Ergodicity in Infinite Hamiltonian Systems with Conservative Noise

Carlangelo Liverani[1]

Stefano Olla[2]

Abstract

We study the stationary measures of an infinite Hamiltonian system of interacting particles in $\mathbb{R}^3$ subject to a stochastic local perturbation conserving energy and momentum. We prove that the translation invariant measures that are stationary for the deterministic Hamiltonian dynamics, reversible for the stochastic dynamics, and with finite entropy density are convex combination of "Gibbs" states. This result implies hydrodynamic behavior for the systems under consideration.

INTRODUCTION

The ergodicity problem in Hamiltonian dynamical systems is at the base of equilibrium statistical mechanics. While, beginning with the celebrated Sinai's paper [Si], some result are known for finite system (see [LW] for a general approach to ergodicity in Hamiltonian systems), very little is known concerning infinite systems (some results are known for special systems with an arbitrary, but finite, number of particles [BLPS]). By ergodicity of an infinite system we mean that convex combinations of Gibbs measures are the only invariant measures, within a reasonably "regular" class.

Furthermore, recent developments in non-equilibrium hydrodynamics (cf. [OVY]) show that the ergodicity of an infinite systems is a main ingredient in the rigorous derivation of Euler equations as a macroscopic description of the conservation laws for the density, the momentum and the energy (at least in the smooth regime of these equations).

Since no results are present for deterministic systems, it is natural to ask if a stochastic

[1] II Università di Roma "Tor Vergata", Dipartimento di Matematica, 00133 Roma, Italy. E-mail: liverani@ccd.utovrm.it .

[2] Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France and Politecnico di Torino, Dipartimento di Matematica, corso Duca degli Abruzzi 24, 10129 Torino, Italy. E–mail: olla@paris.polytechnique.fr .

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perturbation may help in proving ergodicity. The stochastic perturbation should conserve the energy, the momentum and the number of particles of the system, while destroying *locally* the other possible invariant of the motion.

A stochastic perturbation of this type is introduced in [OVY]: any two particles exchange randomly momentum in such a way as to preserve only the total momentum and energy of the two particles. The rate of exchange is assumed to decrease when the distance between the two particles increases, but the range is infinite. Accordingly, any particle is interacting stochastically with any other and the corresponding diffusion on the momenta space of any finite number of particles is elliptic. This permits to characterize the distribution of the momenta of any finite number of particles conditioned to the positions: it must be a uniform measure on the corresponding invariant manifold in the momenta space. The equivalence of ensembles implies that the distribution of the momenta conditioned on the position is a convex combination of "Maxwellians". In addition, one can localize the invariance, under the Hamiltonian dynamics, of the distribution, and prove that the distribution of the positions satisfies the DLR equations with respect to the corresponding interaction.

The purpose of the present paper is to extend the foregoing argument to *finite* range stochastic interactions. Two difficulties arise immediately: one of local and the other of global type.

Locally, restricting oneself to a finite "chain" (or cluster) of particles interacting stochastically, the diffusion on the space of momenta is no longer elliptic; it becomes then necessary to prove that it is, at least, hypoelliptic. This is done quite easily with an inductive argument and in grand generality: only the convexity of the kinetic energy is needed.

The global obstacle is of a more serious nature. The diffusion on the momenta is hypoelliptic only when restricted to chains of interacting particles. But, several clusters of particles, too far apart to interact stochastically, may be present; hence, they could be at "different temperatures". We need the help of the deterministic Hamiltonian dynamics to "connect" distant clusters.

Taking commutators between the vector fields generating the stochastic dynamics and the Hamiltonian generator one obtains a Lie algebra of vector fields large enough to generate all the tangent space to the energy-momentum manifold on the phase space (i.e., position and momentum). This means that our system is invariant for the dynamics generated by these vector fields (that turn out to be local) which enable, after some work, to produce "cluster deformations" that connect any cluster with the others. Proceeding in such a way we can obtain, in each sufficiently large finite box, a "unique cluster" and consequently prove that the momenta are uniformly distributed.

A further difficulty arises if the kinetic energy is quadratic (i.e., the usual "Gaussian case"). In fact, in this case all the above mentioned dynamics preserve also the center of mass of any finite cluster of particles. To complete the argument in this case it would be necessary to perform cluster deformations that conserve the center of mass, hence substantially complicating the above argument. We belive that our program could be carried out for the Gaussian case as well but we stop short of it also in view of the fact that its application to hydrodynamics is unclear (see point (d) in the following discussion). As in [OVY] we consider only stationary measures having finite entropy density with respect to a grancanonical Gibbs measure. This condition seems to characterize a nice class of regular measures. To complete our argument, various extra assumptions are necessary:

(a) The range of the stochastic interaction is finite but must be strictly larger than the one of the deterministic potential.

(b) The invariant measures considered must have sufficiently high particles density. More precisely, we need to be sure that, for almost any configuration, any sufficiently large box contains at least two particles interacting stochastically. The bound on the density we ask here is very rough, and we believe it can be substantially improved by using a more refined argument. Alternatively one can assume that the average potential energy is positive, which will imply that in a box large enough at last two particle interact deterministically, though stochastically. Unfortunately potential energy is not a conserved quantity, so usually one does not have any information about its average value, that is why we prefer a condition on the density, which is stricter but easier to use.

(c) We assume that the measures considered are separately invariant for the deterministic and the stochastic dynamics. Furthermore, they must be reversible for the stochastic dynamics alone. The reversibility with respect to the global stochastic dynamics is a more general condition than the ones needed to derive the hydrodynamic limit: the invariance for each local stochastic dynamics (cf. proposition 2.2) would suffice (cf. [OVY]).

(d) In order to apply our results to obtain hydrodynamic limits following [OVY] we consider kinetic energies that are not quadratic, since [OVY] does not apply to the quadratic case. Nonetheless, we must assume a mild restriction on the kinetic energy function: the local dynamics cannot have undesired invariant (like the center of mass in the Gaussian case, see lemma 2.5). We provide examples of kinetic energy functions that satisfy both our condition and the ones assumed in [OVY] (cf. Appendix 1).

As a consequence of our result, theorem 2.1 of [OVY] is valid for Hamiltonian dynamics with stochastic perturbation of the non-Gaussian type considered in the present paper.

For lattice systems the problem of ergodicity is solved in [FFL] in a more satisfactory way. In fact, there it is not needed condition (c), i.e., only the invariance with respect to the total dynamics (deterministic + stochastic) is required.

Concerning condition (c), notice that we could have asked the invariance for the finite stochastic dynamic in each finite box. We prove indeed that this is equivalent to the global reversibility (cf. proposition 2.2). Proposition 2.2 has an interest in itself: it says that if a stochastic dynamics on a lattice in finite dimension is hypoelliptic then for the corresponding infinite dynamics all the reversible measures are given by Gibbs measures. This generalize a result of M. Zhu (cf.[Z]) to the “hypoelliptic” situation.

The next section contains a more precise description of the results outlined here, together with the plan of the paper.

1. NOTATIONS AND RESULTS

Sample space

A point of $I\!R^3 \times I\!R^3$ will be denoted by (q, p) and the sample space Ω will consist of points $\omega = \{(q_\alpha, p_\alpha)\}$. Any bounded region B in $I\!R^3$ will contain only a finite number of particles, with positions q_α, in addition one can think of p_α as tags, and consider the corresponding finite configuration in $B \times I\!R^3$.

Interaction

We consider a radial repelling finite range smooth pair potential $V(q_\alpha - q_\beta)$ such that:

i) $V(x) = 0 \quad |x| > R_0 \quad$ (finite range)

ii) V is superstable (i.e. it satisfies the superstability inequality: *there exists* $B > 0$ *and* $A > 0$ *such that for any finite box* Λ *and any configuration we have*:

$$\sum_{q_a \in \Lambda} \sum_{\beta} V(q_\alpha - q_\beta) \;\geq \frac{A}{|\Lambda|}|\omega_\Lambda|^2 - B|\omega_\Lambda| \tag{1.1}$$

(see [R])).

iii) $\langle x, \nabla V(x)\rangle \;\equiv \sum_i^3 x_i \frac{\partial V}{\partial x_i}(x) \leq 0 \quad \forall x \in I\!R^3 \quad$ (repelling interaction)

The repelling condition (iii) is of a technical nature and it should be possible to remove it by a more accurate analysis.

Kinetic Energy

It is given by a strictly convex function $\phi(p) \in C^\infty(I\!R^3)$. We consider two cases:

(G) $\phi(p)$ is a quadratic function of p, which is the classical *Gaussian* case.

(NG) $\phi(p) = \sum_{i=1}^3 \varphi(p^i)$ with φ a strictly convex smooth positive function on $I\!R$ with

$$\frac{1}{2}\frac{d^2}{dx^2}(\varphi''(x))^2 = \varphi'''(x)^2 + \varphi^{iv}(x)\varphi''(x) \neq 0 \tag{1.2}$$

apart from, at most, finitely many points. In addition, we require the invariance for reflections, i.e. $\varphi(x) = \varphi(-x)$, and that φ is not too flat near the origin; more precisely, we assume that there exist m such that $\varphi^{(m)}(0) \neq 0$.[1] We will refer to this case as the *non-Gaussian* case.

Notice that, if $\frac{d^2}{dx^2}(\varphi''(x))^2 = 0$ for each x the condition $\varphi(x) = \varphi(-x)$ implies $\varphi''(x) =$constant, i.e., we have the Gaussian case. This shows that, morally, our conditions cover all the possible cases; yet, it could be interesting to carry out a more detailed investigation.

As already mentioned, our main motivation to treat the case (NG) is to apply the present results to the derivation of the hydrodynamic limit. To do so, the kinetic energy function must satisfy the conditions:

$$\left|\frac{\partial \phi}{\partial p^j}\right| \leq C', \quad \left|\frac{\partial^2 \phi}{\partial p^j \partial p^i}\right| \leq C'' \qquad \forall p \in I\!R^3 \tag{1.3}$$

which are clearly not satisfied by the classical case (G).

Hamiltonian Dynamics

[1] This last condition will be needed only in the proof of Lemma 3.2

The Hamiltonian is defined by the formal expression:

$$\mathcal{H}(\omega) = \sum_{\alpha} \phi(p_\alpha) + \frac{1}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} V(q_\alpha - q_\beta)$$

and the Liouville operator by

$$L = \sum_{\alpha} \sum_{i=1}^{3} \left[\partial_{p^i_\alpha} \mathcal{H}\ \partial_{q^i_\alpha} - \partial_{q^i_\alpha} \mathcal{H}\ \partial_{p^i_\alpha} \right].$$

In this paper, we are not concerned with the existence of the dynamics generated by L or its stochastic perturbations. Our aim is simply to characterize the probability measures on Ω that are 'formally' invariant (see Th. 1.1). For a more detailed description of the above objects see [$AGGLM$, §2].

Stochastic perturbation of the dynamics

We will use the notation $v^i_\alpha \equiv \phi_i(p_\alpha) \equiv \partial_{p^i_\alpha} \phi$. In the following smooth will mean always differentiable infinitely many times.

For each smooth function $\eta_{\alpha\beta} : \mathbb{R}^6 \to \mathbb{R}^3$, (i.e. $\eta_{\alpha\beta} = \eta_{\alpha\beta}(p_\alpha, p_\beta)$) we define the vector field

$$X(\eta_{\alpha\beta}) = \langle \eta_{\alpha\beta},\, D_{\alpha\beta} \rangle \equiv \sum_{i=1}^{3} \eta^i_{\alpha\beta} D^i_{\alpha\beta}$$

where $D_{\alpha\beta} = \partial_{p_\alpha} - \partial_{p_\beta}$.

