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The K-Property of Hamiltonian Systems with Restricted Hard Ball Interactions

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1. Introduction

In 1985, Chernov and Sinai suggested the study of a mechanical model of elastic hard balls: the so-called *pencase*, which we will call the Chernov-Sinai pencase (cf. the manuscript [S-Ch(1985)]). Here large hard ball particles move on a torus, elongated in one spatial dimension; since then the cyclic order of the particles remains invariant, each ball can only interact with its two neighbours. Their idea was that, on one hand, because of the restricted form of the interactions, the expected ergodicity of the model could be easier to establish than that of general hard ball systems on tori and, on the other hand, this quasi-one-dimensional model seems to be a quite suitable mechanical one for the study of the hydrodynamic limit transition (cf. [Sin(1985)]).

For systems of elastic hard balls on a torus Yakov Sinai, in 1963, [Sin(1963)] gave a mathematically rigorous version of Boltzmann's celebrated ergodic hypothesis: *the system of an arbitrarily fixed number N of elastic hard balls moving in the $\nu-$torus $\mathbb{T}^\nu$ $(\nu \geq 2)$ is ergodic — of course, on the submanifold of the phase space specified by the trivial conservation laws.*

The aforementioned hypothesis, which we call the *Boltzmann-Sinai Ergodic Hypothesis*, was first established for the case $N = \nu = 2$ in Sinai's celebrated paper

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[Sin(1970)]. After several steps ([S-Ch(1987)]: the case $N = 2, \nu \geq 2$, [K-S-Sz(1991)]: the case $N = 3, \nu \geq 2$, [K-S-Sz(1992)]: the case $N = 4, \nu \geq 3$) the most general result so far was obtained by Simányi [Sim(1992)], who could verify the conjecture for the case $\nu \geq N \geq 2$ (a more detailed survey of the history of Boltzmann's ergodic hypothesis can be found in [Sz(1995)]).

The aim of the present work is to verify the K-property of models closely related to the mechanical system of hard balls. The restriction we make is on the *graph of interactions*: we first assume that it is a tree whose *edge-degree* is bounded from above by a constant D. Let us make here two remarks:

(i) the *graph of interactions* means that only pairs of particles connected by an edge in this graph can interact (in our model they do this through a hard ball collision) whereas non-connected pairs do not interact at all;

(ii) the *degree of an edge* of a graph is the number of edges adjacent to the given edge, and then the *edge-degree* of a graph is the maximum of the degrees of all edges.

(The precise description of our model will be given soon). Then we prove that such a system is a K-mixing one if $\nu \geq D + 2$, and, moreover, for the case $\nu = D + 1$ we obtain that none of the relevant Lyapunov-exponents vanishes and the ergodic components are open.

By strengthening the methods used in treating the aforementioned tree-interactions, (in particular, the geometric-algebraic part (1) of our strategy detailed at the end of this section), we can also treat an interaction graph which is not a tree: a *circle of length* N. This result implies the K-property of the Chernov-Sinai pencase in dimension $\nu = 4$. It is easy to see that in the Chernov-Sinai pencase, in dimensions $\nu \leq 4$, the graph of interactions is a simple cyclic one (i. e. the graph that consists of just one circle of length N). Our methods also imply that the Chernov-Sinai pencase has non-zero Lypaunov-exponents and open ergodic components in our physical space dimension $\nu = 3$ (the pencase as a system of ball-particles loses order invariance if $\nu \geq 5$, and if one wants to maintain this property, systems of particles with other, less natural shapes can be, in fact, introduced and treated by our methods). It is worth mentioning that in the low-dimensional cases (starting from $\nu = 4$) our results provide the first mechanical models where the ergodicity of the system is established for an arbitrary number of degrees of freedom — apart from a model introduced in [B-L-P-S(1992)], where the boundary of the vessel also contributes to the hyperbolic effects.

Furthermore, an important advantage of the hereby presented method for going down to physical dimensions is that it avoids the use of more involved algebraic tools that seem unavoidable in the case of treating hard sphere systems in the general framework.

Let us be more specific: introduce the model and formulate our results.

Assume that, in general, a system of N (≥ 2) balls, identified as $1, 2, \dots, N$ are given in $\mathbb{T}^\nu$, the ν-dimensional unit torus $(\nu \geq 2)$. Denote the phase point of the ith ball by $(q_i, v_i) \in \mathbb{T}^\nu \times \mathbb{R}^\nu$. Assume also that the graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ of interactions is given, where the set of edges $\mathcal{E}$ makes a *non-oriented tree* (i. e. a connected graph without loop) on the set of vertices $\mathcal{V} := \{1, 2, \dots, N\}$. (In the case of the mechanical model of N hard balls on $\mathbb{T}^\nu$ this graph is, of course, the complete one.) Further, for every $\{i, j\} : \; 1 \leq i, \; j \leq N, \; i \neq j$, the potential functions $U_{i,j} : \; \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ are given as follows

$$U_{i,j}(r) = \begin{cases} \infty & 0 \leq r < d_{i,j} \\ 0 & r \geq d_{i,j} \end{cases}$$

where the numbers $d_{i,j} = d_{j,i}$ are nonnegative. They determine the set of edges $\mathcal{E}$: $\{i, j\} \in \mathcal{E}$ if and only if $d_{i,j} > 0$. Moreover, the interaction between the pair $\{i, j\}$ of particles with configuration coordinates $q_i, q_j \in \mathbb{T}^\nu$ is

$$\tilde{U}_{i,j}(q_i, q_j) = U_{i,j}(\|q_i - q_j\|). \tag{1.1}$$

where $\|.\|$ denotes the euclidean distance in the torus $\mathbb{T}^\nu$. The configuration space $\overline{\mathbf{Q}}$ of the N balls is a subset of $\mathbb{T}^{N \cdot \nu}$: for every $\{i, j\} \in \mathcal{E}$ we cut out from $\mathbb{T}^{N \cdot \nu}$ the cylindric scatterer:

$$\overline{C}_{i,j} = \left\{ Q = (q_1, \dots, q_N) \in \mathbb{T}^{N\cdot\nu} : \|q_i - q_j\| < d_{i,j} \right\}. \tag{1.2}$$

In other words $\overline{\mathbf{Q}} := \mathbb{T}^{N\nu} \setminus \bigcup_{\{i,j\}\in\mathcal{E}} \overline{C}_{i,j}$ (one cylindric scatterer $\overline{C}_{i,j}$ can obviously consist of several parallel cylinders, due to the non-uniqueness of the center of mass in a torus). The energy $H = \frac{1}{2}\sum_1^N ||v_i||^2$ and the total momentum $P = \sum_1^N v_i$ are first integrals of the motion. Thus, without loss of generality, we can assume that $H = \frac{1}{2}$, $P = 0$, and, moreover, that the sum of spatial components $B = \sum_1^N q_i$ is zero. (If $P \neq 0$, then the center of mass has an additional conditionally periodic or periodic motion.) For these values of H, P and B, the phase space of the system reduces to the unit tangent bundle $\mathbf{M} := \mathbf{Q} \times S^{N\cdot\nu-\nu-1}$ of the configuration space $\mathbf{Q}$, where

$$\mathbf{Q} := \left\{ Q \in \overline{\mathbf{Q}} : \sum_1^N q_i = 0 \right\}$$

with $d := \dim\mathbf{Q} = N\cdot\nu - \nu$, and further S^k denotes, in general, the k-dimensional unit sphere. It is easy to see that the dynamics of the N particles, determined by their uniform motion with elastic collisions (corresponding to the interaction potentials (1.1)) between pairs belonging to $\mathcal{E}$ on one hand, and the billiard flow $\{S^t : t \in \mathbb{R}\}$ in $\mathbf{Q}$ with specular reflections at $\partial\mathbf{Q}$ on the other hand, conserve the Liouville measure $d\mu = const \cdot dq \cdot dv$, and are isomorphic. These – isomorphic – dynamical systems $\left(\mathbf{M}, S^{\mathbb{R}} := \{S^t : t \in \mathbb{R}\}, \mu\right)$ are called *standard billiard ball systems with $\mathcal{D}$-interactions* where $\mathcal{D} := (d_{i,j})_{i,j=1}^N$. It is evident that $\mathcal{D}$ determines $\mathcal{E}$ and thus the model, too. (These standard billiard ball flows of N particles with $\mathcal{D}$-interactions in $\mathbb{T}^\nu$ are, in fact, *semi-dispersing billiards* $(\mathbf{M}, S^{\mathbb{R}}, \mu)$, i. e. billiards with convex scatterers.) We also recall that a billiard is called *dispersing* if its scatterers are strictly convex (for instance, the standard billiard ball flow in the case of $N = 2$ balls on $\mathbb{T}^\nu$).

The first main result of our paper is the following

Theorem A. *Assume that the edge-degree of the tree of interactions is D, and for* *some* $N \geq 2$, $\nu \geq D+2$ *Int*$\mathbf{Q}$ *(or, equivalently Int*$\mathbf{M}$*) is connected. Then the standard billiard ball system* $(\mathbf{M}, \{S^t\}, \mu)$ *with a* $\mathcal{D}$*-interaction is a K-flow.*

Note 1. In general, if Int$\mathbf{M}$ decomposes into a finite number of connected components, then our methods provide that the standard billiard ball flow with a $\mathcal{D}$-interaction is a K-system on each of these connected components.

