcal{G}$ is defined as*

$$\mathsf{H}[\mathcal{F} \mid \mathcal{G}] = \mathsf{H}[\mathcal{F} \circ \mathcal{G}] - \mathsf{H}[\mathcal{G}].$$

Lemma 4.4

i) $\mathsf{H}[\mathcal{F} \mid \mathcal{G}] \geq 0 \qquad (4.1)$

ii) $\mathsf{H}[\mathcal{F}_1 \circ \mathcal{F}_2 \mid \mathcal{G}] \leq \mathsf{H}[\mathcal{F}_1 \mid \mathcal{G}] + \mathsf{H}[\mathcal{F}_2 \mid \mathcal{G}] \qquad (4.2)$

iii) $\mathsf{H}[\mathcal{F} \mid \mathcal{G}_1 \circ \mathcal{G}_2] \leq \mathsf{H}[\mathcal{F} \mid \mathcal{G}_1] \qquad (4.3)$

Proof: (4.1) is actually the same statement as (3.6). The proof of (4.2) uses the strong subadditivity (3.8) of the entropy and Remark 3.1:

$$\begin{aligned}\mathsf{H}[\mathcal{F}_1 \circ \mathcal{F}_2 \mid \mathcal{G}] &= \mathsf{H}[\mathcal{F}_1 \circ \mathcal{F}_2 \circ \mathcal{G}] - \mathsf{H}[\mathcal{G}] \\ &\leq \mathsf{H}[\mathcal{F}_1 \circ \mathcal{G}] + \mathsf{H}[\mathcal{F}_2 \circ \mathcal{G}] - 2\mathsf{H}[\mathcal{G}] \\ &= \mathsf{H}[\mathcal{F}_1 \mid \mathcal{G}] + \mathsf{H}[\mathcal{F}_2 \mid \mathcal{G}].\end{aligned}$$

Finally (4.3) follows from (4.2) and Remark 3.1:

$$\begin{aligned}\mathsf{H}[\mathcal{F}_1 \mid \mathcal{G}_1 \circ \mathcal{G}_2] &= \mathsf{H}[\mathcal{F} \circ \mathcal{G}_1 \circ \mathcal{G}_2] - \mathsf{H}[\mathcal{G}_1 \circ \mathcal{G}_2] \\ &= \mathsf{H}[\mathcal{F} \circ \mathcal{G}_2 \mid \mathcal{G}_1] + \mathsf{H}[\mathcal{G}_1] - \mathsf{H}[\mathcal{G}_2 \mid \mathcal{G}_1] - \mathsf{H}[\mathcal{G}_1] \\ &\leq \mathsf{H}[\mathcal{F} \mid \mathcal{G}_1] + \mathsf{H}[\mathcal{G}_2 \mid \mathcal{G}_1] - \mathsf{H}[\mathcal{G}_2 \mid \mathcal{G}_1] \\ &= \mathsf{H}[\mathcal{F} \mid \mathcal{G}_1].\end{aligned}$$

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Proof of Theorem 4.2: Obviously, due to Theorem 3.3, $\mathsf{h}_{\mathfrak{B}} \leq \mathsf{h} = \mathsf{h}_{\mathrm{KS}}$. We use the H-density of $\mathfrak{B}$ to obtain for each partition $\mathcal{C}$ of X and each $\epsilon > 0$ an operational partition $\mathcal{G}$ in $\mathfrak{B}$ such that

$$\mathsf{H}[\mathcal{C} \mid \mathcal{G}] \leq \epsilon.$$

We use the properties of lemma 4.4 to get

$$\begin{aligned}&\mathsf{H}[\Theta^{m-1}(\mathcal{C}) \circ \cdots \circ \Theta(\mathcal{C}) \circ \mathcal{C}] \leq \\ &\mathsf{H}[\Theta^{m-1}(\mathcal{G}) \circ \cdots \circ \Theta(\mathcal{G}) \circ \mathcal{G} \circ \Theta^{m-1}(\mathcal{C}) \circ \cdots \circ \Theta(\mathcal{C}) \circ \mathcal{C}] \leq \\ &\mathsf{H}[\Theta^{m-1}(\mathcal{G}) \circ \cdots \circ \Theta(\mathcal{G}) \circ \mathcal{G}] + \sum_{j=0}^{m-1} \mathsf{H}[\Theta^j(\mathcal{C}) \mid \Theta^{m-1}(\mathcal{G}) \circ \cdots \circ \Theta(\mathcal{G}) \circ \mathcal{G}] \leq \\ &\mathsf{H}[\Theta^{m-1}(\mathcal{G}) \circ \cdots \circ \Theta(\mathcal{G}) \circ \mathcal{G}] + \sum_{j=0}^{m-1} \mathsf{H}[\Theta^j(\mathcal{C}) \mid \Theta^j(\mathcal{G})] = \\ &\mathsf{H}[\Theta^{m-1}(\mathcal{G}) \circ \cdots \circ \Theta(\mathcal{G}) \circ \mathcal{G}] + m\,\mathsf{H}[\mathcal{C} \mid \mathcal{G}].\end{aligned}$$