We are interested in vector fields with null divergence, i.e.,

$$\operatorname{div}(X(\eta_{\alpha\beta})) = \langle\, D_{\alpha\beta}, \eta_{\alpha\beta} \rangle = \sum_{i=1}^{3} D^i_{\alpha\beta} \eta^i_{\alpha\beta} = 0. \tag{1.4}$$

Furthermore, we ask that $X(\eta_{\alpha\beta})$ is tangent to the surfaces, $\mathbb{R}^3 \times \mathbb{R}^3$,

$$\begin{cases} p^i_\alpha + p^i_\beta = c^i & i = 1, 2, 3 \\ \phi(p_\alpha) + \phi(p_\beta) = c^0, \end{cases}$$

that is, the orthogonality relation

$$\langle \eta_{\alpha\beta},\, D_{\alpha\beta}(\phi(p_\alpha) + \phi(p_\beta)) \rangle = 0 \tag{1.5}$$

(equivalently, $\langle \eta_{\alpha\beta},\, v_\alpha \rangle = \langle \eta_{\alpha\beta},\, v_\beta \rangle$), which will imply the conservation of energy and momenta with respect to the stochastic dynamics.

Let $X(\eta_{\alpha\beta})^*$ be the adjoint of $X(\eta_{\alpha\beta})$ with respect to the measures

$$e^{\lambda_4(-\phi(p_\alpha) - \phi(p_\beta)) + \lambda \cdot (p_\alpha + p_\beta)}\ dp_\alpha\ dp_\beta$$

for any $\lambda_4 > 0$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, with the restriction that

$$\int \exp(-\lambda_4 \phi(p) + \lambda \cdot p) dp < +\infty \ .$$

We have, because the null divergence and the orthogonality property, that $X(\eta_{\alpha\beta})^* = -X(\eta_{\alpha\beta})$.

We use the previous vector fields to define an operator of the second order that will be the generator of the stochastic perturbation. Consider a finite number $K \geq 3$ of vectors $\{\eta^\theta_{\alpha\beta}\}$ with the properties above. We define the operator

$$\hat{L}_{\alpha\beta} = -\frac{1}{2} \sum_{\theta=1}^{K} X(\eta^\theta_{\alpha\beta})^* X(\eta^\theta_{\alpha\beta}) \ = \ \frac{1}{2} \sum_{\theta} X(\eta^\theta_{\alpha\beta})^2.$$

Moreover, we require that, at each point, the linear combination of $\{\eta^\theta_{\alpha\beta}\}$ spans a two dimensional subspace of $\mathbb{R}^3$ (the maximum compatible with (1.5), eventually apart from a set $\widetilde{\Sigma}^s_{\alpha\beta}$ consisting of the finite union of codimension–two manifolds. Therefore, $\hat{L}_{\alpha\beta}$ is selfadjoint, and elliptic outside $\widetilde{\Sigma}^s_{\alpha\beta}$. For later purposes, we define

$$\widetilde{\Sigma}_{\alpha\beta} = \widetilde{\Sigma}^s_{\alpha\beta} \cup \{(p_\alpha, p_\beta) \mid v_\alpha = v_\beta\}, \tag{1.6}$$

by convexity follows that $\widetilde{\Sigma}_{\alpha\beta}$ is the finite union of smooth manifold with codimension two as well.

Let $\sigma(q)$ be a radial smooth function on $\mathbb{R}^3$, such that $\sigma(q) > 0$ for each $\|q\| < R_1$, and $\sigma(q) = 0$ for each $\|q\| \geq R_1 > 4R_0$. Then we consider the operator

$$\hat{L} = \sum_{\alpha,\beta} \sigma(q_\alpha - q_\beta) \hat{L}_{\alpha\beta}.$$

In the following considerations it will be important that σ is strictly positive for a radius R_1 strictly greater than $4R_0$, i.e. that the range of the stochastic interaction is larger than the one of the 'deterministic' interaction.[2]

Gibbs Measures

Let $\Lambda \subset \mathbb{R}^3$. Each configuration $\omega \in \Omega$ can be written as $\omega = \{\omega_\Lambda, \omega_{\Lambda^c}\}$ where $\omega_\Lambda = \{(q_\alpha, p_\alpha) \in \omega \mid q_\alpha \in \Lambda\}$.

Let $\mathbb{P}$ be a probability measure on Ω. If the $\mathbb{P}$-conditional distribution of ω_Λ given the configuration outside ω_{Λ^c} is proportional to

$$\frac{1}{n!} \exp \left[\lambda_0 n + \sum_{\alpha=1}^{n} \sum_{i=1}^{3} \lambda_i p^i_\alpha - \lambda_4 \mathcal{H}_{\Lambda,n}(\omega_\Lambda, \omega_{\Lambda^c}) \right]$$

[2] The factor 4 is due to technical reasons and plays a role only in section 4.

then $I\!P$ is called Gibbs Measure (or grandcanonical Gibbs measure). In the above expression n is the number of particles in Λ (that we will denote by $|\omega_\Lambda|$) and the local Hamiltonian is defined by

$$\mathcal{H}_{\Lambda,n}(\omega_\Lambda,\omega_{\Lambda^c}) = \sum_{q_\alpha\in\omega_\Lambda}\left[\phi(p_\alpha)+\frac{1}{2}\sum_{q_\beta\in\omega_\Lambda;\ \alpha\neq\beta}V(q_\alpha-q_\beta)+\sum_{q_\beta\in\omega_{\Lambda^c}}V(q_\alpha-q_\beta)\right].$$

Statement of the result

Let Q and P two probability measures on Ω, and let Q_Λ and P_Λ their restriction on a finite box Λ. The relative entropy of Q_Λ with respect to P_Λ is defined by

$$H_\Lambda(Q|P) \;=\; \sup_{F\in\mathcal{F}_\Lambda}\{Q(F)-\log P(\exp(F))\} \tag{1.7}$$

where $\mathcal{F}_\Lambda$ are the smooth functions localized in Λ. For the properties of H_Λ see, for example, [OVY]. In the following Q will be the translation invariant measure under consideration, while $I\!P$ will be any grancanonical Gibbs measure for the interaction V.

Lemma 1.1 . *If there exists a constant C such that for each box Λ,*

$$H_\Lambda(Q|I\!P)\le C|\Lambda|$$

then,

(i) $$Q\left(|\Lambda|^{-2}|\omega_\Lambda|^2\right)\le C_1<\infty$$

(ii) $$Q\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\|p_a\|\right)\le C_2<\infty$$

(iii) $$Q\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\left[\phi(p_\alpha)+\sum_{q_\beta\in\omega}V((q_\alpha-q_\beta)\right]\right)\le C_3<\infty$$

where C_1, C_2, C_3 are constants independent on Λ.

Proof.

The inequalities (ii) and (iii) are consequences of the following entropy inequality:

$$Q(F)\le\frac{1}{\beta}\log I\!P\ (\exp(\beta F))\ +\ \frac{1}{\beta}H(Q|I\!P) \tag{1.8}$$

which is valid for any local function F and any constant $\beta>0$. It follows directly from the definition (1.7). Then for sufficiently small β, we have:

$$Q\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\|p_a\|\right)\le\frac{1}{\beta|\Lambda|}\log P\left(\exp\left(\beta\sum_{q_\alpha\in\Lambda}\|p_a\|\right)\right)\ +\ \frac{1}{\beta}C\le\frac{1}{\beta}C'$$

In a similar way one can prove (iii). While (i) follows by the same argument and the superstability inequality (1.1). □

Define

$$\rho(\omega) \;=\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}|\omega_\Lambda|$$

$$\pi(\omega) \;=\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}\Pi_\Lambda \;\equiv\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \sum_{q_\alpha\in\Lambda} p_a$$

$$e(\omega) \;=\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}E_\Lambda \;\equiv\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \sum_{q_\alpha\in\Lambda} \Bigg[\phi(p_\alpha) + \frac{1}{2} \sum_{q_\beta\in\Lambda;\; \alpha\neq\beta} V(q_\alpha - q_\beta) + \sum_{q_\beta\notin\Lambda} V(q_\alpha - q_\beta)\Bigg]$$

The above Lemma 1.1, and the translation invariance of Q, insures that the limits $\rho(\omega)$, $e(\omega)$, $\pi(\omega)$ exist Q-almost everywhere.

The aim of this paper is to prove the following:

Theorem 1.2 . *Let Q be a translation invariant probability measure on Ω, if*

(i) There exists a constant C such that for each box Λ, $H_\Lambda(Q|\mathbb{P}) \leq C|\Lambda|$;

(ii) $Q\left(\{\omega\in\Omega \mid \rho(\omega) > \rho_*\}\right) \;=\; 1$ where $\rho_* = \frac{3}{4R_1^3\pi}$;

(iii) Q is invariant w.r.t. the dynamics generated by L (the deterministic part), in the sense that, for any smooth local function $F_\Lambda(\omega_\Lambda)$,[3]

$$\mathbb{E}^{Q}(LF_\Lambda) \;=\; 0\;;$$

(iv) Q is reversible with respect to $\hat{L}$ (the stochastic perturbation), i.e., for any two smooth local functions φ and ψ holds

$$\mathbb{E}^{Q}(\psi\hat{L}\varphi) \;=\; \mathbb{E}^{Q}(\varphi\hat{L}\psi)\;;$$

then Q is a convex combination of (gran canonical) Gibbs Measures.

Remark (1.3).

Condition (ii) on the density is a sufficient condition in order to always find at least two particle interacting stochastically. Since the range of the deterministic interaction is smaller than the one of the stochastic interaction, this condition may be replaced by ensuring that the average potential energy is strictly positive, i. e. if we define

$$u(\omega) \;=\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}U_\Lambda \;\equiv\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \sum_{q_\alpha, q_\beta\in\Lambda} V(q_\alpha - q_\beta)$$

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[3] *By $\mathbb{E}^{Q}$ we mean the expectation with respect to the measure Q.*

then the condition reads

$$Q(u(\omega) > 0) \;=\; 1$$

This condition is not practical because $u(\omega)$ does not correspond to a conserved quantity. It will be of no use for the application to hydrodynamics (cf.[OVY]), where we cannot have such information on Q.

The proof of Theorem 1.1 will be carried out in three parts. In the next section we will construct a multitude of local dynamics that leave the finite dimensional restrictions of the measure Q invariant. Section three is dedicated to the characterization of typical configurations for the class of measures Q under consideration. In section four we show that the above mentioned dynamics give a local characterization weaker than the one implied by DLR equations, but sufficient to claim that the global distribution of the momenta, conditioned to the positions, is given by a convex combination of "Maxwellian" (corresponding to the proper ϕ). We conclude the argument in section five, along the line of [OVY], by proving that in the infinite limit the kinetic energy is "invariant" for the deterministic dynamic generated by L. Thus, each component of the convex combination is invariant for L. A classic argument (cf.[GV] and [OV]) shows that invariant distribution for L that have distribution of the momenta conditioned to the position given by a Maxwellian are canonical Gibbs measures.

2. CLUSTERS AND LOCAL DYNAMICS

Given a configuration ω, we call "connected" two particles that are sufficiently close to interact stochastically (α and β are connected if $\sigma(q_\alpha - q_\beta) > 0$, i.e. $|q_\alpha - q_\beta| < R_1$). We call "cluster" a set of particles such that any two can be joined by a chain of connected ones. We call "isolated cluster" a cluster such that no other particle is connected to any particle in the cluster.