Note 2. As it has been proved in recent manuscripts by N. I. Chernov and C. Haskell, [Ch-H(1995)] on one hand, and by D. Ornstein and B. Weiss, [O-W(1995)] on the other hand, the K-mixing property of a semi-dispersing billiard flow actually implies its Bernoulli property, as well.

Our second main result concerns the so-called *cyclic interaction*, where $\mathcal{E}_c := \{\{1,2\},\{2,3\},\dots,\{N-1,N\},\{N,1\}\}$. Let $\mathcal{D}_c$ be a corresponding matrix of interaction ranges.

Theorem B. *The standard billiard ball system with a cyclic interaction is a K-system if* $\nu \geq 4$ *and Int*$\mathbf{Q}$ *is connected.*

As mentioned before, an interesting mechanical realization of the cyclic interaction is the Chernov-Sinai pencase. Here the torus is an elongated one, i. e. (physical) balls of radii r move on a torus $\mathbb{T}^\nu_L := \mathbb{R}^\nu/(L\mathbb{Z}\times\mathbb{Z}^{\nu-1})$ (in other words, $d_{i,j} = 2r$) for every $\{i,j\} \in \mathcal{E}_c$. Now, as a particular case of Theorem B, we obtain

Corollary. *The Chernov-Sinai pencase is a K-system if* $\nu = 4$ *and* $\sqrt{3}/4 < r < 1/2$.

We note that actually the phase space of the pencase consists of exactly $N!$ connected, isomorphic components whenever $\nu \leq 4$ and $\sqrt{\nu-1}/4 < r < 1/2$. Thus, of course, the K-property can only hold, and is meant here, on each of them, separately.

By fortifying the inductive arguments we also obtain the following results:

Theorem C. *Assume that the edge-degree of the graph of interaction is* D. *If* $N \geq 2$, $\nu = D+1$, *then, almost everywhere, none of the relevant Lyapunov-*

exponents of the standard billiard ball system $(\mathbf{M}, \{S^t\}, \mu)$ *with a* $\mathcal{D}$*-interaction vanishes. Furthermore, the ergodic components of the system are open (and thus of positive measure) and on each of them the flow has the K-property.*

Theorem D. *For the models of Theorem B and its Corollary, in dimension* $\nu = 3$*, almost everywhere, none of the relevant Lyapunov exponents vanishes. Furthermore, the ergodic components of the system are open (and thus of positive measure) and on each of them the flow has the K-property.*

General strategy. The basic notion in the theory of semi-dispersing billiards is that of the *sufficiency* of a phase point or, equivalently, of its orbit. The conceptual importance of sufficiency can be explained as follows (for a technical introduction and our prerequisites, see section 2): In a suitably small neighbourhood of a (typical) phase point of a dispersing billiard the system is *hyperbolic*, i. e. its relevant Lyapunov exponents are not zero. For a semi-dispersing billiard the same property holds for sufficient points only! Physically speaking, a phase point is sufficient if its trajectory encounters in its history all possible degrees of freedom of the system. Then the fundamental theorem of semi-dispersing billiards (see the main result of [S-Ch (1987)]) says that — under certain conditions — a suitably small neighbourhood of a sufficient phase point does belong to one ergodic component. As a consequence, if, for instance, almost every phase point of a semi-dispersing billiard is sufficient, then this property implies that the ergodic components are open and, therefore, their number is countable. To obtain, however, (global) ergodicity of the flow, a more stringent property of the subset of non-sufficient points is needed. For that purpose, in our work with A. Krámli [K-S-Sz (1989)] , the topological property of *slimness* (earlier misleadingly called 'residuality'), closely related to that of *topological codimension two*, was suggested and used in later works, too.

Definition 1.3. *We say that a subset of a smooth manifold* M *is* slim *if it can be covered by a countable union of codimension two (i. e. at least two), closed subsets* *of $\mu-$measure zero, where μ is a smooth measure on M.*

Here the dimension of a separable metric space is its topological or Hurewicz dimension, which is any one of the following equivalent notions: the small inductive, the large inductive, or the covering dimension, see, for instance, [H-W(1941)].

It has long been an accepted idea among experts that the right way for proving the K–property of hard ball systems is an induction on the number N of balls. According to our strategy initiated in [K-S-Sz (1991)] and explained in the introductions of [K-S-Sz (1992)] and [Sim(1992)-I] , there are three major parts in such an induction, once a combinatorial property, called *richness* of the symbolic collision sequence of a trajectory, had been suitably defined. (Our general concept of richness, appropriate for hard ball systems, is introduced in Definition 3.2.)

(1) The "geometric–algebraic considerations" on the codimension of the closed algebraic sets describing the non–sufficient trajectory segments with a combinatorially rich symbolic collision structure (This step has no inductive character!);

(2) Proof of slimness (negligibility) of the set of phase points with a combinatorially non–rich collision sequence by using the K–property of less than N hard balls in the torus;

(3) Checking the Chernov–Sinai Ansatz — an important necessary condition for the proof of the Theorem on Local Ergodicity, see Condition (A) before Theorem 5 in [S-Ch (1987)], or Condition 3.1 in [K-S-Sz (1990)] . This step also uses the K–property of less than N balls and step (1) for the system of N hard balls in the torus.

The paper is organized as follows: Section 2 summarizes certain prerequisites. Section 3 provides the proofs of the theorems and their corollaries assuming Main Lemmas 3.4 and 3.7 of geometric-algebraic nature. Then Main Lemma 3.4 related to tree-interactions and used in the proofs of Theorems A and C is established in

Section 4, while Main Lemma 3.7 related to the cyclic interaction and used in the proofs of Theorems B and D gets verified in Section 5. Section 6 contains some further remarks.

2. Prerequisites.

As to the basic notions concerning semi-dispersing billiards we refer to the paper [K-S-Sz (1990)]. For convenience and brevity, we will throughout use the concepts and notations of sections 2 and 3 of the aforementioned work. Here we only summarize some further notions from [K-S-Sz (1991)] , [K-S-Sz (1992)] , [Sim(1992)]. These are either new or their exposition is simpler than that given in the original work.

An often used abbreviation is the shorthand $S^{[a,b]}x$ for the trajectory segment $\{S^t x : a \le t \le b\}$. The natural projections from $\mathbf{M}$ onto its factor spaces are denoted, as usual, by $\pi : \mathbf{M} \to \mathbf{Q}$ and $p : \mathbf{M} \to S^{N\cdot\nu-\nu-1}$ or, sometimes, we simply write $\pi(x) = Q(x) = Q$ and $p(x) = V(x) = V$ for $x = (Q, V) \in \mathbf{M}$. Any $t \in [a, b]$ with $S^t x \in \partial M$ is called *a collision moment or collision time.*

As pointed out in previous works on billiards, the dynamics can only be defined for trajectories where the moments of collisions do not accumulate in any finite time interval (cf. Condition 2.1 of [K-S-Sz(1990)]). An important consequence of Theorem 5.3 of [V(1979)] is that — for semi-dispersing billiards — there are *no trajectories at all with a finite accumulation point of collision moments.*

As a result, for an arbitrary non-singular orbit segment $S^{[a,b]}x$ of the standard billiard ball flow, there is a uniquely defined maximal sequence $a \le t_1 < t_2 < \cdots < t_m \le b : \; m \ge 0$ of collision times and a uniquely defined sequence $\sigma_1 < \sigma_2 < \cdots < \sigma_m$ of "colliding pairs", i. e. $\sigma_k = \{i_k, j_k\}$ whenever $Q(t_k) = \pi(S^{t_k}x) \in \partial\overline{C}_{i_k,j_k}$. The sequence $\Sigma := \Sigma(S^{[a,b]}x) := (\sigma_1, \sigma_2, \ldots, \sigma_m)$ is called the *symbolic collision sequence* of the trajectory segment $S^{[a,b]}x$.

As well known, billiards are dynamical systems with *singularities.* A collision at a point $x \in \partial M$ such that, in $\pi(x)$, at least two smooth pieces of $\partial \mathbf{Q}$ meet is called a *multiple* collision. A collision is called *tangential* if $x \in \partial M$ and $p(x) \in \mathcal{T}_{\pi(x)} \partial \mathbf{Q}$, i. e. $p(x)$ is tangential to $\partial \mathbf{Q}$ at the point of reflection. We shall use the collection $\mathcal{SR}^+$ of all singular reflections:

Definition 2.1. *The set $\mathcal{SR}^+$ is the collection of all phase points $x \in \partial \mathbf{M}$ for which the reflection, occurring at x, is* singular *(tangential or multiple) and, in the case of a multiple collision, x is supplied with the outgoing velocity V^+.*

It is not hard to see that $\mathcal{SR}^+$ is a compact cell–complex in $\mathbf{M}$ and $\dim(\mathcal{SR}^+) = \dim \mathbf{M} - 2 = 2d - 3$.

As usual, we will denote by $\mathbf{M}^*$ the set of phase points $x \in \mathbf{M}$ whose full trajectory contains infinitely many collisions such that at most one of them is singular. The subset of points $x \in \mathbf{M}^*$ whose orbit has no singular collision at all will be denoted by $\mathbf{M}^0$, and finally we denote $\mathbf{M}^1 := \mathbf{M} \setminus \mathbf{M}^0$.