Dividing by m and taking the limit for large m, it follows that $\mathsf{h}_{\mathfrak{B}} \geq \mathsf{h}_{\mathrm{KS}}$.

■

Corollary 4.5 *Let $(\mathsf{X}, \theta, \mu)$ be a dynamical system, then the algebra of finite linear combinations of characteristic functions of measurable sets is H-dense in $\mathcal{L}^\infty(\mathsf{X}, \mu)$.*

Consider a dynamical system $(\mathsf{X}, \theta, \mu)$ where X is a compact Hausdorff space equipped with its Borel σ-algebra $\mathfrak{B}(\mathsf{X})$, θ is an automorphism of X and μ a θ-invariant regular measure on X. We prove that the unital algebra $\mathcal{C}(\mathsf{X})$ of continuous, complex-valued functions on X is H-dense in $\mathcal{L}^\infty(\mathsf{X}, \mu)$.

Theorem 4.6: *Let $(\mathsf{X}, \theta, \mu)$ be a dynamical system on a compact Hausdorff space X with regular, θ-invariant measure μ, then $\mathsf{h}_{\mathcal{C}(\mathsf{X})} = \mathsf{h}_{\mathrm{KS}}$.*

Proof: Fix a partition $\mathcal{C} = (C_0, \ldots, C_{k-1})$ of X into disjoint Borel sets $C_i \in \mathfrak{B}(\mathsf{X})$, $i = 0, \ldots, k-1$. By the regularity of μ and Urysohn's lemma there exist for any $\epsilon > 0$ open sets V_i, compact sets K_i and continuous functions f_i such that

$$K_i \subset C_i \subset V_i, \quad \chi_{K_i} \leq f_i \leq \chi_{V_i}, \quad \text{and} \quad \mu(V_i \setminus K_i) < \epsilon.$$

One readily verifies that $\mathcal{G} = (g_0, \ldots, g_{k-1})$, defined by

$$g_i = \sqrt{\frac{f_i}{\sum_j f_j}} \quad i = 0, \ldots, k-1,$$

is a partition of unity in $\mathcal{C}(\mathsf{X})$.

The density matrix associated to $\mathcal{F} \circ \mathcal{G}$ can be written as

$$\rho[\mathcal{F} \circ \mathcal{G}] = \bigoplus_{j=0}^{k-1} V^j$$

with

$$[V^j]_{j,j} = \mu(K_j) + \int_{C_j \setminus K_j} d\mu\, g_j^2 \tag{4.4}$$

$$[V^j]_{i,i} = \int_{C_j \cap (V_i \setminus C_i)} d\mu\, g_i^2 \quad \text{for} \quad i \neq j. \tag{4.5}$$

$$[V^j]_{p,q} = \int_{C_j \cap V_p \cap V_q} d\mu\, g_p\, g_q \quad \text{for} \quad p \neq q. \tag{4.6}$$

By construction, it is possible to choose the g_i such that (4.4) is arbitrary close to $\mu(C_j)$ and such that (4.5) and (4.6) are arbitrary small. Hence, for any $\epsilon > 0$, there is a partition $\mathcal{G}$ in $\mathcal{C}(\mathsf{X})$ for which $\mathsf{H}[\mathcal{F} \circ \mathcal{G}] - \mathsf{H}[\mathcal{F}] < \epsilon$. Similarly $|\mathsf{H}[\mathcal{G}] - \mathsf{H}[\mathcal{F}]| < \epsilon$.

∎

If the compact Hausdorff space has more structure, such as that of a differentiable manifold, then $C^\infty(\mathsf{X})$, the unital algebra of differentiable functions on X, is H-dense in $\mathcal{L}^\infty(\mathsf{X}, \mu)$. The proof of this assertion follows easily since there is an extension of Urysohn's lemma on a compact differentiable manifold in terms of differentiable functions [BC].