Any configuration ω can be grouped in many isolated clusters $\{\Gamma_i(\omega_q)\}$. These clusters may be finite or infinite. Also, the restriction of any configuration to a finite region ω_Λ is grouped into finite clusters $\{\Gamma_i^\Lambda(\omega_q)\}$, where we have overlooked the connections with the particles outside Λ.

To simplify notations, in this section we denote by $\widetilde{I\!\!E}$ the expectation of Q conditioned to a configuration of positions ω_q.

Consider a cluster Γ_i^Λ, and let n be the number of particles in it. Then $\hat{L}_{\Gamma_i^\Lambda} = \sum_{\alpha\beta\in\Gamma_i^\Lambda} \sigma(q_\alpha - q_\beta)\hat{L}_{\alpha\beta}$ is an operator on $I\!\!R^{3n}$ and it conserves the quantities

$$\sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^1, \qquad \sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^2, \qquad \sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^3, \qquad \sum_{q_\alpha\in\Gamma_i^\Lambda} \phi(p_\alpha).$$

Let us consider the corresponding connected hypersurfaces of dimension $3n-4$:

$$\Sigma_c \equiv \left\{ (p_1,\ldots,p_n) \;\middle|\; \sum_\alpha p_\alpha^1 = c^1, \sum_\alpha p_\alpha^2 = c^2, \sum_\alpha p_\alpha^3 = c^3, T = \sum_\alpha \phi(p_\alpha) = c^4 \right\}, \quad (2.1)$$

and the sets

$$\widetilde{\Sigma}_c \equiv \Big\{ (p_1, \ldots, p_n) \;\Big|\; (p_\alpha, p_\beta) \in \widetilde{\Sigma}_{\alpha\beta} \;\text{ for some } \alpha,\, \beta \in \{1, \ldots, n\} \Big\}$$

where $\widetilde{\Sigma}_{\alpha\beta}$ has been defined by (1.6). Note that $\widetilde{\Sigma}_c$ is the finite union of smooth submanifolds in Σ_c of codimension, at least, two; hence $\Sigma_c \setminus \widetilde{\Sigma}_c$ is connected.

Lemma 2.1. *$L_{\Gamma_i^\Lambda}$ is an hypoelliptic operator on $\Sigma_c \backslash \widetilde{\Sigma}_c$, i.e., the Lie algebra generated by*

$$\left\{ X^\theta_{\alpha\beta} \mid \alpha, \beta \in \Gamma_i \right\}$$

spans the tangent space of Σ_c at each point in $\Sigma_c \setminus \widetilde{\Sigma}_c$.

Proof.

Let us fix our attention on an arbitrary point in $\Sigma_c \backslash \widetilde{\Sigma}_c$.

We can represent a cluster of n particles by a graph $\widetilde{\mathcal{G}}_n$ with n vertices. Each vertex of the graph corresponds to a particle; two vertices are joined by a bond (edge) if and only if they correspond to particles close enough to interact stochastically. In a cluster of n particles there are at least $n-1$ bonds:

$$\sharp\{(\alpha, \beta) \mid \sigma(q_\alpha - q_\beta) < R_1\} \geq \; n - 1.$$

We consider then a minimal acyclical (i.e., the edges do not form any loop) connected sub-graph $\mathcal{G}_n$ of $\widetilde{\mathcal{G}}_n$. This amounts to choose $n-1$ bonds. We choose a set of $2n-2$ linearly independent vector fields $X^{\theta_i}_{\alpha\beta}$, $i \in \{1, 2\}$, $(\alpha\beta)$ being a bond in $\mathcal{G}_n$.

Two bonds are said to be contiguous if they have one vertex in common. There are at least $n-2$ couples of contiguous bonds. If we compute the commutator between two contiguous bonds $\{\alpha\beta\}$ and $\{\gamma\beta\}$ we obtain:

$$[X^{\theta_i}_{\alpha\beta}, X^{\theta_j}_{\gamma\beta}] \; = \langle \eta^{\theta_i}_{\alpha\beta},\, A_{\theta_j} D_{\gamma\beta} \rangle - \langle \eta^{\theta_j}_{\gamma\beta},\, B_{\theta_i} D_{\alpha\beta} \rangle$$

where A_{θ_j} is the matrix whose element kl is given by $D^k_{\alpha\beta}(\eta^{\theta_j}_{\gamma\beta})_l$ (that is, the derivative, with respect to the k component of $p_\alpha - p_\beta$, of the l component of $\eta^{\theta_j}_{\gamma\beta}$) and B_{θ_i} is the matrix $D^k_{\gamma\beta}(\eta^{\theta_i}_{\alpha\beta})_l$.

Let $\mathcal{G}_k$, $k < n$ be a sequence of connected sub-graph of $\mathcal{G}_n$ containing k vertices, $\mathcal{G}_k \subset \mathcal{G}_{k+1}$, and $\mathcal{A}_k$ the Lie algebra generated by the vector fields $X^{\theta_i}_{\alpha\beta}$, $\alpha,\, \beta \in \mathcal{G}_k$. We will show by induction that $\mathcal{A}_n$ contains $3n-4$ linearly independent vector fields.

The fact is clear for $\mathcal{A}_2$. Let us suppose that it is true for $\mathcal{A}_k$ and let us show that $\mathcal{A}_{k+1}$ contains three extra vector fields linearly independent from all the previous ones.

We start by noticing that $\mathcal{G}_{k+1}$ is obtained by adding a particle (vertex), say α, to $\mathcal{G}_k$ and that such particle is connected with the rest of the graph by only one bond. Let us call β the particle to which α is connected and γ a particle, in $\mathcal{G}_k$, connected to β. $\mathcal{A}_{k+1}$ is larger (or equal) than the set of vector fields generated by all the vectors in $\mathcal{A}_k$, $X^{\theta_i}_{\alpha\beta}$

and $[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}]$. If our inductive hypothesis is false, then there exist $\lambda_i \in I\!R$, $\mu_{ij} \in I\!R$ and $Y_{ij} \in \mathcal{A}_k$ such that

$$0 = \lambda_1 X^{\theta_1}_{\alpha\beta} + \lambda_2 X^{\theta_2}_{\alpha\beta} + \mu_{ij}[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}] + Y_{ij}$$

for each $i, j \in \{1, 2\}$. To see that this is impossible we apply the above tangent vector to p_α

$$0 = \lambda_1 \eta^{\theta_1}_{\alpha\beta} + \lambda_2 \eta^{\theta_2}_{\alpha\beta} - \mu_{ij} \left(B_{\theta_i}\right)^T \eta^{\theta_j}_{\gamma\beta}.$$

By multiplying this relation by $D_{\alpha\beta}T$, where T is the kinetic energy defined inside (2.1), and by using the fact that, for points not in $\widetilde{\Sigma}_c$, $D_{\alpha\beta}T \neq 0$, follows

$$0 = \mu_{ij} \langle B_{\theta_i} D_{\alpha\beta} T,\, \eta^{\theta_j}_{\gamma\beta}\rangle.$$

To simplify the previous expression we recall that, by definition, $\langle \eta^{\theta_j}_{\alpha\beta},\, D_{\alpha\beta}T\rangle = 0$ and applying the operator $D_{\gamma\beta}$ to such an equality yields

$$0 = B_{\theta_i} D_{\alpha\beta} T + H_\beta \eta^{\theta_i}_{\alpha\beta},$$

where H_β is the matrix $\partial_{p_\beta}\partial_{p_\beta}T$. It is essential to notice that the Hessian H_β is positive definite, given the convexity of ϕ.

By collecting the previous relations follows

$$0 = \mu_{ij} \langle H_\beta \eta^{\theta_i}_{\alpha\beta},\, \eta^{\theta_j}_{\gamma\beta}\rangle.$$

To conclude the proof it is enough to show that at least one of the scalar products $\langle H_\beta \eta^{\theta_i}_{\alpha\beta},\, \eta^{\theta_j}_{\gamma\beta}\rangle$ is different from zero. Notice that $\{\eta^{\theta_i}_{\alpha\beta}\}$ and $\{\eta^{\theta_j}_{\gamma\beta}\}$ span two two-dimensional planes in $I\!R^3$. Such planes must intersect, at least, at a line. Let $\zeta \neq 0$ be a vector belonging to such a line. Clearly there exists τ_i τ'_i such that $\sum_l \tau_l \eta^{\theta_l}_{\alpha\beta} = \zeta$ and $\sum_l \tau'_l \eta^{\theta_l}_{\alpha\beta} = \zeta$. This yields to the contradiction $0 = \langle H_\beta \zeta,\, \zeta\rangle$. That is, the vectors $X^{\theta_1}_{\alpha\beta}$, $X^{\theta_2}_{\alpha\beta}$, $\sum_{ij} \tau_i \tau'_j [X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}]$ are linearly independent with respect to themselves and with respect to the ones in the algebra $\mathcal{A}_k$. $\square$

Proposition 2.2. *The condition (iv) of reversibility of the measure Q implies that, for any bond corresponding to connected particles $b = \{\alpha, \beta\}$ in the configuration ω_q and any smooth local function ϕ, we have*

$$I\!E^{\,Q}(X^\theta_{\alpha\beta}\phi \mid \omega_q) \equiv \widetilde{I\!E}\,\left(X^\theta_{\alpha\beta}\phi\right) = 0$$

provided that $supp(\phi) \cap \widetilde{\Sigma}_c = \emptyset$.

Proof.

Let φ and ψ be an arbitrary smooth local functions with support in a finite region Λ', and Λ a region containing Λ' and so large that particles inside Λ' cannot interact with particles outside Λ. Since the size of the support of a test function can be assumed arbitrarily small, without loss of generality, we will carry out a local argument; namely we will assume that the support of ψ is contained in a conveniently small neighborhood of an arbitrary configuration not belonging to $\widetilde{\Sigma}_c$ (i.e., the support must be so small as not to intersect $\widetilde{\Sigma}_c$). If the configuration ω_q does not contain any cluster in Λ, then the Proposition is obviously true. Next, we will assume that only one isolated cluster[4] Γ_Λ is present (the case in which several isolated clusters are present can be treated in the same way, as we will remark at the end of the proof). The graph associated to Γ_Λ is finite and it will contain M bonds. The reversibility condition (iv) implies:

$$-\widetilde{\mathbb{E}}\left(\psi \hat{L} \varphi\right) = \sum_{b\in\Gamma_\Lambda} \sum_{\theta} \sigma_b \widetilde{\mathbb{E}}\left(X_b^\theta \psi X_b^\theta \varphi\right)$$

where $b = \{\alpha, \beta\}$ is a generic bond in Γ_Λ, i.e., $\sigma_b = \sigma(q_\alpha - q_\beta) > 0$. Note that the operators $X_{\alpha\beta}$ with q_α or q_β not in Λ do not appear in the right hand side of the above equation, although it is possible that $\sigma(q_\alpha - q_\beta) \neq 0$; this is due to the fact that, since the test functions depend only on the particles in Λ', if $q_\alpha \notin \Lambda$ and $\sigma(q_\alpha - q_\beta) \neq 0$, then $q_\beta \notin \Lambda'$ which implies $X_{\alpha\beta}\psi = 0 = X_{\alpha\beta}\varphi$.

