Towards quantum localisation in Gaussian random potentials

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Abstract. At sufficiently low energies there is no absolutely continuous spectrum for a quantum-mechanical particle moving in multi-dimensional Euclidean space under the influence of certain Gaussian random potentials.

Since the pioneering work of Anderson [1] models of a single quantum-mechanical particle moving under the influence of a random potential have attracted much attention from both physicists [2,3] and mathematicians [4,5]. The central issue is to understand the spectral characteristics of the associated random Schrödinger operators in order to predict physically interesting quantities such as the (integrated) density of states or transport coefficients. Considerable progress has been achieved over the years in proving rigorously the existence of so-called localised states of one- and higher-dimensional lattice operators and of one-dimensional continuum operators. Localisation properties of random Schrödinger operators in d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$, have been derived, however, almost exclusively for random potentials with an underlying lattice structure, see [6] for the early work and [7,8] for recent results and further references. To our knowledge, the only works devoted to the localisation problem with a truly continuum random potential deal with a Poisson potential where an additional randomness in the single-impurity coupling constant had to be assumed [9,7]. Our goal here is to report on a localisation result for another type of a truly continuum random potential. More precisely, we will outline a proof for the absence of the absolutely continuous component in the low-energy spectrum of Schrödinger operators with certain Gaussian random potentials.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $(\omega, x) \mapsto V_\omega(x)$ be a real-valued homogeneous Gaussian random field indexed by $\mathbb{R}^d$ with zero mean, $\int \mathrm{d}P(\omega)\, V_\omega(x) = 0$, and covariance function $C(x) :=$

$\int \mathrm{d}P(\omega)\, V_\omega(x) V_\omega(0)$ obeying, of course, $0 < C(0) < \infty$. In the following we will need three assumptions on C. For the formulation of the third one we introduce some notation. Given a Borel subset $\Lambda \subset \mathbb{R}^d$, we denote by $|\Lambda| := \int_\Lambda \mathrm{d}^d x$ its d-dimensional Lebesgue measure and by $\mathcal{F}_\Lambda \subset \mathcal{F}$ the sub-sigma-algebra of events generated by the set of random variables $\{V_{\cdot}(x) : x \in \Lambda\}$. For two Borel subsets $\Lambda, \Lambda' \subset \mathbb{R}^d$ let $\mathrm{dist}(\Lambda, \Lambda') := \inf\{|x - x'|_\infty : x \in \Lambda, x' \in \Lambda'\}$ be their distance measured in the maximum norm $|x|_\infty := \max\{|x_j| : 1 \le j \le d\}$ of $x =: (x_1, \ldots, x_d) \in \mathbb{R}^d$. Next we introduce the quantity $\kappa(\Lambda, \Lambda') := \sup\{|P(A \cap A') - P(A)P(A')| : A \in \mathcal{F}_\Lambda, A' \in \mathcal{F}_{\Lambda'}\} \le 1/4$ which measures the stochastic dependence of the restrictions of the random field V to Λ and Λ', respectively. Finally, for each pair of reals $L, G > 0$ the so-called strong-mixing coefficient [10] of V is defined as follows

$$\alpha(L, G) := \sup\left\{\kappa(\Lambda, \Lambda') : \Lambda, \Lambda' \subset \mathbb{R}^d;\ \mathrm{dist}(\Lambda, \Lambda') \ge \lambda L;\ |\Lambda|, |\Lambda'| \le \lambda^d G\right\}.$$

Here $\lambda > 0$ denotes some arbitrary but fixed length scale, which serves to make L and G dimensionless. Now we are in a position to formulate the above-mentioned

Assumptions on the covariance.

(A1) $C(x) \ge 0$ for all $x \in \mathbb{R}^d$.

(A2) There exists a pair of constants $b, \beta > 0$ such that

$$C(0) - C(x) \le b\, |x|_\infty^\beta$$

for all x in a neighbourhood of the origin.