Corollary 4.7: *Let $(\mathsf{X}, \theta, \mu)$ be a dynamical system on an infinitely differentiable compact Hausdorff manifold X with θ-invariant, regular measure μ. Then* $\mathsf{h}_{\mathcal{C}^\infty(\mathsf{X})} = \mathsf{h}_{\mathrm{KS}}$.

5. Automorphisms on tori

The aim of this section is to show that the entropy of a simple dynamical system can be computed using only the algebraic description. We will not rely on any standard computation of the entropy or of the Lyapunov exponents on the level of the manifold which supports the dynamics [Ledr]. The proof that we obtain is even shorter than the usual proofs such as given in [P, Wal].

We will denote by $\mathbf{T}$ the one-dimensional torus $\mathbf{R}/2\pi$, which can be thought of as the unit circle in the complex plane. The algebra of continuous, complex-valued functions on $\mathbf{T}$ is the uniform closure of the subalgebra of finite linear combinations of functions

$$W(k) : \mathbf{T} \to \mathbf{C} : x \mapsto \exp ikx, \quad k \in \mathbf{Z}.$$

μ will denote the normalized Lebesgue measure on $\mathbf{T}$. The algebra of continuous functions on the n-dimensional torus $\mathbf{T}^n$ is the uniform closure of the algebra $\mathfrak{D}$ of the finite linear combinations of functions $W(\mathbf{k})$ with $\mathbf{k} \in \mathbf{Z}^n$ where $W(\mathbf{k})$ is the function $\mathbf{x} \in \mathbf{T}^n \mapsto \exp(i\mathbf{k}\mathbf{x})$. The dynamics θ on $\mathbf{T}^n$ is determined by a matrix $T \in \mathsf{SL}(n, \mathbf{Z})$, the group of $n \times n$ matrices with integer entries and determinant 1:

$$\theta\,\mathbf{x} = T\,\mathbf{x} \bmod 2\pi.$$

As $T \in \mathsf{SL}(n, \mathbf{Z})$, θ is a continuous transformation of $\mathbf{T}^n$ with continuous inverse and $\mu \circ \theta = \mu$. On the level of the von Neumann algebra $\mathcal{L}^\infty(\mathbf{T}^n, \mu)$, the evolution becomes $\Theta(f)(\mathbf{x}) = f(\theta(\mathbf{x}))$. From the definition of θ it follows that $\Theta(W(\mathbf{k})) = W(T^\dagger\,\mathbf{k})$ where $T^\dagger$ is the adjoint of T.

The matrix T has n linearly independent eigenvectors ϕ_i corresponding to eigenvalues t_i. The vectors ϕ_i are generally not mutually orthogonal. The spectrum of T is self-adjoint in the sense that if t is eigenvalue then also $\overline{t}$ is an eigenvalue. Furthermore the product of the t_i is equal to 1.

For any measurable subset C of $\mathbf{T}^n$, let Q_C be the orthogonal projection operator from $\mathcal{L}^2(\mathbf{T}^n, \mu)$ on $\mathcal{L}^2(C, \mu)$. It is the multiplication operator on $\mathcal{L}^2(\mathbf{T}^n, \mu)$ by the characteristic function of C. Similarly, for any subset Λ of $\mathbf{Z}^n$, P_Λ is, in the Fourier space of $\mathbf{T}^n$, the projection operator on $\ell^2(\Lambda) \subset \ell^2(\mathbf{Z}^n)$. P_Λ can be written as

$$P_\Lambda = \sum_{\mathbf{k} \in \Lambda} |W(\mathbf{k})\rangle\langle W(\mathbf{k})|.$$

Lemma 5.1

i) For any finite $\Lambda \subset \mathbf{Z}^n$ and any measurable $C \subset \mathbf{T}^n$, $P_\Lambda\, Q_C\, P_\Lambda$ is unitarily equivalent to $Q_C\, P_\Lambda\, Q_C$. Both operators have range of dimension at most $\#(\Lambda)$ and trace $\#(\Lambda)\, \mu(C)$.

ii) Let Λ_n be a sequence of finite subsets of $\mathbf{Z}^n$, increasing to $\mathbf{Z}^n$ and such that for any $\mathbf{k} \in \mathbf{Z}^n$

$$\lim_{n\to\infty} \frac{\#(\Lambda_n \cap (\Lambda_n + \mathbf{k}))}{\#(\Lambda_n)} = 1,$$

then

$$\lim_{n\to\infty} \frac{1}{\#(\Lambda_n)\,\mu(C)}\ \exp \mathsf{S}\Big(\frac{1}{\#(\Lambda_n)\,\mu(C)}\, P_{\Lambda_n}\, Q_C\, P_{\Lambda_n}\Big) = 1.$$