A technical obstacle to our proof is that, in general, the vector fields $\{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}$ are neither linearly independent nor their linear combinations span all the Lie algebra that they generate. Typically, only $L \leq KM$ such vector fields will be linearly independent,[5] while the Lie algebra will be $N \geq L$ dimensional. To overcome such problem we choose, among $\{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}$ and their commutators, a subset of linearly independent vector fields $\{X_1, \ldots, X_N\}$ that form a base of the Lie algebra.[6] In addition, we require

$$\{X_1, \ldots, X_L\} \subset \{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}.$$

Thus, the original KM vector fields can be expressed as linear combinations of the independent vector fields $\{X_1, \ldots, X_L\}$:

$$X_{b_j}^\theta = \sum_{i=1}^{L} \nu_{ji}^\theta X_i \qquad\qquad j = 1, \ldots, KM.$$

Since,

$$[X_{b_j}^\theta,\, X_{b_k}^{\theta'}] = \sum_{l,\,p} [\nu_{jl}^\theta X_l,\, \nu_{kp}^{\theta'} X_p] = \sum_{l,\,p} \left\{ \nu_{jl}^\theta (X_l \nu_{kp}^{\theta'}) X_p - \nu_{kp}^{\theta'} (X_p \nu_{jl}^\theta) X_l + \nu_{jl}^\theta \nu_{kp}^{\theta'} [X_l,\, X_p] \right\}$$

it is clear that $\{X_1, \ldots, X_L\}$ generates the complete Lie algebra under consideration.

[4] Remember that an "isolated cluster" in Λ can still have interactions with particles outside Λ.

[5] KM is the cardinality of $\{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}$; remember that $\theta \in \{1, ..., K\}$.

[6] This is possible provided the support of ψ is sufficiently small.

Let A be the $L \times L$ matrix with elements defined by

$$a_{i,k} = \sum_{j=1}^{M} \sum_{\theta} \sigma_{b_j} \nu_{ji}^{\theta} \nu_{jk}^{\theta}$$

then

$$\sum_{b \in \Gamma_\Lambda} \sum_{\theta} \sigma_b \widetilde{I\!\!E} \left(X_b^{\theta} \psi X_b^{\theta} \varphi \right) = \sum_{i,k=1}^{L} \widetilde{I\!\!E} \left(a_{ik} X_i \psi X_k \varphi \right).$$

It is easy to check that the matrix A is positive defined, and therefore invertible.

According to Lemma 2.1 $\{X_1, \ldots, X_N\}$ span the tangent space of Σ_c (the surfaces associated to the cluster Γ_Λ). Since such surfaces foliate the phase space of the particles contained in Γ_Λ, we can choose coordinates $(c,\, y)$ such that the vector fields $\{Y_i\}_1^N$, associated to the coordinates $\{y_i\}_1^N$, generates the tangent space of Σ_c (i.e., for each c, $\{y\}$ is a system of coordinates for Σ_c). This implies, $\forall i, j$,

$$[Y_i, Y_j] = 0 \quad ; \qquad Y_i^* = -Y_i \quad ; \qquad Y_i y_j = \delta_{ij} \quad .$$

In addition, there exists an invertible $N \times N$ matrix Λ, such that

$$X_i = \sum_{j=1}^{N} \Lambda_{ij} Y_j \ .$$

Let us choose as function φ a coordinate function y_j multiplied by a smooth function with value one on the support of ψ, which, consequently, can be ignored. Applying $\hat{L}$ we have

$$\begin{aligned} \hat{L}\ y_j &= \sum_{k,i} \ X_k\ a_{ki}\ X_i\ y_j \\ &= \sum_{k,i} \ X_k\ a_{ki}\ \Lambda_{ij} \end{aligned}$$

where we have used

$$X_i\ y_j \ = \ \sum_l \ \Lambda_{il}\ Y_l\ y_j \ = \ \Lambda_{ij}.$$

The reversibility relation then gives us:

$$-\sum_{k,i} \widetilde{I\!\!E} \left(\psi X_k \ \left(a_{ki}\ \Lambda_{ij} \right) \right) \ = \ \sum_{k,i} \widetilde{I\!\!E} \left(a_{ki}\ \Lambda_{ij}\ X_k\ \psi \right)$$

which is equivalent to

$$\sum_{k,i} \widetilde{I\!\!E} \left(X_k \left(\psi\ a_{ki}\ \Lambda_{ij} \right) \right) \ = \ 0.$$

Let $V = \Lambda^{-1} I\!R^L \subset I\!R^N$,[7] then $A\Lambda : V \to I\!R^L$ is one to one and onto. Which means that for each $e^k \in I\!R^L$, $e^k = (0, \ldots, 1, \ldots, 0)$, there exists $\alpha^k \in V \subset I\!R^N$ such that $A\Lambda\alpha^k = e^k$. Moreover, in some small neighborhood of any configuration, α^k will vary smoothly.

We can make the following L^2 different choices of ψ

$$\psi_{jh} \;=\; \alpha^h_j \phi$$

where ϕ is a function with sufficiently small support around the configuration we are considering.

Summing over j we obtain

$$0 = \sum_{i,j,k} \widetilde{I\!E}\left(X_k(\alpha^h_j \; \Lambda_{ij} a_{ki} \; \phi)\right) = \sum_k \widetilde{I\!E}\left(X_k e^h_k \phi\right),$$

that is to say

$$\widetilde{I\!E}\,(X_h \; \phi) \;=\; 0 \qquad \forall\, h \in \{0, \ldots, L\}$$

which implies our thesis.

The generalization to the situation where many clusters appear in the region Λ is straightforward since, in the above argument, the coordinate functions y_j are localized on the particular cluster we are considering. Hence, the argument simply factors over the different clusters. □

Up to now we have seen that the measure is invariant with respect to vector fields that generate the tangent space to the surfaces of the momenta of the clusters Γ^Λ_i with constant kinetic energy and momentum. This was done only by using the reversibility of the stochastic dynamics. If in Λ the cluster was unique (like in the case with infinite range stochastic interaction), then this would imply that the measure on the momenta conditioned on the position is microcanonical, i.e. we would have directly lemma 5.1 below. Unfortunately in our case we cannot ignore the existence of isolated clusters. So what we can conclude at this point is that, conditioned on the positions, the distribution of the velocities in each cluster is microcanonical. In order to arrive to the statement of lemma 5.1, we need to somehow exchange the particles between clusters. The only way to do this is to generate, with the help of the Hamiltonian dynamics, other dynamics for which the measure is invariant and that permit such exchanges of particles among clusters. In the rest of the section we will define these dynamics and prove their local properties, and in the section four we will use them to move particles among clusters.

We start by studying the Lie algebra generated by $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta},\, L]\}_{q_\alpha, q_\beta \in \Lambda}$.

Lemma 2.3. *For each region Λ, each local smooth function φ localized in Λ, calling $\mathcal{A}_\Lambda$ the Lie algebra generated by the operators $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta},\, L]\}_{q_\alpha, q_\beta \in \Lambda}$, we have*

$$I\!E^{\,Q}\left(X\varphi \mid |\omega_\Lambda| = n;\ \omega_{\Lambda^c}\right) = 0$$

[7] By $I\!R^L$, here we mean $\{v \in I\!R^N \mid v_i = 0\ \forall i > L\}$.

for each $X \in \mathcal{A}_\Lambda$.

Proof.

The difficulties arise because L does not conserve the number of particles in a finite region. We need to use here the stationarity of Q with respect to L.

Let $\chi_\varepsilon(q)$ be a smooth function equal to one if $q \in \Lambda$, and equal to 0 if the distance between q and Λ is larger than ε. We can then define $N_\varepsilon \equiv \sum_\alpha \chi_\varepsilon(q_\alpha)$ to be an approximation of the number of particles in Λ. Clearly, when ε goes to zero, N_ε tends to the number of particles contained in the closure of Λ, which, since Q is locally absolutely continuous, equals almost everywhere the number of particles contained in the interior.

Let h be a smooth function on $\mathbb{R}^+$ with compact support and φ any smooth local function with support contained in the interior of Λ; in addition, we consider arbitrary smooth functions $\psi_{\alpha\beta} : \mathbb{R}^6 \to \mathbb{R}$ with support in $\Lambda \times \Lambda$ and we use them to define the local operators $X(\psi) \equiv \sum\limits_{\alpha\beta} \psi_{\alpha\beta}(q_\alpha,\, q_\beta)\sigma(q_\alpha - q_\beta)X_{\alpha\beta}$ (clearly all these operators are part of the Lie algebra $\mathcal{A}_\Lambda$). Using the previous definitions, since $X(\psi)h(N_\varepsilon) = 0$, we have,

$$0 = \mathbb{E}\left([L,\, X(\psi)]\varphi h(N_\varepsilon)\right) = \mathbb{E}\left(h[L,\, X(\psi)]\varphi\right) - \mathbb{E}\left(\varphi X(\psi) L h\right).$$

Since $Lh(N_\varepsilon) = h'(N_\varepsilon) \sum\limits_{\gamma \in \Lambda^c} \langle p_\gamma,\, \nabla\chi_\varepsilon(q_\gamma)\rangle$, we have that $X(\psi)Lh = 0$. So we conclude that

$$0 = \mathbb{E}\left(h(N_\varepsilon)[L,\, X(\psi)]\varphi\right).$$

Letting $\varepsilon \to 0$ proves that it is possible to condition with respect to the number of particles in Λ; a similar computation shows that it is possible to condition with respect to the configuration outside Λ as well. □

Lemma 2.3 shows that $\mathcal{A}_\Lambda$ has interesting local properties, these are further clarified by the following Lemma. Consider configurations with n particles in Λ and define Π_Λ, E_Λ like in the equations above theorem 1.2.

Lemma 2.4. *The Lie Algebra* $\mathcal{A}_\Lambda$ *is tangent to the surface* Π_Λ*=constant,* E_Λ*=constant, and acts only on observables depending on the coordinates of the particles inside* Λ*.*

Proof.

Given two particles $\alpha,\, \beta \in \Gamma$ we have

$$\begin{aligned}
&X^\theta_{\alpha\beta}\Pi_\Lambda = 0\\
&X^\theta_{\alpha\beta}E_\Lambda = 0\\
&[X^\theta_{\alpha\beta},\, L]\Pi_\Lambda = X^\theta_{\alpha\beta}\sum_\gamma \frac{\partial T}{\partial q_\gamma} = 0\\
&[X^\theta_{\alpha\beta},\, L]E_\Lambda = X^\theta_{\alpha\beta}\left[\sum_\gamma \langle \frac{\partial \mathcal{H}}{\partial p_\gamma},\, \frac{\partial E_\Lambda}{\partial q_\gamma}\rangle - \langle \frac{\partial \mathcal{H}}{\partial q_\gamma},\, \frac{\partial E_\Lambda}{\partial p_\gamma}\rangle\right]
\end{aligned}$$

Letting $\Delta = \mathcal{H} - E_\Lambda = \sum_{\gamma \notin \Lambda} \phi(p_\gamma) + \frac{1}{2} \sum_{q_\gamma \notin \Lambda} \sum_{q_\delta \notin \Lambda;\, \delta \neq \gamma} V(q_\gamma - q_\delta)$, and $H_\gamma = \left(\frac{\partial^2 \phi(p_\gamma)}{\partial p^i_\gamma \partial p^j_\gamma} \right)$, we can rewrite the last equation as

$$
\begin{aligned}
[X^\theta_{\alpha\beta},\, L] E_\Lambda &= X^\theta_{\alpha\beta} \left[\sum_\gamma \langle \frac{\partial \Delta}{\partial p_\gamma},\, \frac{\partial E_\Lambda}{\partial q_\gamma} \rangle - \langle \frac{\partial \Delta}{\partial q_\gamma},\, \frac{\partial E_\Lambda}{\partial p_\gamma} \rangle \right] \\
&= -\langle \frac{\partial \Delta}{\partial q_\alpha},\, H_\alpha \eta^\theta_{\alpha\beta} \rangle + \langle \frac{\partial \Delta}{\partial q_\beta},\, H_\beta \eta^\theta_{\alpha\beta} \rangle = 0
\end{aligned}
$$

since Δ does not depend on q_α or q_β.