Neutral Subspaces, Advance and Sufficiency

Consider a *non–singular* trajectory segment $S^{[a,b]}x$. Suppose that a and b are *not moments of collision.* Before defining the neutral linear space of this trajectory segment, we note that the tangent space of the configuration space $\mathbf{Q}$ at interior points can be identified with the common linear space

(2.1) $$\mathcal{Z} = \left\{ (w_1, w_2, \dots, w_N) \in (\mathbb{R}^\nu)^N : \sum_{i=1}^N w_i = 0 \right\}.$$

Definition 2.2. *The neutral space $\mathcal{N}_0(S^{[a,b]}x)$ of the trajectory segment $S^{[a,b]}x$ at time zero ($a < 0 < b$) is defined by the following formula:*

$$\begin{aligned} \mathcal{N}_0(S^{[a,b]}x) = \big\{ W \in \mathcal{Z} :\ & \exists (\delta > 0)\ s.\ t.\ \forall \alpha \in (-\delta, \delta) \\ & p\left(S^a\left(Q(x) + \alpha W, V(x)\right)\right) = p(S^a x)\ \& \\ & p\left(S^b\left(Q(x) + \alpha W, V(x)\right)\right) = p(S^b x) \big\}. \end{aligned}$$

It is known (see (3) in Section 3 of [S-Ch (1987)]) that $\mathcal{N}_0(S^{[a,b]}x)$ is a linear subspace of $\mathcal{Z}$ indeed, and $V(x) \in \mathcal{N}_0(S^{[a,b]}x)$. The neutral space $\mathcal{N}_t(S^{[a,b]}x)$ of the segment $S^{[a,b]}x$ at time $t \in [a,b]$ is defined as follows:

$$\mathcal{N}_t(S^{[a,b]}x) = \mathcal{N}_0\left(S^{[a-t,b-t]}(S^t x)\right). \tag{2.3}$$

It is clear that the neutral space $\mathcal{N}_t(S^{[a,b]}x)$ is canonically identified with $\mathcal{N}_0(S^{[a,b]}x)$ by the usual identification of the tangent spaces of $\mathbf{Q}$ along the trajectory $S^{(-\infty,\infty)}x$ (see, for instance, Section 2 of [K-S-Sz (1990)]). (Intuitively speaking, the neutral subspace is the orthocomplement of the positive subspace of the second fundamental form of the image of a parallel beam of light).

Our next definition is that of the *advance*. Consider a non-singular orbit segment $S^{[a,b]}x$ with a symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_m)$ $(m \geq 1)$ as at the beginning of the present section. For $x = (Q,V) \in \mathbf{M}$ and $W \in \mathcal{Z}$, $\|W\|$ sufficiently small, denote $T_W(Q,V) := (Q+W,V)$.

Definition 2.4. *For any $1 \leq k \leq m$ and $t \in [a,b]$, the advance $A(\sigma_k) : \mathcal{N}_t(S^{[a,b]}x) \to \mathbb{R}$ is the unique linear extension of the linear functional defined in a sufficiently small neighbourhood of the origin of $\mathcal{N}_t(S^{[a,b]}x)$ in the following way:*

$$A(\sigma_k)(W) := t_k(x) - t_k(S^{-t}T_W S^t x).$$

It is now time to bring up the basic notion of *sufficiency* of a trajectory (segment). This is the utmost important necessary condition for the proof of the Theorem on Local Ergodicity for Semi–Dispersing billiards, see Condition (ii) of Theorem 3.6 and Definition 2.12 in [K-S-Sz (1990)] .

Definition 2.5.

(1) *The non–singular trajectory segment $S^{[a,b]}x$ (a and b are supposed not to be moments of collision) is said to be sufficient if and only if the dimension of $\mathcal{N}_t(S^{[a,b]}x)$ ($t \in [a,b]$) is minimal, i.e. dim $\mathcal{N}_t(S^{[a,b]}x) = 1$.*

(2) *The trajectory segment* $S^{[a,b]}x$ *containing exactly one singularity is said to be sufficient if and only if both branches of this trajectory segment are sufficient.*

For the notion of trajectory branches see, for example, the end of Section 2 in [Sim(1992)-I].

Definition 2.6. *The phase point* $x \in \mathbf{M}$ *with at most one singularity is said to be sufficient if and only if its whole trajectory* $S^{(-\infty,\infty)}x$ *is sufficient, which means, by definition, that some of its bounded segments* $S^{[a,b]}x$ *is sufficient.*

In the case of an orbit $S^{(-\infty,\infty)}x$ with exactly one singularity, sufficiency requires that both branches of $S^{(-\infty,\infty)}x$ be sufficient.

Weak Lemma on Avoiding of Balls

The next lemma is a measure-theoretic version of Lemma 3 in [K-S-Sz(1989)], and is slightly more generally formulated here than Lemma 2.16 of [K-S-Sz(1991)]. We consider a mixing semi-dispersing billiard in a d-dimensional compact domain $\mathbf{Q}$, where $\mathbf{Q} \subset \mathbb{R}^d$ or $\mathbf{Q} \subset \mathbf{T}^d$. Let $\emptyset \neq B \subset \mathbf{Q}$ be an open subset of $\mathbf{M}$ and let $H \subset \mathbb{R}$ be a subset of $\mathbb{R}$ such that $\sup H = +\infty$. We introduce the notation

$$A_H(B) := \{x \in \mathbf{M} : S^t x \notin B \text{ for every } t \in H\}.$$

Here $A_H(B)$ is the set of those phase points of $\mathbf{M}$ whose trajectory avoids the ball B at every time $t \in H$.

Lemma 2.7. *(Weak lemma on avoiding of balls):*

$$\mu(A_H(B)) = 0,$$

where μ is the usual invariant measure of the billiard system $(M, \{S^t\}, \mu)$.

Decompositions and sub-billiards

It is intuitively clear that trajectory segments along which the system of balls decomposes into at least two non-interacting subclasses can not gather all degrees of freedom or, in a more technical wording, the segment cannot be sufficient. Throughout the paper we will be using the following notations: Assume that a system $\mathcal{N} = \{1, 2, \dots, N\}$ of N (≥ 2) balls is decomposed into two non-empty classes: $\mathcal{N} = P_1 \cup P_2,\ P_1 \cap P_2 = \emptyset, |P_i| \neq 0\ (i = 1, 2)$. We say that the trajectory segment $S^{(a,b)}x$ *is partitioned* by $P = \{P_1, P_2\}$ if all of its non-tangential collisions occur among particles of the same class of P, only. In such a case, the action of S^t on $x = \{\tilde{x}_{P_1}, \tilde{x}_{P_2}\}$ $(t \in [a, b])$ is certainly the product of two independent *subdynamics*: $\tilde{S}^t_{P_1}\tilde{x}_{P_1}$ and $\tilde{S}^t_{P_2}\tilde{x}_{P_2}$.

To be more exact, we express $S^t x$ in detail as a direct product. To this end decompose

$$x = \{\{(q_j, v_j) :\ j \in P_1\}, \{(q_j, v_j) :\ j \in P_2\}\} := \{\tilde{x}_{P_1}, \tilde{x}_{P_2}\}.$$

Denote

$$\sum_{j \in P_i} q_j = \bar{q}_i, \quad \sum_{j \in P_i} v_j = I_i, \quad \frac{1}{2}\sum_{j \in P_i} v_j^2 = E_i$$

for i=1,2. By a standard change of coordinates $K_{(\bar{q}_i, I_i, E_i)}$, to every $\tilde{x}_{P_i}$ there corresponds an element $x_{P_i} = K_{(\bar{q}_i, I_i, E_i)}\tilde{x}_{P_i} \in M_{P_i}$, (the phase space of a billiard with $|P_i|$ balls), and $S^t x$ can be understood as

$$S^t x = \{K^{-1}_{(\bar{q}_i + tI_i, I_i, E_i)} S^t_{P_i} K_{(\bar{q}_i, I_i, E_i)}\tilde{x}_{P_i} :\ i = 1, 2\}.$$

The dynamics S^t_i or $K^{-1}_{(\bar{q}_i + tI_i, I_i, E_i)} S^t_{P_i} K_{(\bar{q}_i, I_i, E_i)}$ acting on $\tilde{x}_{P_i}$ will be called *subdynamics* or a *sub-billiard.*

3. Proof of the Theorems Assuming Main Lemmas 3.4 and 3.7

Definition 3.1. *We say that the symbolic collision sequence* $\Sigma = (\sigma_1, \dots, \sigma_m)$ *is connected if the collision graph of this sequence:* $\mathcal{G}_\Sigma := (\mathcal{V} = \{1, 2, \dots, N\}, \mathcal{E}_\Sigma := \{\{i_k, j_k\} :$ *where* $\sigma_k = \{i_k, j_k\},\ 1 \leq k \leq m\})$ *is connected.*

Definition 3.2. *We say that the symbolic collision sequence* $\Sigma = (\sigma_1, \dots, \sigma_m)$ *is* C*-rich, with* C *being a natural number, if it can be decomposed to at least* C *consecutive, disjoint collision sequences in such a way that each of them is connected.*

Proof of Theorem A. Let us fix a system satisfying the conditions of the Theorem where the number of particles is $N \geq 2$. The proof of the K-property of the standard billiard flow with $\mathcal{D}$-interaction will go on by induction on the number N of particles. Assume thus that the standard billiard ball system with every permitted $\mathcal{D}$-interaction and in every permitted dimension has the K-property for any number of particles $n < N$ whenever the interior of the configuration space is connected. For $N = 2$ this is known to be the case by [Sin(1970)] , [B-S (1973)] ($\nu = 2$) and by [S-Ch (1987)] ($\nu \geq 3$).