Proof: The unitary equivalence of $P_\Lambda\, Q_C\, P_\Lambda$ and $Q_C\, P_\Lambda\, Q_C$ follows from the polar decomposition of $P_\Lambda\, Q_C$ and implies immediately that the dimension of the range of both operators is at most $\#(\Lambda)$ because P_Λ has range of dimension $\#(\Lambda)$. The trace can easily be computed:

$$\mathrm{Tr}\, P_\Lambda\, Q_C\, P_\Lambda = \sum_{\mathbf{k}\in\Lambda} \langle W(\mathbf{k}), Q_C\, W(\mathbf{k})\rangle = \sum_{\mathbf{k}\in\Lambda} \int_C d\mu = \#(\Lambda)\,\mu(C).$$

¿From the definition of the entropy we obtain

$$\mathsf{S}\Big(\frac{1}{\#(\Lambda)\,\mu(C)}\, P_\Lambda\, Q_C\, P_\Lambda\Big) = \log \#(\Lambda) + \log \mu(C) + \frac{1}{\#(\Lambda)\,\mu(C)}\, \mathrm{Tr}\, \eta(P_\Lambda\, Q_C\, P_\Lambda).$$

The lemma will be proven if we show that

$$\lim_{n\to\infty} \frac{1}{\#(\Lambda_n)}\, \mathrm{Tr}\, \eta(P_{\Lambda_n}\, Q_C\, P_{\Lambda_n}) = 0.$$

For any $\epsilon > 0$ there is a $C(\epsilon) > 0$ such that

$$0 \le \eta(x) \le \epsilon + C(\epsilon)\, x\,(1-x), \quad 0 \le x \le 1.$$

As $0 \le P_\Lambda\, Q_C\, P_\Lambda \le \mathbb{1}$ and as the dimension of the range of $P_\Lambda\, Q_C\, P_\Lambda$ is equal to $\#(\Lambda)$ we can make the following estimate

$$\frac{1}{\#(\Lambda)}\, \mathrm{Tr}\, \eta(P_\Lambda\, Q_C\, P_\Lambda) \le \epsilon + C(\epsilon)\, \frac{1}{\#(\Lambda)} \Big(\mathrm{Tr}\,\big(P_\Lambda\, Q_C\, P_\Lambda\big) - \mathrm{Tr}\,\big(P_\Lambda\, Q_C\, P_\Lambda\big)^2\Big).$$

We rewrite the last term:

$$\begin{aligned}
\frac{1}{\#(\Lambda)} &\Big(\mathrm{Tr}\,\big(P_\Lambda\, Q_C\, P_\Lambda\big) - \mathrm{Tr}\,\big(P_\Lambda\, Q_C\, P_\Lambda\big)^2\Big) = \\
&= \mu(C) - \frac{1}{\#(\Lambda)} \sum_{\mathbf{k},\mathbf{k}'\in\Lambda} |\langle W(\mathbf{k}), Q_C\, W(\mathbf{k}')\rangle|^2 = \\
&= \mu(C) - \sum_{\mathbf{k}\in\Lambda} \frac{\#(\Lambda \cap (\Lambda + \mathbf{k}))}{\#(\Lambda)} \left|\int_C d\mu(\mathbf{x})\ \exp i\mathbf{k}\mathbf{x}\right|^2.
\end{aligned}$$

As $n \to \infty$, the functions

$$\mathbf{k} \mapsto \frac{\#(\Lambda_n \cap (\Lambda_n + \mathbf{k}))}{\#(\Lambda_n)}$$

tend to 1 and are evidently bounded by 1. Because $\mathbf{k} \mapsto \left|\int_C d\mu(\mathbf{x}) \, \exp i\mathbf{k}\mathbf{x}\right|^2$ is absolutely summable with sum $\mu(C)$, we can apply the dominated convergence theorem and conclude that

$$\lim_{n\to\infty} \sum_{\mathbf{k}\in\Lambda_n} \frac{\#(\Lambda_n \cap (\Lambda_n + \mathbf{k}))}{\#(\Lambda_n)} \left|\int_C d\mu(\mathbf{x}) \, \exp i\mathbf{k}\mathbf{x}\right|^2 = \mu(C).$$

$\blacksquare$

Remark 5.2 Lemma 5.1 has an obvious quantum mechanical interpretation. Both unnormalized density matrices $P_\Lambda \, Q_C \, P_\Lambda$ and $Q_C \, P_\Lambda \, Q_C$ are possible quantum candidates for a microcanonical ensemble for a free particle on the torus localized in C and with momentum in Λ. Statement *ii)* of Lemma 5.1 claims that the standard picture of the entropy as the logarithm of the phase space volume occupied by the system is asymptotically correct.