Similarly, a direct computation shows that, if $q_\gamma \notin \Lambda$, then

$$
\begin{aligned}
[X^\theta_{\alpha\beta},\, L] q_\gamma &= 0 \\
[X^\theta_{\alpha\beta},\, L] p_\gamma &= 0\,.
\end{aligned}
$$

□

At this point we have to distinguish between the Gaussian and the non-Gaussian case. The difference is that in the gaussian case the center of mass is always conserved. Define

$$
\Theta_\Lambda \;=\; \sum_{q_\alpha \in \Lambda} q_\alpha
$$

Lemma 2.5. *If ϕ is quadratic, the Lie Algebra $\mathcal{A}_\Lambda$ is tangent to the surface Π_Λ=constant, E_Λ=constant, and Θ_Λ=constant.*

Proof.

All we need to compute is

$$
\begin{aligned}
&X^\theta_{\alpha\beta} Q_\Lambda = 0 \\
&[X^\theta_{\alpha\beta},\, L] Q_\Lambda = X^\theta_{\alpha\beta} \sum_{\gamma \in \Lambda} \frac{\partial \phi}{\partial p_\gamma} = (H_\alpha - H_\beta) \eta^\theta_{\alpha\beta} = 0
\end{aligned}
$$

since, in the present case, $H_\alpha = H_\beta$=constant. □

This means that, in the Gaussian case, the vector fields we are considering conserve the center of mass, even if this is not conserved by L; accordingly, the Lie Algebra generated by $\{X^\theta_{\alpha\beta},\, [X^\theta_{\alpha\beta},\, L]\}$, for some $\alpha, \beta \in \Gamma$ (Γ being some cluster in Λ), can be at most five dimensional.[8] We prove that the algebra has the largest possible dimension.

[8] Here and in the following for dimension of a Lie Algebra we mean the minimal dimension of it when restricted to the tangent spaces at different points.

Lemma 2.6. *If ϕ is quadratic, and $\alpha, \beta \in \Gamma$ are connected, then the Lie Algebra generated by $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta},\, L]\}$ is five dimensional.*

Proof.

Applying the vector fields to q_α we have

$$\begin{aligned}[X^\theta_{\alpha\beta},\, L]q_\alpha =& H_\alpha \eta^\theta_{\alpha\beta}\\ [X^\theta_{\alpha\beta},\, [X^\theta_{\alpha\beta},\, L]]q_\alpha =& H_\alpha D_{\alpha\beta}(\eta^\theta_{\alpha\beta})\eta^\theta_{\alpha\beta}\end{aligned}$$

This vectors span a three dimensional vector space and are linearly independent with respect with the vectors $X^\theta_{\alpha\beta}$. To see this, it is sufficient to consider a generic linear combination, equal it to 0, and multiply it by $H_\alpha^{-1} D_{\alpha\beta} E$, then

$$0 = \sum_i \mu_i \langle D_{\alpha\beta}E,\, \eta^{\theta_i}_{\alpha\beta}\rangle + \nu \langle D_{\alpha\beta}E,\, D_{\alpha\beta}(\eta^{\theta_1}_{\alpha\beta})\eta^{\theta_1}_{\alpha\beta}\rangle.$$

Next, remember that $\langle D_{\alpha\beta}E,\, \eta^\theta_{\alpha\beta}\rangle = 0$, differentiating such an expression by $D_{\alpha\beta}$ one gets

$$(H_\alpha + H_\beta)\eta^\theta_{\alpha\beta} + D_{\alpha\beta}(\eta^\theta_{\alpha\beta})^T D_{\alpha\beta}E = 0$$

and, multiplying it by $\eta^\theta_{\alpha\beta}$,

$$\langle \eta^\theta_{\alpha\beta},\, (H_\alpha + H_\beta)\eta^\theta_{\alpha\beta}\rangle = -\langle D_{\alpha\beta}E,\, D_{\alpha\beta}(\eta^\theta_{\alpha\beta})\eta^\theta_{\alpha\beta}\rangle.$$

Using the above equalities we obtain

$$\nu\langle \eta^{\theta_1}_{\alpha\beta},\, (H_\alpha + H_\beta)\eta^{\theta_1}_{\alpha\beta}\rangle = 0$$

that is $\nu = 0$. From this follows $\mu_i = 0$. □

In the non-Gaussian case the center of mass is not conserved by the vector fields we are considering, and we have no other obvious conserved quantity. We impose a condition on the noise to make sure that there are no conserved quantities, beside those considered in Lemma 2.4. More precisely we require the following:

Condition on the Noise. *For each two particles α, β, interacting stochastically, we require that the Lie algebra generate by the vectors $X^\theta_{\alpha\beta}$ and $[X^\theta_{\alpha\beta},\, L]$ is eighth dimensional at each point of every surface with fixed total energy and total momentum except, at most, for the finite union of smooth manifolds of codimension two $\widetilde{\Sigma}_{\alpha\beta}$.*

In Appendix 1 we will show that if ϕ satisfy (NG) then the above condition is satisfied.

We introduce two family of surfaces in $I\!R^{6n}$,

$$\begin{aligned}\Xi(n,\, \Pi,\, E,\, \omega_c) = \Bigg\{ (q,\, p) \in I\!R^{6n} \,\Bigg|\, \sum_\alpha p_\alpha = \Pi;\ \sum_\alpha \phi(p_\alpha) + \frac{1}{2}\sum_{\alpha,\beta} V(q_\alpha - q_\beta)\\ + \sum_\alpha \sum_{\beta\in\omega_c} V(q_\alpha - q_\beta) = E \Bigg\}\end{aligned}$$

$$\Xi(n,\, \Theta,\, \Pi,\, E,\, \omega_c) = \Big\{ (q,\, p) \in \Xi(n,\, \Pi,\, E,\, \omega_c) \,\Big|\, \sum_\alpha q_\alpha = \Theta \Big\}$$

and let $\widetilde{\Xi}$ be the union of the sets for which $(p_\alpha, p_\beta) \in \widetilde{\Sigma}_{(\alpha,\beta)}$, for some $\alpha \neq \beta$. By hypotheses $\widetilde{\Xi}$ has at least codimension two in Ξ, in additions it has zero Lebesgue measure.

Lemma 2.7. *For all $n \in \mathbb{N}$, for almost all Π, E, Θ, for each $X \in \mathcal{A}_\Lambda$, and any local function φ with support contained in Λ and disjoint from $\widetilde{\Xi}$*

$$\mathbb{E}\left(X\varphi \mid \omega_\Lambda \in \Xi,\, \omega_{\Lambda^c}\right) = 0.$$

In addition, if Λ contains a unique cluster, then the Lie algebra $\mathcal{A}_\Lambda$ contains all the tangent space of Ξ at each point of $\Xi \backslash \widetilde{\Xi}$.

Proof.

The first condition follows from lemma 2.3 and lemma 2.4 (or lemma 2.5 for the Gaussian case). To address the second part of the lemma we start an induction argument similar to the one used in lemma 2.1. We want to generate a 6n-7 dimensional Lie algebra in the Gaussian case and a 6n-4 dimensional Lie algebra in the non-Gaussian case. In both cases we need at each step of the induction argument, i.e., for every particle α that we add to a cluster, six new independent vector fields.

From the proof of Lemma 2.1 we have already three independent vector fields generated by $\{X^{\theta^i}_{\alpha\beta}, X^{\theta^j}_{\beta\gamma}\}$. All these are acting only in the direction of the momenta, so all we need is to look at the action of the new vector fields on the positions direction to establish their linear independence.

Define

$$\widetilde{L}^{\theta_k}_{\alpha\beta} = [X^{\theta_k}_{\alpha\beta},\, L]$$

$$\overline{L}^{\theta_k\theta_l}_{\alpha\beta\gamma} = [X^{\theta_l}_{\beta\gamma}, \widetilde{L}^{\theta_k}_{\alpha\beta}].$$

Applying these vector fields to q_α we have:

$$\begin{aligned}\widetilde{L}^{\theta_k}_{\alpha\beta} q_\alpha &= X^{\theta_k}_{\alpha\beta} L q_\alpha = H_\alpha \eta^{\theta_k}_{\alpha\beta}\\ \overline{L}^{\theta_k\theta_l}_{\alpha\beta\gamma} q_\alpha &= X^{\theta_l}_{\beta\gamma} H_\alpha \eta^{\theta_k}_{\alpha\beta} = H_\alpha \left(D_\beta \eta^{\theta'}_{\alpha\beta}\right)^T \eta^{\theta}_{\beta\gamma}.\end{aligned}$$

It is then enough to prove that the vectors $X^{\theta_i}_{\alpha\beta}$, $[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}]$, $\widetilde{L}^{\theta_i}_{\alpha\beta}$, $\sum_{ij} \xi_{ij} \overline{L}^{\theta_i\theta_j}_{\alpha\beta\gamma}$, for some choice of ξ_{ij}, and Y (where Y belongs to the lie algebra generated by the vectors already considered during the induction procedure) are linearly independent. Again we assume that it is not so, i.e.,

$$0 = \sum_{k=1}^{2} \Lambda_k X^{\theta_k}_{\alpha\beta} + \mu_{ij}[X^{\theta_i}_{\alpha\beta},\, X^{\theta_j}_{\gamma\beta}] + \sum_{k=1}^{2} \nu_k \widetilde{L}^{\theta_k}_{\alpha\beta} + \tau \sum_{kl} \xi_{kl} \overline{L}^{\theta_k\theta_l}_{\alpha\beta\gamma} + Y,$$

for some Λ_i, μ_{ij}, ν_i, τ, Y. We apply the previous expression to, q_α and obtain

$$0 = \sum_{k=1}^{2} \nu_k H_\alpha \eta^{\theta_k}_{\alpha\beta} + \tau \sum_{kl} \xi_{kl} H_\alpha \left(D_\beta \eta^{\theta_l}_{\alpha\beta}\right)^T \eta^{\theta_k}_{\gamma\beta}.$$

If we multiply by $H_\alpha^{-1} D_{\alpha\beta} E$, recalling the properties of η, we obtain

$$0 = \tau \sum_{kl} \xi_{kl} \langle H_\alpha \eta^{\theta_k}_{\alpha\beta},\, \eta^{\theta_l}_{\gamma\beta} \rangle.$$

Which shows that, out of $\widetilde{\Sigma}$, it is always possible to choose ξ_{kl} such that the sum is different from zero. This implies $\tau = 0$ and allows us to conclude the proof in complete analogy with lemma 2.1. $\square$

As promised, we have found a bundle of local dynamics preserving the measure Q (or, more precisely, its local conditional measures), i.e. the dynamics generated by the vector fields in the Lie algebra $\mathcal{A}_\Lambda$.

3. CONDITIONING TO TYPICAL CONFIGURATIONS .

Using the entropy bound and large deviations estimates, we will show here that certain configurations have probability 0 for any measure Q satisfying our hypotheses. We will need to exclude these configurations from the considerations of the next section.

First of all, we want to disregard configurations with locally big barriers of potential, so we are going to analyse those configurations with local high density.