It is important to note here that, in fact, the Bernoulli property of the standard billiard flow $(\mathbf{M}, \{S^t\}, \mu)$ follows from its ergodicity, as it has been recently shown by Chernov–Haskell and Ornstein–Weiss, cf. [C-H (1995)] and [O-W(1995)]. For a brief summary of how the K-property follows, see – for instance – Section 7 of [Sim(1992)-I] , or [K-S(1986)].

Richness will be understood in the sense of Defintion 3.2 by choosing $C := C(N)$ prescribed by the forthcoming Main Lemma 3.4. We stress again that the proof of Main Lemma 3.4 is independent of the inductive assumption.

The whole strategy outlined in section 1 is based on the Theorem of Local Ergodicity for semi–dispersing billiards. In fact, the proof of global ergodicity uses its form given in [S-Ch (1987)] , whereas the inductive proof of part (3) of the strategy requires its more general, transversal form as phrased in [K-S-Sz (1990)] . For an easier identification of its conditions we refer to the work [K-S-Sz (1990)] .

According to the results of [V (1979)] and [G (1981)], in a semi-dispersing billiard system there are no orbits with a finite accumulation point of collision moments. Consequently, Condition 2.1 of [K-S-Sz (1990)] is fulfilled automatically. Condition 3.1, the so-called Chernov-Sinai Ansatz, will be verified at the end of the section.

Condition 3.2 on the regularity of the set of degenerate tangencies is automatic for cylindric billiards. The argument given in [K-S-Sz (1991)] for Condition 3.3 on the regularity of the set of double singularities for the case $N = 3$ also works in our general case. What is more, Lemma 4.15 of [K-S-Sz (1991)] asserts that the set of orbits containing two or more singular collisions is a countable union of codimension-2 submanifolds of $\mathbf{M}$. Thus, it is sufficient to concentrate our proof on the set $\mathbf{M}^*$ of orbits containing at most one singular collision, and to show that, outside a slim set, any $x \in \mathbf{M}^*$ has an open neighbourhood belonging to one ergodic component. This is what we will do.

Return first to the fundamental theorem (Theorem on Local Ergodicity, Theorem 5 of [S-Ch (1987)] or Corollary 3.12 of [K-S-Sz (1990)]) which is applicable to our system of N particles, saying that *a suitably small, open neighborhood of any sufficient phase point with at most one singularity belongs to one ergodic component.*

Since the complement of a slim set in a connected manifold is always connected (see Property 2.10 from [K-S-Sz (1991)]), our goal is to show that, apart from a slim set of points, every phase point with at most one singularity is sufficient. To do so, let us follow our three step strategy outlined in the introduction. Let us recall first Theorem 5.1 of [Sim(1992)-I] , whose straightforward application to billiard ball systems with our restricted interactions is

Propostion 3.3. *(Theorem 5.1 of [Sim(1992)-I]) Consider a system of N (≥ 3) particles on the unit torus $\mathbb{T}^\nu$ $(\nu \geq 2)$ satisfying the conditions of Theorem A. Assume that for all $n < N$ the statement of Theorem A holds true. Let P be a given, two–class partition of the N particles. Then the set*

$$F_+ = \left\{x \in \mathbf{M} : S^{[0,\infty)}x \text{ is partitioned by } P\right\}$$

is a closed, zero set with codimension at least two (i.e. a closed slim set).

This statement immediately settles part (2) of the aforementioned strategy and, in fact, would allow to fix $C(N)$ arbitrarily large.

The (non-inductive) part (1) of the strategy, detailed in the introduction, will be established by the forthcoming Main Lemma whose proof will be given in Section 4.

Main Lemma 3.4. *Assume that the graph of interactions is a tree with edge-degree D. If the symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_m)$ $(\sigma_k \in \mathcal{E}(\mathcal{G}))$ of the non-singular orbit segment $S^{[a,b]}x_0$ is 3-rich, then there exist an open neighbourhood U $(x_0 \in U \subset M)$ and a closed algebraic subset $\mathcal{N} \subset U$ such that*

(i) *for every $y \in U \setminus \mathcal{N}$ the orbit segment $S^{[a,b]}y$ is sufficient;*

(ii)

$$codim_U \mathcal{N} \geq \nu - D.$$

Consider now a rich point $x \in \mathbf{M}^0$. (By Proposition 3.3, we can assume the richness of x.) If $x \in \mathbf{M}^0$ is not sufficient, then by Main Lemma 3.4, there is an open ball-like neighbourhood U of x and a CW-complex (more accurately, a closed algebraic subset) $\mathcal{N} \subset U$, $x \in \mathcal{N}$, such that $\mathrm{codim}_U \mathcal{N} \geq 2$, and all non-sufficient points of U are contained in $\mathcal{N}$. Consequently, by the Theorem on Local Ergodicity and by the connectedness of the set $U \setminus \mathcal{N}$, the neighborhood U belongs to one ergodic component (modulo zero sets, of course).

Let us finally handle the phase points $x \in \mathbf{M}^1$. We can, and do assume that x belongs, for instance, to the set $\mathcal{SR}^+$. Here we recall Theorem 6.1 of [Sim(1992)-I] , whose straightforward application to our situation is

Proposition 3.5. *(Theorem 6.1 of [Sim(1992)-I]) Suppose that for all $n < N$ the statement of Theorem A holds true and, moreover, Main Lemma 3.4 has been proved for billiard flows satisfying the conditions of Theorem A. Then, for any given two–class partition P of the N particles, any positive number t_0 and any cell $C \subset \mathcal{SR}^+$ of $\mathcal{SR}^+$ with the maximum dimension $(2d-3)$ the subset*

$$C(t_0, P) = \left\{ x \in C : \; S^{[t_0,\infty)}x \text{ is partitioned by } P \right\} \tag{3.6}$$

has measure zero with respect to the Riemannian volume μ_C *of the manifold* C.

(We note that the inductive proof of Theorem 6.1 of [Sim(1992)-I] just used the transversal fundamental theorem and the K-property, both for a smaller number of balls (in our case, of particles)). Note that - since the set of doubly singular phase points is a countable union of codimension-two submanifolds, as said before - the condition of regularity of the orbit $S^{[0,\infty)}x$ can be omitted from the definition of $C(t_0, P)$. (It was present in the original publication.) Since the set $C(t_0, P)$ in this theorem is closed and, therefore, it is a *slim* subset of $\mathbf{M}$, we may assume that x does not belong to any of these sets, i. e. the (non-singular) positive semi-trajectory $S^{(0,\infty)}x$ contains infinitely many consecutive, connected collision graphs. By Main Lemma 3.4, this positive orbit is sufficient, if only x does not belong to some codimension-one submanifolds of $\mathcal{SR}^+$, but such sets are slim (negligible) in the whole phase space $\mathbf{M}$. In this way we have proved two things:

(a) the Chernov-Sinai Ansatz,

and the fact that

(b) apart from a slim subset of the phase space, the trajectory of every phase point $x \in \mathbf{M}^1$ is sufficient.

The inductive proof of the Theorem A is now complete. □

Proof of Theorem B. The proof of Theorem B is completely analogous to that of Theorem A. Our first remark is that there the information on the graph of interactions was essentially used in the proof of Main Lemma 3.4, only. Observe next that, if we want to obtain the statement of Theorem B for a given N, then the statements of Propositions 3.3 and 3.5 will still be true for our cyclic interaction. The reason is that, if a semi-trajectory of our system decomposes into two or more non-interacting subsystems, then the interaction-graphs of these subsystems are simple trees (in fact, just paths). The only remaining task is to replace Main Lemma 3.4, and this is done by

Main Lemma 3.7. *Assume that the graph of interactions consists of one circle of length N. If the symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_m)$ $(\sigma_k \in \mathcal{E}(\mathcal{G}))$ of the non-singular orbit segment $S^{[a,b]}x_0$ is 4-rich, then there exist an open neighbourhood U $(x_0 \in U \subset M)$ and an algebraic set $\mathcal{N} \subset U$ such that*

(i) *for every $y \in U \setminus \mathcal{N}$ the orbit segment $S^{[a,b]}y$ is sufficient;*

(ii)

$$codim_U \mathcal{N} \geq \nu - 2.$$

Main Lemma 3.7 will be demonstrated separately in Section 5. Hence Theorem B. $\square$

Proof of Theorem C. Our arguments will essentially follow the line of the proof of Theorem A, but an additional difficulty arises: since the statement is weaker so is the inductive hypothesis. As remarked earlier, the basic pillars of the induction are Proposition 3.3, Main Lemma 3.4 and Proposition 3.5. The codimension-one statement of Main Lemma 3.4 is sufficient for our purposes in the neighbourhoods of rich points and, consequently, we can concentrate on replacing Propositions 3.3 and 3.5.