Theorem 5.3 *The algebra $\mathfrak{D}$ of finite linear combinations of functions $W(\mathbf{k})$, $\mathbf{k} \in \mathbf{Z}^n$ is* H*-dense in $\mathcal{L}^\infty(\mathbf{T}^n, \mu)$.*

Proof: Fix an arbitrary partition $\mathcal{C} = (C_0, \ldots, C_{k-1})$ of X into disjoint measurable subsets. We prove that the partition $\mathcal{G}$ of unity with

$$\mathcal{G} = \Big\{ \frac{1}{\sqrt{\#(\Lambda)}} \, W(\mathbf{k}) \Big| \mathbf{k} \in \Lambda \Big\}$$

satisfies $\mathsf{H}[\mathcal{C} \circ \mathcal{G}] - \mathsf{H}[\mathcal{G}] \leq \epsilon$ for Λ sufficiently large. Using the same notation as above for the projection operators P_Λ and Q_C, we compute

$$\begin{aligned} \mathsf{H}[\mathcal{C} \circ \mathcal{G}] &= \mathsf{S}\Big(\frac{1}{\#(\Lambda)} \sum_{j=0}^{k-1} Q_{C_j} \, P_\Lambda \, Q_{C_j} \Big) \\ &= \sum_{j=0}^{k-1} \mu(C_j) \, \mathsf{S}\Big(\frac{1}{\#(\Lambda) \, \mu(C_j)} Q_{C_j} \, P_\Lambda \, Q_{C_j} \Big) + \sum_{j=0}^{k-1} \eta\big(\mu(C_j)\big). \end{aligned}$$

Remark that the formula of above is a special case of (3.9), for density matrices with mutually orthogonal ranges (3.9) indeed becomes an equality [Weh]. We can now write the difference between $\mathsf{H}[\mathcal{C}\circ\mathcal{G}]$ and $\mathsf{H}[\mathcal{G}]$ as:

$$\begin{aligned}\mathsf{H}[\mathcal{C}\circ\mathcal{G}]-\mathsf{H}[\mathcal{G}]&=\sum_{j=0}^{k-1}\mu(C_j)\,\mathsf{S}\Big(\frac{1}{\#(\Lambda)\,\mu(C_j)}Q_{C_j}\,P_\Lambda\,Q_{C_j}\Big)+\sum_{j=0}^{k-1}\eta\big(\mu(C_j)\big)-\mathsf{H}[\mathcal{G}]\\&=\sum_{j=0}^{k-1}\mu(C_j)\,\mathsf{S}\Big(\frac{1}{\#(\Lambda)\,\mu(C_j)}Q_{C_j}\,P_\Lambda\,Q_{C_j}\Big)+\sum_{j=0}^{k-1}\eta\big(\mu(C_j)\big)-\log\#(\Lambda)\\&=\sum_{j=0}^{k-1}\mu(C_j)\,\Big(\mathsf{S}\Big(\frac{1}{\#(\Lambda)\,\mu(C_j)}Q_{C_j}\,P_\Lambda\,Q_{C_j}\Big)-\log\mu(C_j)-\log\#(\Lambda)\Big).\end{aligned}$$

Using Lemma 5.1, we can find a $\Lambda\subset\mathbf{Z}^n$ such that this difference becomes arbitrarily small.

■

Let us introduce as in [AFTA] the following notation. For $f,g\in\ell^0(\mathbf{Z}^n)$, the complex sequences indexed by $\mathbf{Z}^n$ with only a finite number of elements different from 0, we put

$$W(f)=\sum_{\mathbf{k}\in\mathbf{Z}^n}f(\mathbf{k})\,W(\mathbf{k}).$$

Clearly $\mathfrak{D}=\{W(f)\mid f\in\ell^0(\mathbf{Z}^n)\}$. Furthermore, $\overline{W(f)}=W(f^\dagger)$ with $f^\dagger(\mathbf{k})=\overline{f(-\mathbf{k})}$, $W(f)\,W(g)=W(f*g)$ with

$$f*g(\mathbf{k})=\sum_{\mathbf{k}'\in\mathbf{Z}^n}f(\mathbf{k}-\mathbf{k}')\,g(\mathbf{k}'),$$