Lemma 3.1. *Let $\Lambda \subset I\!R^3$ and $\Delta \subset \Lambda$ be a box of size R_1, consider the following configurations*

$$\Omega^\varepsilon_\Lambda = \{\omega \mid \exists \Delta \subset \Lambda \,:\, |\omega_\Delta| \geq \varepsilon^{-1} |\Lambda|^{\frac{1}{2}}\}. \tag{3.1}$$

If Q satisfies condition (i) of theorem 1.2 (entropy bound), then there exists $C > 0$ such that:

$$Q(\Omega^\varepsilon_\Lambda) \leq C\varepsilon^2.$$

Proof.

By the entropy inequality

$$Q(\Omega^\varepsilon_\Lambda) \leq \frac{\log 2 + H_\Lambda(Q|I\!P)}{\log\left(1 + I\!P(\Omega^\varepsilon_\Lambda)^{-1}\right)}$$

(which is a consequence of (1.8)), and condition (i) of theorem 1.2, we need only to prove that for a given grancanonical measure $I\!P$

$$I\!P(\Omega^\varepsilon_\Lambda) \leq C_2 |\Lambda| \exp(-C_1 |\Lambda| \varepsilon^{-2})$$

for some constants $C_1, C_2 > 0$ independent from ε and Λ.

Since the measure $I\!P$ is translation invariant

$$I\!P(\Omega^\varepsilon_\Lambda) \leq \frac{C_1}{R_1} |\Lambda| I\!P(\{|\omega_\Delta| > \varepsilon^{-1} |\Lambda|^{\frac{1}{2}}\}).$$

Accordingly, (setting $\Gamma = \int_{I\!R^3} e^{-\lambda_4\phi(p)+\sum_{i=1}^3 \lambda_i p_i} dp$)

$$
\begin{aligned}
I\!P\left(|\omega_\Delta| > \varepsilon^{-1}|\Lambda|^{1/2}\right) =& Z_\Delta^{-1} \sum_{n\geq \varepsilon^{-1}|\Lambda|^{1/2}}^{\infty} \frac{e^{\Lambda_0 n}\Gamma^n}{n!} \int_{\Delta^n} e^{-\lambda_4 V_\Delta} \\
\leq & Z_\Delta^{-1} \sum_{n\geq \varepsilon^{-1}|\Lambda|^{1/2}}^{\infty} \frac{e^{\lambda_0 n}\Gamma^n}{n!} \exp\left[-\lambda_4 A \frac{\varepsilon^{-2}|\Lambda|}{\Delta} + \lambda_4 B \varepsilon^{-1}|\Lambda|^{1/2}\right] |\Delta|^n \\
\leq & C_2 e^{-C_3 \varepsilon^{-2}|\Lambda|}
\end{aligned}
$$

where we have used the explicit form of the grand canonical measures, the positivity and the superstability of the potential. □

Another information needed in the following arguments is a bound on the total kinetic energy shared by a large number of particles.

Lemma 3.2. *Let $a > 0$, and $\varepsilon > 0$ sufficiently small, $\Lambda \subset I\!R^3$ and consider the following configurations*

$$
\widetilde{\Omega}_\Lambda^\varepsilon = \left\{ \omega \,\middle|\, \exists \{\alpha_i\}_{i=1}^{a|\Lambda|} \,:\, q_{\alpha_i} \in \Lambda,\ \sum_{i=1}^{a|\Lambda|} \left[\phi(p_{\alpha_i}) - \phi\left(\frac{\sum_{i=1}^{a|\Lambda|} p_{\alpha_i}}{a|\Lambda|} \right) \right] < \varepsilon a |\Lambda| \right\}, \tag{3.2}
$$

then there exists $C > 0$:

$$
\lim_{|\Lambda|\to\infty} Q(\widetilde{\Omega}_\Lambda^\varepsilon) \leq \frac{C}{a\ln(\varepsilon^{-1})}.
$$

Proof.

We will use the same entropy bound as in the previous lemma. In order to simplify notations, we choose a grancanonical measure $I\!P$ corresponding to the parameters $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and λ_4 such that $\Gamma_\Lambda = \int e^{-\lambda_4\phi(p)} dp = 1$.

Let us define

$$
Y_m = \frac{1}{m}\sum_{i=1}^m \phi(p_i) - \phi\left(\frac{1}{m}\sum_{i=1}^m p_i\right)
$$

and observe that since ϕ is convex Y_m is non–negative. Then we have

$$
I\!P\left(\widetilde{\Omega}_\Lambda^\varepsilon\right) = Z_\Lambda^{-1} \sum_{n\geq a|\Lambda|} \frac{e^{n\lambda_0}}{n!} \binom{n}{[a|\Lambda|]} J_{[a|\Lambda|]} \int e^{-\lambda_4 V_\Lambda} = e^{c|\Lambda|} J_{[a|\Lambda|]} I\!P\left(\{\omega||\omega_\Lambda| \geq [a|\Lambda|]\}\right)
$$

where c is some constant depending on $I\!P$, and

$$
J_m = \int_{Y_m<\varepsilon} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p
$$

and in the following $m = [a|\Lambda|]$. By exponential Chebicheff inequality, for any $\beta > 0$

$$J_m \leq e^{\beta\varepsilon m} \int e^{-m\beta Y_m} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)} \, d^m p.$$

By large deviation asymptotic (cf.[V])

$$\lim_{m\to\infty} \frac{1}{m} \log \int e^{-m\beta Y_m} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)} \, d^m p = \sup_{\mu} \left\{ \beta \left[\phi(\bar{\mu}) - \widehat{\phi}(\mu) \right] - I(\mu) \right\},$$

where $\mu(p)$ are probability densities on $I\!R^3$ (with respect to $e^{-\lambda_4 \phi} dp$),

$$\bar{\mu} = \int p\mu(p) e^{-\lambda_4 \phi} dp, \qquad \widehat{\phi}(\mu) = \int \phi(p)\mu(p) e^{-\lambda_4 \phi} dp$$

and

$$I(\mu) = \int \mu(p) \log(\mu(p))\, e^{-\lambda_4 \phi(p)} dp \ .$$

Since $\phi(x) = \phi(-x)$ the variational problem can be explicitly solved and the maximizing μ is given by

$$\frac{e^{-\beta\phi(p)}}{\int e^{-(\beta+\lambda_4)\phi(p')} dp'}.$$

We can then compute

$$\sup_{\mu} \left\{ \beta \left[\phi(\bar{\mu}) - \widehat{\phi}(\mu) \right] - I(\mu) \right\} = \log \int e^{-(\beta+\lambda_4)\phi(p)} dp.$$

Optimizing on β we obtain

$$\lim_{m\to\infty} \frac{1}{m} \log J_m \leq \inf_{\beta>0} \left[\beta\varepsilon + \log \int e^{-(\beta+\lambda_4)\phi(p)} dp \right].$$

By hypothesis there exists $k \in I\!N^+$:

$$\begin{aligned} \phi(p) &\geq c_1 \|p\|^k \quad \forall \|p\| \leq 1 \\ \phi(p) &\geq c_2 \|p\| \quad \forall \|p\| > 1, \end{aligned}$$

it follows

$$\int e^{-\nu\phi(p)} dp \leq \int_{\|p\|\leq 1} e^{-\nu c_1 \|p\|^k} + \int_{\|p\|\geq 1} e^{-\nu c_2 \|p\|} \leq c_3 \nu^{-\frac{1}{k}},$$

for each $\nu > 1$. Using the above estimates and minimizing over β the lemma follows. $\square$

4. CLUSTERING .

Before getting into the technicalities of the clusters deformations, let us pause here to explain our strategy.

As we already mentioned in section 2, from lemma 2.1 and 2.2 follow that the measure Q on a box Λ_0 conditioned on the positions, on the total momentum and on total kinetic energy:

$$Q_{\Lambda_0}\left(dp_1,\ldots,dp_n\middle|q_1,\ldots,q_n;\sum_{\alpha=1}^{n}p_\alpha,\sum_{\alpha=1}^{n}\phi(p_\alpha)\right) \tag{4.1}$$

is Microcanonical only for the p's corresponding to the particles in the same cluster, in particular this measure is symmetric for exchange of momentum between particles of the same cluster (by "exchange of momentum" we mean any transfer of momentum between two particles that conserves the total kinetic energy). If we could show that a measure is symmetric for exchange of momentum between clusters, it would follow that such a measure is Microcanonical, i.e. lemma 5.1 below (see appendix II for details). One way to achieve this could be to find a transformation on the phase space, for which the measure Q is invariant, that brings a particle α from a cluster Γ_1 in "contact" to another cluster Γ_2, then exchanges the momenta with a particle β of Γ_2, then brings back α to the initial position in the cluster Γ_1. We cannot do exactly this, but we will exchange momenta between the clusters performing more complicated transformations for which our measure Q is still invariant.

Given a box Λ_0 and a configuration $\omega \in \Omega$, let $T_{\alpha,\beta}\omega$ the configuration obtained exchanging momenta between the particle α and particle β, where α and β are two particles with position in Λ_0 (fix any amount $\eta \in I\!R^3$ of momenta to be exchanged compatible with the conservation of the total kinetic energy of the two particles). Observe that only momenta is exchanged while positions are unchanged. Furthermore such operation does not change the total momenta in Π_{Λ_0}, nor the total kinetic energy K_{Λ_0} in the region Λ_0. All we have to prove is that

$$\int \sum_{q_\alpha,q_\beta\in\Lambda_0} \left[F(T_{\alpha,\beta}\omega) - F(\omega)\right]\, dQ(\omega) = 0 \tag{4.2}$$

for any local smooth function $F(\omega)$.

It is very easy to see why (4.2) implies the symmetry of the measure on the momenta (4.1). Choose $F(\omega) = F_1(p_{\Lambda_0})F_2(q_{\Lambda_0},\Pi_{\Lambda_0},K_{\Lambda_0})$. Since $T_{\alpha,\beta}$ leaves invariant F_2, one can condition the relation (4.2) on the quantities on which F_2 depends and obtain

$$\int \left[F_1(T_{\alpha,\beta}p_{\Lambda_0}) - F_1(p_{\Lambda_0})\right]\, dQ(p_{\Lambda_0}|q_{\Lambda_0},\Pi_{\Lambda_0},K_{\Lambda_0}) = 0 \tag{4.3}$$

i.e. that the measure defined by (4.1) is invariant for exchange of momenta between particles.

What we already know is that (4.3) is true if α and β are in the same cluster (defined by the configuration q_{Λ_0} on which we have conditioned).

By condition (ii)[9] of our main theorem, we can choose $a > 0$ such that $\rho(\omega) > \rho_* + 2a$ with Q–probability 1. For any $\varepsilon > 0$ small enough, and $\Lambda \supset \Lambda_0$ large enough, with linear size L, define the set of good configurations

$$\widehat{\Omega}^{\Lambda,\varepsilon} = \left\{ \omega : \left| \frac{|\omega_{\Lambda_1}|}{|\Lambda_1|} - \rho(\omega) | \right| \le a \ ; \ \left| \frac{|\omega_{\Lambda}|}{|\Lambda|} - \rho(\omega) | \right| \le a \right\} \cap (\Omega^{\varepsilon}_{\Lambda})^c \cap (\widetilde{\Omega}^{\varepsilon}_{\Lambda})^c,$$

where Λ_1 is a box concentric to Λ of linear size $L/2$. Then by lemma 3.1 and 3.2

$$\lim_{\varepsilon \to 0} \lim_{|\Lambda| \to \infty} Q\left((\widehat{\Omega}^{\Lambda,\varepsilon})^c \right) \ = \ 0 \ .$$

So it is enough to show that, for any $\varepsilon > 0$ we can find Λ large enough such that

$$\int_{\widehat{\Omega}^{\Lambda,\varepsilon}} \sum_{q_\alpha, q_\beta \in \Lambda_0} [F(T_{\alpha,\beta}\omega) - \ F(\omega)] \ dQ(\omega) \ = \ 0 \tag{4.4}$$

for any bounded function F localized in Λ_0.