Let us start with the first one:

Proposition 3.8. *Consider a system of N (≥ 3) particles on the unit torus $\mathbb{T}^\nu$ $(\nu = D+1)$ satisfying the conditions of Theorem C. Assume that for all $n < N$ the statement of Theorem C holds true. Let P be a given, two–class partition of the N particles. Then the set*

$$F_+ = \left\{ x \in \mathbf{M} : S^{[0,\infty)}x \text{ is partitioned by } P \right\}$$

has measure zero.

Proof of Proposition 3.8. The proof borrows its approach from the proof of Theorem 5.1 of [Sim (1992-I)], and therefore our exposition will be concise. Assume that the two classes are $P_1 = \{1, \ldots, n\}$ and $P_2 = \{n+1, \ldots, N\}$. The cases a)$|P_1|, |P_2| \geq 2$ and b) *either* $|P_1|$ *or* $|P_2| = 1$ can be treated similarly and thus we only consider the first one.

It is, of course, sufficient to show that for an arbitrary small, open, ball-like set $U_0 \subset \mathbf{M}$ one has $\mu(U_0 \cap F_+) = 0$. Every point $x \in U_0$ can be characterized locally by the following coordinates in a unique way:

(1) $\pi_{P_1}(x) \in \mathbf{M}_1$, (2) $\pi_{P_2}(x) \in \mathbf{M}_2$,

(3) $C_1(x) = \frac{1}{n}\sum_{i=1}^{n} q_i(x) \in \mathbb{T}^\nu$, (4) $I_1(x) = \frac{1}{n}\sum_{i=1}^{n} v_i(x)$,

(5) $E_1(x) = \frac{1}{2}\sum_{i=1}^{n} v_i^2(x)$.

Assume that, according to the above splitting of the data, U_0 is of the form

$$U_0 = B_1 \times B_2 \times B_3 \times B_4 \times B_5,$$

where $B_1 \subset \mathbf{M}_1, B_2 \subset \mathbf{M}_2, B_3 \subset \mathbb{T}^\nu, B_4 \subset \mathbb{R}^\nu, B_5 \subset \mathbb{R}_+$ are balls of the corresponding coordinate spaces. Choose and fix any $b_i \in B_i$: $i = 3, 4, 5$. The relation $\mu(F_+) = 0$ will certainly follow if we show that for almost every such choices of the b_is

$$\mu_{b_3,b_4,b_5}(F_+(b_3, b_4, b_5)) = 0, \tag{3.9}$$

where

$$F_+(b_3, b_4, b_5) = \{(y_1, y_2) \in B_1 \times B_2 \colon (y_1, y_2, b_3, b_4, b_5) \in F_+\}$$

and μ_{b_3,b_4,b_5} denotes the conditional measure of μ under the conditions corresponding to fixing the values of b_3, b_4, b_5. (This conditional measure is equivalent to the Liouville measure on $\mathbf{M}_1 \times \mathbf{M}_2$.) Indeed, we will show that (3.9) holds whenever the components of $b_4 = I_1(x)$ are independent over the rationals.

The relation $x \in F_+$ is equivalent to saying that for every pair $i \in P_1, j \in P_2$, $\{i, j\} \in \mathcal{E}$ and every $t \geq 0$

$$\varrho(q_i^t(x), q_j^t(x)) \geq d_{ij}, \ . \tag{3.10}$$

where $\varrho(.,.)$ denotes the euclidean distance in the torus $\mathbb{T}^\nu$.

Now we will be considering the subdynamics corresponding to our two-class partition (cf. the end of Section 2), and will denote them, for simplicity, by S_1 and S_2, respectively (their phase spaces are $\mathbf{M}_1$ and $\mathbf{M}_2$, of course). Denote by $\widetilde{q}_i^t(x)$ the ith configuration vector coordinate of $S_1^t\left(\pi_{P_1}(x)\right)$ if $i \in P_1$, and similarly by $\widetilde{q}_j^t(x)$ the jth configuration vector coordinate of $S_2^t\left(\pi_{P_2}(x)\right)$ if $j \in P_2$. Since for $i \in P_1$

$$q_i^t(x) = \widetilde{q}_i^t(x) + C_1(x) + t\, I_1(x) \qquad\qquad (\mathrm{mod}\ \mathbb{Z}^\nu),$$

and for $j \in P_2$

$$q_j^t(x) = \widetilde{q}_j^t(x) - \frac{n}{N-n} C_1(x) - \frac{nt}{N-n}\, I_1(x) \qquad\qquad (\mathrm{mod}\ \mathbb{Z}^\nu),$$

(3.10) can be replaced by

$$\varrho\left(\widetilde{q}_i^t(x) - \widetilde{q}_j^t(x), -\frac{N}{N-n} C_1(x) - \frac{Nt}{N-n} I_1(x)\right) \geq d_{ij}\ ,$$

where needless to say that the second argument should be understood mod $\mathbb{Z}^\nu$. We also note that $C_1(x)/(N-n)$ is not defined uniquely, but our arguments are independent of the choice of its value.

Consider the product dynamics

$$S^t\left(\pi_{P_1}(x)\right) \times S^t\left(\pi_{P_2}(x)\right),$$

and choose an arbitrary pair $G_1 \subset \mathbf{M}_1$, $G_2 \subset \mathbf{M}_2$ of open subsets such that the factors of the previous product are K-flows on G_1 and G_2, respectively. As a consequence, the product is a mixing flow on the open subset $G_1 \times G_2 \subset \mathbf{M}_1 \times \mathbf{M}_2$. Let us denote by $\tilde{G}_1$ the projection of G_1 to the configuration vector coordinate $\tilde{q}_i$ living on $\mathbb{T}^\nu$ and similarly by $\tilde{G}_2$ the projection of G_2 into $\mathbb{T}^\nu$ corresponding to $\tilde{q}_j$. Of course, both $\tilde{G}_1$ and $\tilde{G}_2$ are non-empty, open subsets of $\mathbb{T}^\nu$, and so is $\tilde{G}_1 - \tilde{G}_2$. Choose an open ball $G^* \subset \tilde{G}_1 - \tilde{G}_2$ whose radius is less than $d_{i,j}/2$. Having fixed the values of $b_3 \in B_3, b_4 \in B_4, b_5 \in B_5$ (with b_4 having rationally independent coordinates), we can find an infinite sequence $0 < t_1 < t_2 < \cdots < t_k < \dots$, $t_k \to \infty$, of time moments such that

$$-\frac{N}{N-n}C_1(x) - \frac{Nt_k}{N-n}I_1(x) \ \in G^* \qquad (\mathrm{mod} \quad \mathbb{Z}^\nu)$$

for all $k = 1, 2, \dots,$. By the weak lemma on avoiding of balls (Lemma 2.7), the μ_{b_3,b_4,b_5}-measure of those phase points of our aforedefined product dynamics, for which $\tilde{q}_i - \tilde{q}_j$ never enters the set G^* in the time moments $t_k: \ k = 1, 2, \dots$, is zero.

Hence (3.9) and Proposition 3.8. $\square$

Finally, we formulate a proposition to replace Proposition 3.5.

Proposition 3.11. *Consider a system of* $N \quad (\geq 3)$ *particles on the unit torus* $\mathbb{T}^\nu \quad (\nu = D+1)$ *satisfying the conditions of Theorem C. Assume, moreover, that for every* $n < N$ *the statement of Theorem C holds true. Then for any given two-class partition* P *of the* N *particles, any positive number* t_0 *and any cell* $C \subset \mathcal{SR}^+$ *of* $\mathcal{SR}^+$ *with the maximum dimension* $(2d-3)$ *the subset* $C(t_0, P)$ *defined by (3.6) has measure zero with respect to the Riemannian volume* μ_c *of the manifold* C.

Proof of Proposition 3.11. The proof given for Theorem 6.1 of Sim (1992-I) also works under our conditions. The basic argument for that is that the inductive assumption ensures that the factors corresponding to the subdynamics defined by the two-class partition have positive ergodic components, and the whole proof can be resctricted to products of these positive components. On these product sets the basic hyperbolic structure, fundamental for the proof, is the same as in the case of the product of K-flows.

Thus Proposition 3.11 and, consequently, Theorem C is established. $\square$

Finally, the proof of Theorem D can be obtained from that of Theorem C just like Theorem B was established after the proof of Theorem A. $\square$

4. Combinatorial Richness Implies Sufficiency. Trajectories With a Restricted Collision Graph. The Proof of Main Lemma 3.4

The content of this paragraph is the proof of Main Lemma 3.4 which heavily uses the upcoming Lemma 4.2.

We denote by

$$A_k : \mathcal{N}_a\left(S^{[a,b]}x_0\right) \to \mathbb{R} \tag{4.1}$$

the advance (time shift) corresponding to the k-th collision σ_k, see Section 2. It is known (see, for instance, Section 2 of [Sim(1992)-I]) that A_k is a linear functional on the real vector space $\mathcal{N}_a\left(S^{[a,b]}x_0\right)$.