$\mu(W(f))=f(\mathbf{0})$ and $\mu(\overline{W(f)}\,W(g))=\langle f,g\rangle$. For any partition $\mathcal{F}=\big(W(f_0),\ldots,W(f_{k-1})\big)$ in $\mathfrak{D}$, we will denote

$$\mathsf{Supp}(\mathcal{F})=\bigcup_{j=0}^{k-1}\mathsf{Supp}(f_j).$$

The density matrix $\rho[\mathcal{F}]$ has matrix elements $\rho[\mathcal{F}]_{ij}=\langle f_j\mid f_i\rangle$ and therefore by (2.1)

$$\mathsf{H}[\mathcal{F}]=\mathsf{S}\big(\rho[\mathcal{F}]\big)=\mathsf{S}\big(\big[\langle f_j\mid f_i\rangle\big]_{ij}\big)=\mathsf{S}\Big(\sum_j\mid f_j\rangle\langle f_j\mid\Big)\le\log\#\big(\mathsf{Supp}(\mathcal{F})\big).\tag{5.1}$$

Lemma 5.4 *For any partition $\mathcal{F}\in\mathfrak{D}$ we have* $\mathsf{h}[\mathcal{F}]\le\sum_{\substack{j\\1\le|t_j|}}\log|t_j|$

Proof: Let $\mathcal{F}^{(m)}$ be the partition generated by applying $m-1$ times the dynamics to $\mathcal{F}$. As $\mathcal{F}$ is a partition in $\mathfrak{D}$ it can be written as $\mathcal{F} = \big(W(f_0), \ldots, W(f_{k-1})\big)$ with $f_i \in \ell^0(\mathbf{Z}^n)$. The support of the partition $\mathcal{F}^{(m)}$, which also belongs to $\mathfrak{D}$, satisfies:

$$\mathsf{Supp}(\mathcal{F}^{(m)}) \subset \sum_{j=0}^{m-1} (T^\dagger)^j \, \mathsf{Supp}(\mathcal{F}).$$

From (5.1) it follows that

$$\mathsf{h}[\mathcal{F}] \leq \lim_{m\to\infty} \frac{1}{m} \log \#(\mathsf{Supp}(\mathcal{F}^{(m)})).$$

It therefore remains to estimate the number of multi-integers contained in $\mathsf{Supp}(\mathcal{F}^{(m)})$. Let $\mathbf{k} \in \mathsf{Supp}(\mathcal{F})$ and write $\mathbf{k} = \sum_i k_i \, \phi_i$, where ϕ_i is the eigenvector of $T^\dagger$ corresponding to the eigenvalue t_i. Clearly

$$(T^\dagger)^j \, \mathbf{k} = \sum_i (t_i)^j \, k_i \, \phi_i,$$

but this implies that for large m

$$\begin{aligned} \#\Big(\mathsf{Supp}(\mathcal{F}^{(m)})\Big) &= \#\Big(\sum_{j=0}^{m-1} \sum_{\mathbf{k}\in\mathsf{Supp}(\mathcal{F})} (T^\dagger)^j \, \mathbf{k}\Big) \\ &\leq \text{Constant} \prod_{\substack{j \\ 1<|t_j|}} |t_j|^m. \end{aligned}$$

Taking the logarithm, dividing by m, and taking the limit $m \to \infty$ finishes the proof.

■

Lemma 5.5 $\mathsf{h}_{\mathfrak{D}} \geq \sum_{1\leq|t_j|} \log|t_j|$ *or equivalently, for any* $\epsilon > 0$ *there is a partition* $\mathcal{F}$ *in* $\mathfrak{D}$ *such that* $\mathsf{h}[\mathcal{F}] \geq \sum_{1\leq|t_j|} \log|t_j| - \epsilon$.