Let $\Xi_\Lambda(n, \Pi, E, \omega_c) \subset I\!R^{6n}$ be the surface on which n particles have positions in Λ, total momentum Π, and total energy E (note that the total energy inside Λ is affected by ω_c). Because of the boundaries ω_c, this surface may have many different connected components $\Xi^j_\Lambda(n, \Pi, E, \omega_c) \subset I\!R^{6n}$.

Proposition 4.1. *For any $\varepsilon > 0$ there exists Λ large enough such that the measure Q restricted to $\Xi^j_\Lambda(n, \Pi, E, \omega_c) \cap \widehat{\Omega}^{\Lambda,\varepsilon}$, is proportional to the Microcanonical measure*[10] *for almost all Π, E and ω_c.*

It is easy to see that (4.2) follows from proposition 4.1. In fact, ω and $T_{\alpha,\beta}\omega$ belong always to the same connected component (connected components can be distinguished only by the positions q's), and Microcanonical measures are invariant for exchanges of momenta between the particles.

The rest of the section will be then dedicated to the proof of proposition 4.1. We will fix now the box Λ, and we will drop the index Λ when this will not create confusion; moreover, in the rest of the paragraph we will drop the index j and Ξ will refer to a fixed connected component.

---

[9] If, as noted in remark 1.3, the condition is on the potential energy, just substitute the definition of $\widehat{\Omega}^{\Lambda,\varepsilon}$ with

$$\widehat{\Omega}^{\Lambda,\varepsilon} = \{\omega : U_{\Lambda_1} > 0, U_\Lambda > 0\} \cap (\Omega^{\varepsilon}_{\Lambda})^c \cap (\widetilde{\Omega}^{\varepsilon}_{\Lambda})^c.$$

and the rest of the argument of this section will remain essentially unchanged.

[10] To define the Microcanonical measure consider that $(\Lambda \times I\!R^3)^n$ is foliated by the surfaces $\Xi(E, \Pi)$ when varying E and Π. Accordingly it is possible to define the conditioning of the Lebesgue measure on $(\Lambda \times I\!R^3)^n$ to almost all the above mentioned surfaces. Such a conditional measure is exactly the Microcanonical measure on Ξ. This Microcanonical measure is also the only one invariant for the action of every element of the tangent space.

What we have proven in the previous section is that our Lie algebra $\mathcal{A}$ generates the tangent space of $\Xi(n,\,\Pi,\,E,\,\omega_c)$ only at those points corresponding to a unique cluster.

Let us call $d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p)$ the measure Q conditioned on surface $\Xi(n,\,\Pi,\,E,\,\omega_c)$ i.e.

$$\int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} f(q,\,p)\,d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p)$$
$$= \mathbb{E}^{Q}\left(f(\omega_\Lambda)\;\middle|\;|\omega_\Lambda| = n,\,\Pi_\Lambda(\omega) = \Pi,\,E_\Lambda(\omega) = E,\,\omega_c\right).$$

Since all the quantities we have conditioned on, in the definition of μ, are conserved by the vector fields of the Lie subalgebra generated only by the particles in Λ, the conditional measure $d\mu$ is invariant for such a subalgebra (Lemma 2.7); moreover, the subalgebra is composed by null divergence vector fields. This implies that, in a sufficiently small neighborhood B of a point corresponding to a configuration with a unique cluster, the measure $d\mu_n$ is proportional to the Microcanonical measure. More precisely consider an open set $B \subset \Xi$ with a constant cluster structure and let χ be the characteristic function of such a set. If all the configurations in B have a **unique** cluster and $v_i \neq v_j$ for every $i,\,j$, it follows

$$\int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} \chi(q,\,p) F_\Lambda(q,\,p)\,d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p)$$
$$= Z(n,\,\Pi,\,E,\,\omega_c) \int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} \chi(q,\,p) F_\Lambda(q,\,p)\,dM(q,\,p)$$

where dM is the Microcanonical measure on Ξ and Z is a normalization constant. In fact, the Microcanonical measure is invariant with respect to $\mathcal{A}$. Moreover, there exists vector fields $\{Y_i\}_{i=1}^m$ from $\mathcal{A}$ that span all the tangent space of Ξ at each point of B (provided B is chosen small enough). Hence, $d\mu$ must be an invariant measure for the elliptic operator $\sum_{i=1}^m Y_i^* Y_i$. The claim follows since it is well known that such an elliptic operator has a unique invariant measure.

If in the configurations in B are present several not interacting clusters $\{\Gamma_i\} = \widetilde{\Gamma}$, then from section 2 follows that the Lie algebra $\mathcal{A}$, restricted to Ξ, does not necessarily span all the tangent space. Yet, for each $\Gamma_i \in \widetilde{\Gamma}$, we can consider the surface $\Xi(\Gamma_i)$ obtained by fixing the positions of the particles not in Γ_i.[11] From section 2 follows then that the Lie Algebra $\mathcal{A}_\Lambda$, restricted to the surface $\Xi(\Gamma_i)$ spans all its tangent space. Thus, the simple application of the invariance with respect to the available vector fields yields the weaker result

$$\int_{\Xi_j(n,\,\Pi,\,E,\,\omega_c)} \chi(q,\,p) F_\Lambda(q,\,p)\,d\mu_{n,\,\Pi,\,E,\,j,\,\omega_c}(q,\,p)$$
$$= Z(n,\,\Pi,\,E,\,j,\,\omega_c) \int_{\Xi_j(n,\,\Pi,\,E,\,\omega_c)} \chi(q,\,p) F_\Lambda(q,\,p)\,dM_{\widetilde{\Gamma}}(q,\,p)$$

[11] To be more precise, suppose that Γ_i consists of m particles. Fix the position and velocities of all the particles in Λ not belonging to Γ_i and call their total energy E_1 and their total momentum Π_1. Then, $\Xi(\Gamma_i)$ is the surface in $\mathbb{R}^m$ defined by $\sum_{\alpha\in\Gamma_i} p_\alpha = \Pi - \Pi_1 \equiv \Pi'$ and $\sum_{\alpha\in\Gamma_i} \phi(p_\alpha) + \frac{1}{2}\sum_{\alpha,\,\beta\in\Gamma_i} V(q_\alpha - q_\beta) + \sum_{\alpha\in\Gamma_i,\,\beta\notin\Gamma_i} V(q_\alpha - q_\beta) = E - E_1 \equiv E'$. Notice that we are not writing explicitly the dependence on E' and Π', since this does not create ambiguities.

where

$$M_{\widetilde{\Gamma}}(q,\,p)(\cdot \mid (q_j,\,p_j) \notin \Gamma_i) = M_{\Gamma_i}((q,\,p) \in \Gamma_i)$$

M_{Γ_i} being the Microcanonical measure for the particles belonging to Γ_i.

Yet, it is possible to use the dynamics generated by the vector fields in order to get a better result. We will show that one can construct maps, connected to cluster deformations, with the property of preserving both the measures $d\mu$ and dM. To be more concrete we need to define precisely what is meant by deforming a cluster.

Recall that $\widetilde{\widetilde{\Xi}} = \{(q,\,p) \in \Xi \mid (p_i,\,p_j) \in \widetilde{\Sigma}_{ij}$ for some $i,\,j\}$. Moreover, given a partition $\mathcal{P}$ of the particles (i.e., $\cup_{P\in\mathcal{P}} P = \{1,\,...,\,n\}$) we will say that a measure is Microcanonical with respect to the partition $\mathcal{P}$ if for each $P \in \mathcal{P}$ conditioning the measure to all the particles not in P one obtains the Microcanonical measure for the particles in P. (From now on, with an evident abuse of notations, we will use $M_{\mathcal{P}}$ to designate any measure which is Microcanonical with respect to $\mathcal{P}$.)

Furthermore, by $\mathcal{A}_{\delta,\,\mathcal{P}}$ we will mean the Lie algebra generated by the vector fields associated to bonds in which the particles are closer than $R_1 - \delta$, for some fixed δ smaller than $R_1 - R_0$, and belongs to the same element of the partition $\mathcal{P}$; finally, by $\mathcal{A}_{\delta,\,\mathcal{P}}(\xi)$ we designate the restriction of $\mathcal{A}_{\delta,\,\mathcal{P}}$ at $\mathcal{T}_\xi\Xi$.[12]

Definition 4.2. *By "allowed deformation" with respect to a partition $\mathcal{P}$ and a tolerance δ, we mean a piecewise smooth curve $\gamma : [0,\,1] \to \Xi\backslash\widetilde{\widetilde{\Xi}}$ with the property that, for some $\delta \in I\!R^+$ and for each $s \in [0,\,1]$, $\gamma'(s) \in \mathcal{A}_{\delta,\,\mathcal{P}}(\gamma(s))$.*

Note that, in a given configuration, the clusters form a partition.

Definition 4.3. *Given a set $B \in \Xi$ we call "$\mathcal{P}(B)$" the coarsest partition of $\{1,\,...,\,n\}$ finer than the partitions produced by the isolated clusters of each ξ in B.*

Proposition 4.4. *Given a configuration $\xi \equiv (q,\,p) \in \Xi\backslash\widetilde{\widetilde{\Xi}}$, let $r = \sup_{\alpha,\beta:\,|q_\alpha - q_\beta| < R_1} |q_\alpha - q_\beta|$ and $\delta \le R_1 - r$, suppose that there exists a neighborhood B of ξ in which the measure μ is Microcanonical with respect to some partition $\mathcal{P}$ (coarser than $\mathcal{P}(\{\xi\})$) and an allowed deformation with respect to $\mathcal{P}$ and δ connecting ξ to some other configuration $\xi_1 \in \Xi\backslash\widetilde{\widetilde{\Xi}}$, then there exists a neighborhood $B_1 \subset B$ of ξ such that*

$$\mu|_{B_1} = M_{\mathcal{P}\wedge\mathcal{P}(\{\xi_1\})}|_{B_1}.$$

The proof of proposition 4.4 is the content of subsection 4.1.

Proposition 4.4. shows that we can have the wanted result, provided we can generate enough allowed deformation. In sub–section 4.2, we will prove the existence of enough cluster deformations, more precisely we will prove proposition 4.1.

4.1 Cluster deformation.

Proof of Proposition 4.4.

The following is a useful auxiliary Lemma.

[12] Clearly $\mathcal{A}_{\delta,\,\mathcal{P}}(\xi)$ is a linear subspace of $\mathcal{T}_\xi\Xi$.

Lemma 4.5. *If the measure is Microcanonical with respect to a partition $\mathcal{P}_1$ and, at the same time, with respect to a partition $\mathcal{P}_2$, then it is Microcanonical with respect to the partition $\mathcal{P}_1 \wedge \mathcal{P}_2$.*

Proof.