(A3) There exists a triple of constants $a, \gamma, \delta > 0$ obeying $\gamma < 1 + \delta/[\delta + 4(d-1)]$ such that

$$\alpha(l^\gamma/4, l^d) \le a l^{-\delta} \qquad \text{for all } l > 1.$$

Remarks.

i) The Hölder condition at the origin (A2) implies [11, Thm 4.1.1] the continuity of the realisations of V almost surely with respect to the probability measure P.

ii) The algebraic bound (A3) implies the strong-mixing property [10] of V. More precisely, $\lim_{L\to\infty} L^\mu\, \alpha(L, G) = 0$ for all $G > 0$ and $\mu < \delta/\gamma$. The strong-mixing property is considerably stronger than the usually required ergodicity. It seems to be an open problem to characterise the covariance functions of strongly mixing Gaussian random fields indexed by $\mathbb{R}^d$, $d \ge 2$.

iii) Taken together, (A2) and (A3) require a compromise between local dependence and global independence of V.

iv) In one dimension an example fulfilling (A1) – (A3) is $C(x) = C(0)\exp\{-|x|/\xi\}$, $\xi > 0$. While the first two assumptions are obviously satisfied, the third one follows from the general theory developed in [12, ch VI, Thm 6]. A simple class of allowed covariance functions for all $d \geq 1$ may be constructed as follows. Pick a function $U : \mathbb{R}^d \to \mathbb{R}_+$ which is non-negative, compactly supported and Hölder continuous of order $\beta > 0$. Then define $C(x) := \int \mathrm{d}^d x'\, U(x+x')U(x')$. Again, it is easy to see that the first two assumptions are satisfied. The third one follows from the fact that C has compact support.

Given a Gaussian random field V of the above type, the associated random Schrödinger operator – acting in the Hilbert space $L^2(\mathbb{R}^d)$ of complex-valued functions on $\mathbb{R}^d$ which are square-integrable with respect to $\mathrm{d}^d x$ – is then given as

$$H_\omega := -\frac{1}{2}\Delta + V_\omega\,. \tag{1}$$

Here $\Delta := \sum_{j=1}^d \partial^2/\partial x_j^2$ is the d-dimensional Laplacian and V appears as an additive random potential which acts as a multiplication operator. Note that we have chosen physical units, where the square of Planck's constant equals the mass of the point particle whose dynamics is described by (1). According to standard arguments [13,4,5], the Schrödinger operator (1) enjoys the following properties:

i) It is almost surely essentially self-adjoint on the dense subspace $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, consisting of arbitrarily often differentiable complex-valued functions with compact support in $\mathbb{R}^d$.

ii) Its spectrum $\sigma(H_\omega)$ equals almost surely the real line $\mathbb{R}$. The spectral components, the absolutely continuous, the singular continuous and the pure point spectrum, arising in the Lebesgue decomposition $\sigma(H_\omega) = \sigma_{\mathrm{ac}}(H_\omega) \cup \sigma_{\mathrm{sc}}(H_\omega) \cup \sigma_{\mathrm{pp}}(H_\omega)$ are also almost surely non-random sets.

iii) Its integrated density of states exists almost surely as a non-random distribution function with a Gaussian low-energy tail. The topological support of the corresponding measure is the entire real line $\mathbb{R}$.

Now we are prepared to state the main result.

Theorem.
For the multi-dimensional random Schrödinger operator H_ω given by (1) with a Gaussian random potential satisfying assumptions (A1) – (A3), there exists an energy $E_0 < 0$ such that

$$\sigma_{\mathrm{ac}}(H_\omega) \cap \,]-\infty, E_0] = \emptyset \qquad \text{almost surely.}$$

Outline of proof.
The heart of the proof is a multi-scale analysis in the spirit of [14]. Its technical realisation is patterned after [15] and [7] in order to cope with a correlated random potential and a continuous space, respectively. Here, we will illustrate only the key points where special properties of Gaussian random fields require an adaption of the usual strategy and how assumptions (A1) – (A3) enter.