Proof: We will consider partitions $\mathcal{F}$ generated by $f_j \in \ell^0(\mathbf{Z}^n)$ that live on distinct single points of $\mathbf{Z}^n$. For such a partition the density matrix $\rho[\mathcal{F}]$ is diagonal with equal weights $1/|\mathcal{F}|$ on the diagonal and therefore its entropy is precisely equal to $\log|\mathcal{F}|$. As we wish to compare for $r = 1, 2, \ldots$ the entropies of the dynamical systems $(\mathfrak{D}, \Theta, \mu)$ and $(\mathfrak{D}, \Theta^r, \mu)$ we will explicitly include a subscript referring to the dynamics. So, $\mathsf{h}_\Theta(\mathcal{F})$ will denote the entropy of the partition $\mathcal{F}$ generated by the dynamics Θ. Note that Θ^r is generated by the matrix T^r in $\mathsf{SL}(n, \mathbf{Z})$. ¿From the monotonicity (3.6) it follows that

$$\mathsf{H}\big(\Theta^{r(m-1)}(\mathcal{F}) \circ \Theta^{r(m-2)}(\mathcal{F}) \circ \cdots \circ \mathcal{F}\big) \leq \mathsf{H}\big(\Theta^{r(m-1)}(\mathcal{F}) \circ \Theta^{r(m-1)-1}(\mathcal{F}) \circ \cdots \circ \Theta(\mathcal{F}) \circ \mathcal{F}\big)$$

and hence:

$$\mathsf{h}_{\Theta^r}(\mathcal{F}) \leq r \, \mathsf{h}_\Theta(\mathcal{F}). \tag{5.2}$$

Let $T^\dagger$ be a matrix in $\mathsf{SL}(n, \mathbf{Z})$ with eigenvectors ϕ_i and corresponding eigenvalues t_i. If t_i is real, then the corresponding ϕ_i may be chosen real. If t_i is complex then $\overline{t_i}$ is also an eigenvalue and its corresponding eigenvector is the complex conjugate of ϕ_i. We have to distinguish between the case of a real and that of a pair of adjoint, complex eigenvalues of $T^\dagger$.

Assume first that t is a real eigenvalue of $T^\dagger$. Let $\Delta(t)$ be the centered line segment in $\mathbf{R}^n$ of length 1 in the direction of ϕ:

$$\Delta_t = \{a\,\phi \mid a \in \mathbf{R},\ \|a\,\phi\| \le 1/2\}.$$

Under $T^\dagger$, $\Delta(t)$ is mapped into $|t|\,\Delta_t$.
Assume next that t is complex with non-zero imaginary part. Let Δ_t be the non-degenerate, centered ellipse with major axis of length 1 in the two-dimensional subspace of $\mathbf{R}^n$ spanned by the vectors $z\,\phi + \overline{z\,\phi}$, $z \in \mathbf{C}$:

$$\Delta_t = \Big\{ z\,\phi + \overline{z\,\phi} \ \Big|\ z \in \mathbf{C},\ |z| \le 1/\sqrt{2(\|\phi\|^2 + |\langle \phi, \overline{\phi} \rangle|)} \Big\}.$$

Again Δ_t is mapped under $T^\dagger$ into $|t|\,\Delta_t$.

Let $\mathbf{k}$ be an arbitrary vector in $\mathbf{Z}^n$. We write $\mathbf{k} = \sum_t P_t\,\mathbf{k}$, where P_t is either the one-dimensional, generally not orthogonal, projection operator onto the subspace spanned by an eigenvector ϕ corresponding to a real eigenvalue t, or a projection operator on the two-dimensional real subspace spanned by $z\,\phi + \overline{z\,\phi}$, $z \in \mathbf{C}$ corresponding to a complex eigenvalue. In the summation we implicitly assume that, for complex eigenvalues, only one of the eigenvalues t or $\overline{t}$ appears.

Consider for $r \in \mathbf{N}$ large enough, the solid in $\mathbf{R}^n$

$$\Lambda(r) = \sum_{1<|t|} (|t|^r - 1)\,(\Delta_t/2\sqrt{n}) + \Big\{ \psi = \sum_{1 \ge |t|} c_t\,\phi_t \ \Big|\ \|\psi\| \le 1/2 \Big\}.$$

For $r \in \mathbf{N}$ sufficiently large, a large number of points of $\mathbf{Z}^n$ will be contained in the interior of $\Lambda(r)$: there exists a constant C, independent of r such that

$$\#(\Lambda(r)) \ge C \prod_{1<|t|} |t|^r. \tag{5.3}$$

Assume for the time being, that for all $m \in \mathbf{N}$, the maps

$$(\mathbf{k}_0, \ldots, \mathbf{k}_m) \in \mathbf{Z}^n \cap \Lambda(r) \mapsto \mathbf{k}_0 + (T^\dagger)^r\,\mathbf{k}_1 + \cdots + (T^\dagger)^{rm}\,\mathbf{k}_m \tag{5.4}$$

are injective. The density matrix $\rho[\mathcal{F}^{(m)}]$ corresponding to the partition generated by applying $m-1$ steps of the dynamics $(T^\dagger)^r$ to the partition

$$\mathcal{F} = \{W(\mathbf{k})/\sqrt{|\Lambda(r)|} \mid \mathbf{k} \in \Lambda(r) \cap \mathbf{Z}^n\}$$

is diagonal. Therefore

$$\mathsf{h}_{\Theta^r}[\mathcal{F}] \geq \log \#(\Lambda(r)).$$

It now suffices to use (5.2) and (5.3) and to take the limit $r \to \infty$ in order to get the result.