Lemma 4.2. *If $1 \le p < q < r \le m$, $\sigma_p = \sigma_r$, $\sigma_p \ne \sigma_q$, and $\sigma_p \cap \sigma_q \ne \emptyset$, then $A_p = A_q = A_r$ modulo a closed algebraic set of exceptional phase points x_0, such that the codimension (in $\mathbf{M}$, as always!) of this exceptional set is at least $\nu - D$.*

Proof. Induction on the number $r-p$. For $r-p=1$ the statement is vacuous. (Or, in other words, if we considered $r-p=2$ as the starting point of the induction, its proof would be exactly the same as the inductive step.) Let now $r-p>1$, and suppose the inductive hypothesis for smaller values of $r-p$.

Plainly, we can assume that $\sigma_k \ne \sigma_p$ for $k = p+1, \dots, r-1$. Write down the Connecting Path Formula (CPF, Lemma 2.9 of [Sim(1992)-II]) for the new advance A_r in terms of A_p and the advances A_k with $p<k<r$ and $\sigma_k \cap \sigma_p \ne \emptyset$:

$$\begin{aligned} A_r\left(v^-_{i_p}(t_r) - v^-_{j_p}(t_r)\right) &= A_p\left(v^+_{i_p}(t_p) - v^+_{j_p}(t_p)\right) + \\ \sum_{\substack{p<k<r \\ i_p\in\sigma_k}} A_k\left(v^+_{i_p}(t_k) - v^-_{i_p}(t_k)\right) &- \sum_{\substack{p<k<r \\ j_p\in\sigma_k}} A_k\left(v^+_{j_p}(t_k) - v^-_{j_p}(t_k)\right). \end{aligned} \tag{4.3}$$

According to our inductive hypothesis, the advances A_k in (4.3) only depend on the edge σ_k, and they are independent of the actual value of k. Thus, the terms in

(4.3) with the same σ_k can be collected together, so we obtain

$$A_r\left(v_{i_p}^-(t_r)-v_{j_p}^-(t_r)\right)=A_p\left(v_{i_p}^+(t_p)-v_{j_p}^+(t_p)\right)+$$

$$\sum_{\substack{\sigma\in\mathcal{E}\\ i_p\in\sigma}}A_\sigma\cdot\sum_{\substack{p<k<r\\ \sigma_k=\sigma}}\left(v_{i_p}^+(t_k)-v_{i_p}^-(t_k)\right)-\sum_{\substack{\sigma\in\mathcal{E}\\ j_p\in\sigma}}A_\sigma\cdot\sum_{\substack{p<k<r\\ \sigma_k=\sigma}}\left(v_{j_p}^+(t_k)-v_{j_p}^-(t_k)\right). \tag{4.4}$$

Suppose now that, to the contrary of the lemma, $A_p\neq A_r$. Evaluate (4.4) at a neutral vector $n_0\in\mathcal{N}_a\left(S^{[a,b]}x_0\right)$ for which, say, $A_p(n_0)=0$ but $A_r(n_0)\neq 0$. (This can always be achieved by adding to n_0 an appropriate multiple of the flow direction.) Formula (4.4) evaluated at such $n_0\in\mathcal{N}_a\left(S^{[a,b]}x_0\right)$ yields

$$A_r(n_0)\left(v_{i_p}^-(t_r)-v_{j_p}^-(t_r)\right)=\sum_{\substack{\sigma\in\mathcal{E}\\ i_p\in\sigma}}A_\sigma(n_0)\Gamma_\sigma-\sum_{\substack{\sigma\in\mathcal{E}\\ j_p\in\sigma}}A_\sigma(n_0)\Gamma_\sigma, \tag{4.5}$$

where the vector Γ_σ is defined by the formula

$$\Gamma_\sigma=\sum_{\substack{p<k<r\\ \sigma_k=\sigma\\ \sigma\cap\sigma_p=\alpha}}\left(v_\alpha^+(t_k)-v_\alpha^-(t_k)\right). \tag{4.6}$$

Of course, in the equations (4.4-4.5) we only include the terms $A_\sigma\cdot\Gamma_\sigma$ with collisions σ for which the sum in (4.6) is non-empty. Correspondingly, in the remaining part of the proof of Lemma 4.2, when speaking about A_σ or Γ_σ, we always mean by σ a collision with a non-empty sum (4.6).

By our assumption on the graph $\mathcal{G}$, the number of terms on the right-hand-side does not exceed D. We observe that the vectors Γ_σ from (4.6) are determined the evolution of two disjoint systems evolving between the collisions σ_p and σ_r, and these two systems get connected by the collision $\sigma_r=\{i_r,j_r\}=\{i_p,j_p\}$. Thus, the relative velocity $\left(v_{i_p}^-(t_r)-v_{j_p}^-(t_r)\right)$ on the left-hand-side of (4.5) can be chosen independently of the vectors Γ_σ. Therefore, the event that $\left(v_{i_p}^-(t_r)-v_{j_p}^-(t_r)\right)$ belongs to the linear subspace of $\mathbb{R}^\nu$ generated by the vectors Γ_σ (which is the assertion of (4.5)) can only occur on a codimension-$(\nu-D)$ subset of the phase space.

Hence $A_p=A_r$ modulo a codimension-$(\nu-D)$ closed algebraic set.

Let us turn now to the proof of $A_p = A_q$. It requires a bit more detailed analysis of the vectors Γ_σ on the right-hand-side of (4.4). Namely, the inductive assumption not only guarantees that the advance A_k of σ_k merely depends on the unordered pair $\sigma_k = \sigma$, but, also, it says that $A_{\sigma^*} = A_{\sigma^{**}}$ whenever there exist integers k_1^*, k_2^*, k^{**} among the integers $p+1, \dots, r-1$, and there exists an element $\alpha \in \{i_p, j_p\}$ such that $\alpha \in \sigma^* \cap \sigma^{**}$, $k_1^* < k^{**} < k_2^*$, $\sigma^* = \sigma_{k_1^*} = \sigma_{k_2^*}$, and $\sigma^{**} = \sigma_{k^{**}}$. On the set of edges (collisions)

$$E = \{\sigma \in \mathcal{E} \,|\, \sigma \cap \{i_p, j_p\} \neq \emptyset \text{ and } \exists k,\, p < k < r,\, \sigma = \sigma_k\} \tag{4.7}$$

we define the equivalence relation $\sim$ as the smallest equivalence relation on E making every pair of σ^* and σ^{**} above equivalent. Let $E = C_1 \cup \dots \cup C_s$ be the splitting of E into the equivalence classes of $\sim$. Thanks to the inductive hypothesis, we can pull together the terms $A_\sigma \cdot \Gamma_\sigma$ of (4.4) corresponding to the same class C_i of $\sim$ as follows:

$$A_r \left(v_{i_p}^-(t_r) - v_{j_p}^-(t_r)\right) = A_p \left(v_{i_p}^+(t_p) - v_{j_p}^+(t_p)\right) + \sum_{i=1}^{s} A(C_i) \cdot \Gamma(C_i), \tag{4.8}$$

where $A(C_i) = A_\sigma$ for any $\sigma \in C_i$, and $\Gamma(C_i) = \sum_{\sigma \in C_i} \Gamma_\sigma$ if $i_p \in \sigma$ for the elements σ of C_i, while $\Gamma(C_i) = -\sum_{\sigma \in C_i} \Gamma_\sigma$ otherwise. The next sub-lemma shows that the advantage of the expansion (4.8) is that the terms $\Gamma(C_i) \in \mathbb{R}^\nu$ become independent of each other.

Sub-lemma 4.9. *Let C_i and C_j $(1 \le i < j \le s)$ be two $\sim$ equivalence classes with the property*

$$\bigcap_{\sigma \in C_i \cup C_j} \sigma \neq \emptyset,$$

i. e. the edges of C_i and C_j are all adjacent to i_p or they are all adjacent to j_p. Then either for every pair $k, l \in \{p+1, \dots, r-1\}$ the relations $\sigma_k \in C_i$ and $\sigma_l \in C_j$ imply $k < l$, or for every pair $k, l \in \{p+1, \dots, r-1\}$ the relations $\sigma_k \in C_i$ and $\sigma_l \in C_j$ imply $k > l$, that is, there is a natural time ordering among the classes C_i containing the edges adjacent to i_p (j_p).