i) Assumption (A2) enables us to control the unbounded fluctuations of the Gaussian random potential restricted to a box $\Lambda_l(x) := \{x' \in \mathbb{R}^d : |x' - x|_\infty \leq \lambda l/2\}$ centred at $x \in \mathbb{R}^d$ with edges of length $\lambda l > 0$. For sufficiently large l one has [11, Thm 4.1.1]

$$P\left(\left\{\omega \in \Omega : \sup_{x \in \Lambda_l(0)} |V_\omega(x)| \geq v\right\}\right) \leq 4^{d+1} \exp\left\{-\frac{v^2}{2C(0)\left(1 + 20\sqrt{\ln l}\right)^2}\right\} \tag{2}$$

for all $v \geq 0$. This Lévy-type of inequality is needed in the base clause of the multi-scale analysis to estimate the probability that the norm of the resolvent of the operator H_ω restricted to a box of initial length scale λl_0 exceeds some given value. It is needed again in the recursion clause to bound a similar probability involving the commutator term which arises from the geometric resolvent equation.

ii) The rôle of the important weak-dependence assumption (2.2) in [15] is taken over by our assumption (A3) together with the estimate

$$P(A \cap A') \leq P(A)P(A') + \alpha\left(\mathrm{dist}(\Lambda, \Lambda'), \max\{|\Lambda|, |\Lambda'|\}\right) \tag{3}$$

for the probability of the joint occurrence of two events $A \in \mathcal{F}_\Lambda$ and $A' \in \mathcal{F}_{\Lambda'}$.

iii) The third ingredient to make the multi-scale analysis work is a Wegner-type of estimate. For a given triple of constants $E_w \in \mathbb{R}$, $\varepsilon_w > 0$ and $l_w > 0$ there exists a constant $0 < W < \infty$ such that

$$P\left(\left\{\omega \in \Omega : \mathrm{dist}\left(\sigma(H_{\omega,\Lambda_l(0)}), \{E\}\right) \leq \varepsilon\right\}\right) \leq \varepsilon l^{2d} W \tag{4}$$

for all $E \leq E_w$, $0 \leq \varepsilon \leq \varepsilon_w$ and $l \geq l_w$. Here $H_{\omega,\Lambda}$ denotes the Dirichlet restriction of H_ω to a box $\Lambda \subset \mathbb{R}^d$. The first steps in our derivation of (4) are borrowed from Wegner's original work for lattice models with independent random potentials [16], see also, for example, [6]. Due to (A1) the occurring average of the partial functional derivative $\delta/\delta V_\omega(x)$ of the (mollified) integrated density of states of $H_{\omega,\Lambda_l(0)}$ may be bounded from above in such a way that a functional integration by parts with respect to the Gaussian probability measure [17, Thm 6.3.1] can be performed. A final appeal to the existence and non-randomness of the integrated density of states then leads to (4).

Having adapted the multi-scale analysis, we follow the proof of [7, Thm 2.1] to bound the infinite-volume resolvent locally in the usual operator norm $||\cdot||$. More precisely, we get that there exists a constant $E_0 < 0$ such that

$$\sup_{\varepsilon>0} ||(H_\omega - E - \mathrm{i}\varepsilon)^{-1} 1\!\!1_\Lambda|| < \infty \tag{5}$$

for all compact $\Lambda \subset \mathbb{R}^d$ and $\mathrm{d}E \otimes \mathrm{d}P(\omega)$-almost all $(E, \omega) \in]-\infty, E_0] \times \Omega$. The outline of the proof of the Theorem is now completed by referring to the de La Vallée-Poussin theorem, see, for example, [4, pp 16,17] and [5, Thm A.10].

Remarks.

i) Unfortunately, we have not yet been able to rule out the singular continuous spectrum for sufficiently low energies, even though we can show an algebraic decay in $\mathrm{dist}(\Lambda, \Lambda')$ of $\sup_{\varepsilon>0} ||1\!\!1_{\Lambda'}(H_\omega - E - \mathrm{i}\varepsilon)^{-1} 1\!\!1_\Lambda||$ for $\mathrm{d}E \otimes \mathrm{d}P(\omega)$-almost all $(E, \omega) \in]-\infty, E_0] \times \Omega$.

ii) The above Theorem remains true for the case where H_ω is generalised to include a periodic potential and/or a constant magnetic field. For more stringent localisation results in these cases with, however, a random potential having an underlying lattice structure, see [8,18–20]. The Theorem should also remain true in the presence of a homogeneous and ergodic random vector potential which is sufficiently smooth. The almost-sure constancy of the spectral components, the existence and certain asymptotic properties of the integrated density of states in this case have recently been proven in [21,22].

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