It still remains to show the injectivity of the maps (5.4). Consider two sequences $(\mathbf{k}_0, \ldots, \mathbf{k}_m)$ and $(\mathbf{k}'_0, \ldots, \mathbf{k}'_m)$ in $\Lambda(r) \cap \mathbf{Z}^n$ and assume, without loss of generality, that $\mathbf{k}_m \neq \mathbf{k}'_m$. At least one of the $P_t\, \mathbf{k}$ with $|t| > 1$ has a length not less than $\|\mathbf{k}\|/2\sqrt{n}$. We now estimate the difference of the projections of the first m terms. By the construction of Δ_t the maximal distance between points in $P_t\, \Lambda$ will not exceed $(|t|^r - 1)/2\sqrt{n}$ and, after j applications of $(T^\dagger)^r$, it will not exceed $|t|^{jr}/2\sqrt{n}$. Therefore

$$\begin{aligned}
\left\| \sum_{j=0}^{m-1} P_t \left({T^\dagger}^{jr} \left(\mathbf{k}_j - \mathbf{k}'_j \right) \right) \right\| &\leq \sum_{j=0}^{m-1} \left\| P_t \left((T^\dagger)^{jr} \left(\mathbf{k}_j - \mathbf{k}'_j \right) \right) \right\| \\
&\leq \sum_{j=0}^{m-1} \left(|t|^r - 1 \right) |t|^{jr}/2\sqrt{n} \\
&< |t|^{mr}/2\sqrt{n}.
\end{aligned}$$

Clearly the first m terms can never compensate for the minimal difference due to the $(m+1)$-th term.

■

Theorem 5.6 *For the dynamical system* $(\mathbf{T}^n, \Theta, \mu)$ *described above*

$$\mathsf{h}_{\mathrm{KS}} = \mathsf{h}_{\mathfrak{D}} = \mathsf{h} = \sum_{\substack{i \\ |t_i|>1}} \log |t_i|$$

Proof: In Theorem 5.3 it was shown that $\mathfrak{D}$ is H-dense in $\mathcal{L}^\infty(\mathbf{T}^n, \mu)$. The proof follows immediatly from Lemma 5.4 and Lemma 5.5.

■

Remark 5.7 There is a natural quantization of the dynamical systems on the n-dimensional tori that we have considered. The commutative algebra generated by the $W(\mathbf{k})$ is deformed by introducing a symplectic form σ and imposing the relations

$$W(\mathbf{k})\, W(\mathbf{k})^* = \mathbb{1} \quad \text{and} \quad W(\mathbf{k})\, W(\mathbf{k}') = \exp(i\sigma(\mathbf{k}, \mathbf{k}'))\, W(\mathbf{k} + \mathbf{k}') \quad \mathbf{k}, \mathbf{k}' \in \mathbf{Z}^n.$$

On the level of the C*-algebra generated by such $W(\mathbf{k})$, the dynamics is given by the automorphism $W(\mathbf{k}) \mapsto W(T^\dagger\, \mathbf{k})$ and the invariant state ω by $\omega(W(\mathbf{k})) = 0$ for $\mathbf{k} \neq \mathbf{0}$. The computation of the quantum dynamical entropy $\mathsf{h}_{\mathfrak{D}}$ with respect to partitions in the algebra $\mathfrak{D}$ of finite linear combinations of $W(\mathbf{k})$ can be performed along the same lines as that in

Theorem 5.6 and it leads to the natural generalization of the result obtained in [AFTA]. We conjecture that $\mathsf{h}_{\mathfrak{D}} = \mathsf{h}$, where h is the entropy computed allowing general partitions in the von Neumann algebra $\mathfrak{M}$ of the representation of ω. The ergodic properties and Lyapunov exponents of dynamical systems on non-commutative tori have been considered in [ENTS]. It is obvious that there is for these models a connection between these exponents and the dynamical entropy. The intrinsic connection between Lyapunov exponents and entropy for quantum dynamical systems should certainly be investigated.

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