Proof. A straightforward consequence of the definition of the equivalence relation $\sim$. $\square$

We can now finish the proof of Lemma 4.2. Suppose that, contrary to the statement of the lemma, $A_q(n_0) \neq 0$, $A_p(n_0) = A_r(n_0) = 0$ with some vector $n_0 \in \mathcal{N}_a \left(S^{[a,b]}x_0\right)$. Evaluate (4.8) at this vector n_0:

$$\sum_{i=1}^{s} A(C_i)[n_0] \cdot \Gamma(C_i) = 0. \tag{4.10}$$

Since $A(C_i)[n_0] \neq 0$ for $\sigma_q \in C_i$, the equation (4.10) means that the vectors

$$\Gamma(C_1), \dots, \Gamma(C_s) \ (\in \mathbb{R}^\nu)$$

are linearly dependent. According to the previous sub-lemma and the fact that the graph $\mathcal{G}$ is a tree, the velocity changes $\Gamma(C_1), \dots, \Gamma(C_s)$ may be chosen independently. Taking into account the inequality $s \leq D$, we have that the linear dependence (4.10) can only occur on a set of phase points x_0 with codimension at least $\nu + 1 - D$. Hence Lemma 4.2. $\square$

Finishing the proof of Main Lemma 3.4

Suppose that the subgraphs $\Sigma^{(1)} = (\sigma_1, \dots, \sigma_k)$, $\Sigma^{(2)} = (\sigma_{k+1}, \dots, \sigma_l)$, and $\Sigma^{(3)} = (\sigma_{l+1}, \dots, \sigma_m)$ of Σ are all connected on the vertex set $\mathcal{V} = \{1, 2, \dots, N\}$, i. e. the set of edges of $\Sigma^{(j)}$ $(j = 1, 2, 3)$ is the set $\mathcal{E}$ of all edges of $\mathcal{G}$. The fact that $\mathcal{G}$ is a tree and Lemma 4.2 straightforwardly imply that all advances $A_1, \dots, A_m$ must be equal modulo a closed algebraic set of exceptional phase points x_0 with codimension at least $\nu - D$. The identity of these advances, however, precisely means that the orbit segment $S^{[a,b]}x_0$ is sufficient, see also Lemma 2.13 of [Sim(1992)-II]. This finishes the proof of Main Lemma 3.4. $\square$

5. Trajectories with a Circular Collision Graph. The Proof of Main Lemma 3.7

In this section we slightly generalize the results of the previous paragraph by considering the following graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$ of the *circular* interaction:

$$\mathcal{E}_c = \{\{i, i+1\} \,|\, 1 \le i \le N-1\} \cup \{\{1, N\}\}\,. \tag{5.1}$$

For this graph $\mathcal{G}_c$, which is a circle of length N, the edge degree D is equal to two. The actual importance of this circular interaction is explained by the fact that this is just the interaction of the particles in the so called *pencase model* introduced by Chernov and Sinai, see [S-Ch(1985)] or Section 1.

The content of this section is exclusively the proof of Main Lemma 3.7.

Before proving anything, however, it is worth noting that the whole issue of proving sufficiency by the use of combinatorial richness is totally insensitive of the uniform velocity dilations of the billiard flow. Therefore, it is not necessary to assume the usual energy normalization $\sum_{i=1}^N v_i^2 = 1$, so we will not do that.

First of all, we need the following

Lemma 5.2. *Suppose that the graph of allowed collisions is the circle $\mathcal{G}_c$ and $S^{[a,b]}x_0$ is a non-singular orbit segment with the symbolic sequence $\Sigma = (\sigma_1, \dots, \sigma_m)$. If $1 \le p < q < r \le m$, $\sigma_p = \sigma_r$, $\sigma_p \ne \sigma_q$, and $\sigma_p \cap \sigma_q \ne \emptyset$, then $A_p = A_q = A_r$ modulo a closed algebraic set of exceptional phase points x_0 such that the codimension of this set is at least $\nu - 2$.*

Proof of Lemma 5.2. This is again an induction on the number $r-p$. For $r-p=1$ there is nothing to prove. Suppose that $r-p>1$ and the statement is true for smaller values of $r-p$. Just like in the previous section, we can assume that $\sigma_k \ne \sigma_p$ for $k = p+1, \dots, r-1$, and $\sigma_p = \{1, N\}$, i. e. $i_p = 1$, $j_p = N$. The useful, short form of the Connecting Path Formula for the new advance A_r in terms of the earlier advances is now

$$
\begin{aligned}
(5.3)\qquad & A_r \cdot \left(v_1^-(t_r) - v_N^-(t_r)\right) = A_p \cdot \left(v_1^+(t_p) - v_N^+(t_p)\right) \\
& + A_{\{1,2\}} \cdot \sum_{\substack{k \\ p<k<r \\ \sigma_k=\{1,2\}}} \left(v_1^+(t_k) - v_1^-(t_k)\right) - A_{\{N-1,N\}} \cdot \sum_{\substack{k \\ p<k<r \\ \sigma_k=\{N-1,N\}}} \left(v_N^+(t_k) - v_N^-(t_k)\right) \\
& = A_p \cdot \left(v_1^+(t_p) - v_N^+(t_p)\right) + A_{\{1,2\}} \cdot \left(v_1^-(t_r) - v_1^+(t_p)\right) - A_{\{N-1,N\}} \cdot \left(v_N^-(t_r) - v_N^+(t_p)\right).
\end{aligned}
$$

Here again we used, as a corollary of the inductive hypothesis, the fact that the advance A_k of a collision σ_k $(p < k < r)$ merely depends on σ_k and not on the actual value of k.

First we re-write the equation (5.3) with such neutral vectors $n \in \mathcal{N}_a\left(S^{[a,b]}x_0\right)$ for which $A_r(n) = 0$. Recall that any neutral vector can be modified by a scalar multiple of the flow direction in such a way that one achieves $A_r(n) = 0$. Taking also into account that the multiplier of $A_{\{1,2\}}$ in (5.3) is the total change of velocity of the first ball between σ_p and σ_r, and that similar statement is true for the multiplier of $A_{\{N-1,N\}}$, we obtain

$$
\begin{aligned}
(5.4)\qquad & A_p(n)\left(v_1^+(t_p) - v_N^+(t_p)\right) + A_{\{1,2\}}(n)\left(v_1^-(t_r) - v_1^+(t_p)\right) \\
& - A_{\{N-1,N\}}(n)(v_N^-(t_r) - v_N^+(t_p)) = 0,
\end{aligned}
$$

for $n \in \mathcal{N}_a\left(S^{[a,b]}x_0\right)$, $A_r(n) = 0$.

In order to finish the inductive proof of Lemma 5.2, it is just enough to prove that all three advances in (5.4) must be zero apart from a set of codimension at least $\nu - 2$. This is what we do.

We may assume that every edge of the path $\mathcal{E}_c \setminus \{\{1, N\}\}$ occurs among the edges $\sigma_{p+1}, \dots, \sigma_{r-1}$, otherwise the statement of Lemma 5.2 directly follows from Lemma 4.2.

In order to show that $A_p = A_{\{1,2\}} = A_{\{N-1,N\}}$ modulo a codimension-$(\nu-2)$ set, use either the inductive hypothesis of Lemma 5.2, or just the assertion of Lemma 4.2 to construct an equivalence relation $\sim$ on the set of integers $\{p+1, \dots, r-1\}$ as follows: $i \sim j$ $(p+1 \le i < j \le r-1)$ if and only if $A_i = A_j$ logically follows from the statement of Lemma 4.2. The first observation is that the validity of $i \sim j$ merely depends on the collisions σ_i and σ_j, and it is independent of the actual value of i and j. This means that, in fact, the equivalence relation $\sim$ descends to an equivalence relation (also denoted by $\sim$) on the set of edges $\{\sigma_{p+1}, \dots, \sigma_{r-1}\} = \mathcal{E}_c \setminus \{\{1, N\}\}$. Two adjacent edges $\{i-1, i\}$ and $\{i, i+1\}$ are $\sim$ equivalent ($2 \le i \le N-1$) if and only if there are three integers $p+1 \le k_1 < k_2 < k_3 \le r-1$ such that either $\sigma_{k_1} = \sigma_{k_3} = \{i-1, i\}$, $\sigma_{k_2} = \{i, i+1\}$, or $\sigma_{k_1} = \sigma_{k_3} = \{i, i+1\}$, $\sigma_{k_2} = \{i-1, i\}$. The equivalence relation $\sim$ is then the transitive hull of the above defined $\sim$ between adjacent edges.

A ball (vertex) $i \in \{2, \dots, N-1\}$ is called a separating one if $\{i-1, i\} \not\sim \{i, i+1\}$. Let

$$2 \le i_1 < i_2 < \cdots < i_s \le N-1$$

be the collection of all separating vertices. It follows from the equation

$$A_{\{i_{k-1}, i_{k-1}+1\}} = A_{\{i_{k-1}+1, i_{k-1}+2\}} = \cdots = A_{\{i_k-1, i_k\}}$$

($1 \le k \le s+1$, $i_0 = 1$, $i_{s+1} = N$) that the subsystem of balls $\{i_{k-1}, \dots, i_k\}$ has a sufficient trajectory segment on the closed time interval $[t', t'']$, where t' is the time of the first occurence of any collision σ_n ($p+1 \le n \le r-1$) connecting two vertices of the subsystem $\{i_{k-1}, \dots, i_k\}$, while t'' is the time of the last such occurence.

The assertion of Lemma 5.2 is obviously true for $s = 0$. Thus, we assume that $s \ge 1$.

Consider the first separating vertex i_1 ($2 \le i_1 \le N-1$). In order to prove the lemma, it is enough to show that $A_{\{1,2\}} = 0$ in (5.4) modulo a codimension-($\nu - 2$) set, because the proof of $A_{\{N-1,N\}} = 0$ is analogous, and the coefficient $A_p(n)$ cannot be the sole non-zero coefficient in (5.4).

We assume that every collision of type $\{i_1 - 1, i_1\}$ (in the time interval (t_p, t_r), as always!) precedes every collision of type $\{i_1, i_1 + 1\}$ of the time interval (t_p, t_r). (The handling of the opposite case is analogous.) Choose, therefore, a separating time t^* for which every occurence of the collision $\{i_1 - 1, i_1\}$ takes place before t^* and, correspondingly, every occurence of the collision $\{i_1, i_1 + 1\}$ takes place after t^*. For a while we assume that $i_1 \geq 3$. After the proof of the equation $A_{\{1,2\}} = 0$ in this case, we shall handle the case $i_1 = 2$ separately.

Execute a *small* phase perturbation (i. e. a configuration *and* velocity perturbation) at time t^* such that the relative positions $q_k - q_l$ and the relative velocities $v_k - v_l$ are unchanged for $i_1 \leq k < l \leq N$, otherwise these small perturbations may be arbitrary. Distinguish such a varying, perturbed system from the unperturbed one by the superscript $\tilde{}$ over the kinetic data. Then a linear dependence (5.4) with non-zero coefficient $A_{\{1,2\}}(n)$ can be re-written for the perturbed system as

(5.5)

$$\tilde{v}_1^-(\tilde{t}_r) - \tilde{v}_1^+(\tilde{t}_p) = \alpha\left(\tilde{v}_1^+(\tilde{t}_p) - \tilde{v}_{i_1}(t^*)\right) + \alpha\left(\tilde{v}_{i_1}(t^*) - \tilde{v}_N^+(\tilde{t}_p)\right) + \beta\left(\tilde{v}_N^-(\tilde{t}_r) - \tilde{v}_N^+(\tilde{t}_p)\right)$$

with some α, $\beta \in \mathbb{R}$. The advantage of the form (5.5) is that the second and third terms on the right-hand-side of it do not change under the considered perturbations, so we can omit the superscript $\tilde{}$ from them:

(5.6)

$$\tilde{v}_1^-(\tilde{t}_r) - \tilde{v}_1^+(\tilde{t}_p) = \alpha\left(\tilde{v}_1^+(\tilde{t}_p) - \tilde{v}_{i_1}(t^*)\right) + \alpha\left(v_{i_1}(t^*) - v_N^+(t_p)\right) + \beta\left(v_N^-(t_r) - v_N^+(t_p)\right)$$

with some α, $\beta \in \mathbb{R}$.

Denote by $\overline{S}^{[t_p,t_r]}y_0$ the time evolution of the varying, perturbed billiard subsystem $\{1, \dots, i_1\}$ during the closed time interval $[t_p, t_r]$, when all (in principle) possible collision of type $\{i_1, i_1+1\}$ or $\{1, N\}$ are cancelled. Also, this subsystem is observed – by definition – from a moving reference coordinate system attached to the center of mass of the particles $1, 2, \dots, i_1$. (The center of mass can be defined locally in a continuous manner.) Denote the kinetic data of this subsystem (observed from the abovementioned reference system) by $\overline{q}_j(t)$ and $\overline{v}_j(t)$, $j = 1, \dots, i_1$. The linear dependence (5.6) can then be re-written as

(5.7)

$$\overline{v}_1^-(t_r) - \overline{v}_1^+(t_p) = \alpha\left(\overline{v}_1^+(t_p) - \overline{v}_{i_1}^-(t_r)\right) + \alpha\left(v_{i_1}(t^*) - v_N^+(t_p)\right) + \beta\left(v_N^-(t_r) - v_N^+(t_p)\right)$$

with some α, $\beta \in \mathbb{R}$.

We remind the reader here that

(i) the subsystem $\overline{S}^{[t_p,t_r]}y_0$ is sufficient,

and

(ii) the perturbations $\sim$ defined at time t^* make it possible to vary the subsystem's phase point $\overline{S}^{t^*}y_0$ on an open subset of the phase space of that subsystem.

The next sub-lemma is very easy to verify, and its proof will be left to the reader:

Sub-lemma 5.8. *Let $T^{(c,d)}z_0$ be a non-singular and sufficient orbit segment of a standard billiard flow of n balls, where $n \geq 3$. Let us execute all possible (but small!) pure configuration perturbations of this system at time c that keep the velocity of the k-th ball ($k \neq 1$) at time d fixed. Then the velocity $v_1(d)$ of the first ball fills up an open subset of $\mathbb{R}^\nu$ with maximum rank ν if $n \geq 4$, and $v_1(d)$ fills up an open subset of a $(\nu-1)$-dimensional sphere with maximum rank $\nu - 1$ if $n = 3$.*

The properties (i)-(ii) above and Sub-lemma 5.8 (applied to the subsystem of balls $1, \dots, i_1$) immediately ensure that a linear dependence of sort (5.7) can only hold on a set of codimension at least $\nu - 2$. The proof of the equation $A_{\{1,2\}}(n) = 0$ (modulo a codimension-$(\nu-2)$ set) is now proved in the case of $i_1 \geq 3$.

Let us now take up the similar problem in the case of $i_1 = 2$. Then one cannot use Sub-lemma 5.8, so it must be replaced by the following

Sub-lemma 5.9. *Let $T^{(c,d)}z_0$ be a non-singular and sufficient (that is, containing at least one collision) orbit segment of a standard billiard flow of 2 balls, where – as said before – we do not use the energy normalization. Then the only constraint interrelating the relative velocities $v_1(d) - v_1(c)$ and $v_1(c) - v_2(d) = v_1(c) + v_1(d)$ is that these two vectors must be orthogonal to each other.*

Proof. A simple inspection.

The consequence of this sub-lemma is that in the case of $i_1 = 2$ either both vectors $v_1^+(t_p) - v_N^+(t_p)$ and $v_N^-(t_r) - v_N^+(t_p)$ are perpendicular to the vector $v_1^+(t_p) -$

$v_2(t^*)$ or, if not so, then the validity of a linear dependence (5.7) implies that the relative velocity on the left-hand-side of (5.7) must belong to a line if the relative velocities on the right-hand-side of (5.7) are fixed. As a direct inspection shows, however, either possibility can only occur on a codimension-$(\nu-2)$ subset of the phase space. This shows that $A_{\{1,2\}}(n)=0$ (modulo a codimension-$(\nu-2)$ set) in the case of $i_1=2$.

The proof of Lemma 5.2 is now finished. $\square$

Finishing the proof of Main Lemma 3.7.

Let the symbolic collision sequence $\Sigma=(\Sigma_1,\dots,\Sigma_4)$ be composed of the four consecutive, connected subsequences Σ_j, $j=1,2,3,4$, and study the non-singular orbit segment $S^{[a,b]}x_0$ of the billiard flow with the restricted $\mathcal{G}_c$-interaction such that the symbolic collision sequence of $S^{[a,b]}x_0$ is Σ.

Due to the connectedness, the subsequence Σ_2 contains the edges of a full path (of length $N-1$) on the vertex set $\{1,2,\dots,N\}$. By re-labelling the particles we may assume that $\{i,i+1\}\in\Sigma_2$ for $i=1,\dots,N-1$.

Consider an integer k, $2\le k\le N-1$, and show that $A_{\{k-1,k\}}=A_{\{k,k+1\}}$. If either of the edges $\{k-1,k\}$, $\{k,k+1\}$ is in the intersection $\Sigma_1\cap\Sigma_3$, then $A_{\{k-1,k\}}=A_{\{k,k+1\}}$ by Lemma 5.2. Thus we can assume that $\{k-1,k\}\notin\Sigma_1\cap\Sigma_3$ and $\{k,k+1\}\notin\Sigma_1\cap\Sigma_3$. Thanks to the connectedness of Σ_1 and Σ_3, any of these two graphs contains one of the edges $\{k-1,k\}$ and $\{k,k+1\}$. Therefore, we may assume that $\{k-1,k\}\notin\Sigma_3$ and $\{k,k+1\}\notin\Sigma_1$. By analogous reasoning $\{k-1,k\}\notin\Sigma_4$.

Consider now the common edge set $\mathcal{E}_c\setminus\{\{k-1,k\}\}$ of Σ_3 and Σ_4. Since $\Sigma_1\cup\Sigma_2$ contains the entire edge set $\mathcal{E}_c$ of $\mathcal{G}_c$, again by Lemma 5.2 we have that the advances of any two adjacent edges of Σ_3 are equal. However, this implies the sufficiency, especially $A_{\{k-1,k\}}=A_{\{k,k+1\}}$, what we just wanted to show.

Since these arguments only break down at phase points outside a codimension-$(\nu-2)$ subset of the phase space, the proof of Main Lemma 3.7 is now complete. $\square$

6. Final Remarks

1. It is worth mentioning that the methods of [Sim(1992)] provide the K-property of N hard balls with an arbitrary graph of interactions if the dimension ν of the torus satisfies the inequality $\nu \geq N$. For the mechanical model of the *Rayleigh gas*, when one of the balls has radius $r > 0$ whereas the remaining $N-1$ ones are point particles, the tree of interactions is star-like, i. e. $\mathcal{E} = \{\{1, i\} : 2 \leq i \leq N\}$. In this case $D = N - 2$, and our result just requires the constraint $\nu \geq N$ for the K-property, thus we can not improve on Simányi's lower bound.

2. On the other hand, it is easy to see that the arguments used to prove Theorems C and D are also applicable to an arbitrary connnected graph of interaction. Consequently, these arguments combined with the results of [Sim(1992)] provide that, in general, a system of N balls with a connected graph of interactions has open ergodic components whenever $\nu \geq N - 1$.

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