Start with the following observation: let P and G two element respectively of $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $P \cap Q \neq \emptyset$. Then since the measure is invariant for exchanges of momenta between particles inside P and inside Q, then it is invariant for exchanges in $P \cup Q$ (cf. appendix II). This implies that it is Microcanonical in $P \cup Q$. By definition of $\mathcal{P}_1 \wedge \mathcal{P}_2$, it follows that it is microcanonical with respect to this coarser partition. $\square$

We start by constructing explicitly a map Υ that leaves invariant μ and such that $\Upsilon(\xi) = \xi_1$. The idea is to define a one parameter family of vector fields in a neighborhood of the allowed deformation γ; the wanted map will be obtained by integrating such a family in a neighborhood of the curve γ.

We parameterize γ by arc-length. By definition, $\gamma \subset \Xi \backslash \widetilde{\widetilde{\Xi}}$. Let $L = \text{length}(\gamma)$ then $\gamma(0) = \xi$ and $\gamma(L) = \xi_1$. Moreover, for each $s \in [0, L]$, $\gamma'(s)$ can be written as $\sum_{i=1}^{m} a_i(s) X_i$ where $a_i(s)$ are smooth functions depending only on s and X_i are vector fields, in the Lie algebra $\mathcal{A}_{\delta, \mathcal{P}}(\gamma(s))$, that leave Q invariant;[13] here m is some integer depending only on γ. Then we define a time dependent vector field by

$$V(q,\, p,\, s) = \sum_{i=1}^{m} a_i(s) X_i(q,\, p)$$

and the corresponding flow Φ by[14]

$$\begin{aligned} &\frac{d}{ds}\Phi(q,\, p,\, s) = V(\Phi(q,\, p,\, s),\, s) \\ &\Phi(q,\, p,\, 0) = (q,\, p). \end{aligned}$$

By continuity there exists ε such that for each $s \in [0,\, 1]$, setting $\gamma(s) = (q(s),\, p(s))$, if $\widetilde{q} \in \Lambda^n$, $\widetilde{p} \in I\!R^{3n}$, $\|\widetilde{q}_i - q_i(s)\| < \varepsilon$, then $V(\widetilde{q},\, \widetilde{p},\, s) \in \mathcal{A}_{0, \mathcal{P}}((\widetilde{q},\, \widetilde{p}))$. By choosing the initial conditions in a sufficiently small neighborhood B_0 of ξ, we can insure that the solutions of the differential equations, with initial conditions in B_0, are closer than $\frac{\varepsilon}{2}$ to the curve γ.

We define Υ on B_0 by:

$$\Upsilon(\xi) = \Phi(\xi,\, L).$$

Since V is always tangent to Ξ it follows that Υ is a well defined function from $B_0 \subset \Xi \backslash \widetilde{\widetilde{\Xi}}$ to $\Xi \backslash \widetilde{\widetilde{\Xi}}$.

$\Upsilon(\xi) = \xi_1$ follows immediately from the construction ($\gamma(s)$ is a solution of the differential equation). In addition, for each smooth function F, with $\text{supp}F \subset B_0$, $I\!E^{\,Q}(F \circ \phi_s^{-1}) = I\!E^{\,Q}(F)$; that is $\mu|_{B_0}$ is invariant with respect to Υ.

[13] The fact that the a_i can be chosen smooth follows from our requirement that $\gamma' \in \mathcal{A}_\delta(\gamma)$, and is indeed the reason of such a requirement.

[14] Here we are abusing notations and using the same symbol to designate both the vector field and its coordinates with respect to the basis $\{\partial_q,\, \partial_p\}$.

Next, it is easy to check that the vector fields X_i have zero divergence and are tangent to the surfaces of constant momentum and energy of the various elements in $\mathcal{P}$. Accordingly, any measure that is Microcanonical with respect to a partition coarser than $\mathcal{P}$ will be left invariant by the flow; that is $M_{\mathcal{P}}$ is invariant with respect to Υ.

Next choose an neighborhood $B_1 \subset B_0$ of ξ such that $\mathcal{P}(\Upsilon B_1)$ is equal to $\mathcal{P}(\xi_1)$. Let χ be the characteristic function of ΥB_1 and F_Λ a smooth local function with support disjoint from $\widetilde{\Xi}$.

$$
\begin{aligned}
&\int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} \chi(q,\,p) F_\Lambda(q,\,p) d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p) \\
&= \int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} \chi \circ \Upsilon(q,\,p) F_\Lambda \circ \Upsilon(q,\,p) d\mu_{n,\,\Pi,\,E,\,\omega_c}(q,\,p) \\
&= \int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} \chi \circ \Upsilon(q,\,p) F_\Lambda \circ \Upsilon(q,\,p) dM_{\mathcal{P}}(q,\,p) \\
&= \int_{\Xi(n,\,\Pi,\,E,\,\omega_c)} \chi(q,\,p) F_\Lambda(q,\,p) dM_{\mathcal{P}}(q,\,p)
\end{aligned}
$$

Since $M_{\mathcal{P}}(\widetilde{\Xi}) = 0$, for each partition $\mathcal{P}$, and μ_{n,Π,E,ω_c} is absolutely continuous with respect to M for almost all Π, E, ω_c, the equality holds true for each F_Λ. Note that we have used the invariance, with respect to the map Υ, both of the measure μ and of the Microcanonical measures $M_{\mathcal{P}}$.

The consequence of the above chain of equalities is that μ, restricted to ΥB_1, must be Microcanonical with respect to $\mathcal{P}$. But we know from section 2 that it must be Microcanonical with respect to $\mathcal{P}(\{\xi_1\})$ as well. Hence, by Lemma 4.5, it must be Microcanonical with respect to $\mathcal{P} \wedge \mathcal{P}(\{\xi_1\})$ in ΥB_1. Using again the invariance of the Microcanonical measure, with respect to partitions coarser than $\mathcal{P}$, under the map Υ, the result follows. $\square$

Our condition on the density insures that in the box Λ there is, at least, a two particle cluster. If all the clusters in a given configuration could be connected, via an allowed deformation, to each nearby cluster, the argument would be easily concluded. Unfortunately, this is not always possible. The reason for this is that in order to extract two particles from a cluster may be needed more energy than it is available in the cluster itself.[15] Such clusters are "locked" and, in principle, it may be impossible to remove them.

It is then painfully clear that our argument can be concluded only via a discussion of such a pathological behavior; this the task of the next section.

4.2 Killing locked clusters.

This section is dedicated to the proof of Proposition 4.1.

Let $\xi_0 \in (\Xi \backslash \widetilde{\Xi}) \bigcap \widehat{\Omega}_{\Lambda,\,\varepsilon}$, consider a neighborhood $B_0 \subset \Xi \backslash \widetilde{\Xi}$, contained in a ball of sufficiently small radius δ such that $\mathcal{P}_0 = \mathcal{P}(\{\xi_0\})$ is equal to $\mathcal{P}(B_0)$. Our task will be to

[15] E.g., think of a cluster at a corner with a concave shape and zero kinetic energy; in addition the boundary condition may prevent the cluster from sliding along a side of the box.

show that there is an allowed deformation that, together with Proposition 4.4, can be used to show that μ is Microcanonical in a neighborhood of ξ_0 contained in B_0.

Given a partition $\mathcal{P}$ let us define

$$\Pi(\xi, \mathcal{P}) = \left\{ (q', p') \in \Xi \backslash \hat{\Xi} \;\middle|\; q' = q;\ \forall P \in \mathcal{P}\ \ \sum_{j \in P} p_j = \sum_{j \in P} p'_j;\ \sum_{j \in P} \phi(p_j) = \sum_{j \in P} \phi(p'_j) \right\}.$$

We will call $\mathcal{P}$-complete a configuration ξ that can be connected via an allowed deformation with respect to $\mathcal{P}$ to each configuration in $\Pi(\xi, \mathcal{P})$.[16] By definition, ξ_0 is $\mathcal{P}_0$-complete.

We intend to produce a sequence of configurations ξ_i and coarser and coarser partitions $\mathcal{P}_i$, where ξ is connected to ξ_{i+1} by an allowed deformation with respect to P_{i+1}, such that $\mu = M_{\mathcal{P}_i}$ in some neighborhood of ξ_i. From this it will follow that $\mu = M_{\mathcal{P}_i}$ also in some neighborhood of ξ_0.

We will insure that the configurations ξ_i that we are going to construct are all $\mathcal{P}_i$-complete and that $\xi_i \in \widehat{\Omega}_{\Lambda, \varepsilon}$.

In order to start we need that a cluster of two or more particles exists in Λ. But in the configurations under consideration we cannot have all the particles isolated due to the bounds on the density.

We start dealing with clusters that do not interact with particles outside Λ (if they exist). Thanks to our hypothesis that the potential is repelling, it is clear that any such cluster will have two particles that can be "extracted"[17] from the cluster always decreasing the energy along the deformation; that is: clusters that do not touch the boundary cannot be locked. We choose any cluster that is away from the boundary of Λ and perform the following allowed transformations: first select two particles, interacting between themselves, with the above mentioned property and extract them from the cluster as a two–particle cluster (the excess of energy so create is stored in extra kinetic energy of the other particles in the cluster) and move the two particles to touch a neighbor cluster (without interacting deterministically). Clearly, this deformation involves only the vector fields relative to the two particles, hence the deformation is allowed relative to $\mathcal{P}_0$. From proposition 4.4 follows that there exists a neighborhood $B_1 \subset B_0$ of $\xi_1 \equiv \xi_0$ such that $\mu|_{B_1} = M_{\mathcal{P}_1}|_{B_1}$, where $\mathcal{P}_1$ is the partition obtained by $\mathcal{P}_0$ by joining the element from which the two particles have been extracted with the element with which they have been brought into contact.

It remains to show that ξ_1 is $\mathcal{P}_1$ complete. This is done by exhibiting allowed deformations relative to $\mathcal{P}_1$ that connect ξ_1 to any other configuration in $\Pi(\xi_1, \mathcal{P}_1)$. Such deformations are constructed as follows: transfer to the two particles mentioned before any needed amount of energy and momentum compatible with the conservation of total energy and momentum in the cluster, after extract them from the cluster (the excess of energy

[16] The reason why we are interested in this concept is that, later on in the argument, we will need to be able to allocate the kinetic energy, of any element of the partition, among its particles in an arbitrary fashion.

[17] By "extracted" we mean that the particles can be moved out of the region defined by the convex hall of the support of the deterministic potential of the particles belonging to the cluster.

so create is stored in extra kinetic energy of the other particles in the cluster) and move the two particles to touch the same neighbor cluster as before (without interacting deterministically), at this point energy and momentum can be exchanged, finally move back the two particles in their original position. In this way we can show that ξ_1 is complete with respect to the partition $\mathcal{P}_1$. Although the fact seems quite intuitive the proof is not immediate (due to the limitations on the momentum that can be exchanged conserving the kinetic energy) and can be found in Appendix II.

We can proceed in a similar fashion producing coarser and coarser partitions $\mathcal{P}_i$. The process will stop only if we obtain a partition $\mathcal{P}_i$ with all the elements extending till the boundary of Λ or consisting of only 1 particles. Note that up to now we are considering always the same reference configuration $\xi_i \equiv \xi_0$.

The result of the above discussion can be conveniently summarized by the following.

Lemma 4.6. *For each partition $\mathcal{P}_i$, constructed as before ξ_0 is $\mathcal{P}_i$-complete.*

If at this point we have obtained a trivial partition we are done. But sometimes elements may be separated by big clusters with little kinetic energy. At this point we need to use the properties of the good configurations in $\widehat{\Omega}_{\Lambda,\,\varepsilon}$. The idea is that we will do further transformations that will produce somewhere a high energy two–particle cluster, i.e. a two particle cluster with enough kinetic energy to go anywhere in Λ.

Definition 4.7. *We will call "free two particle cluster" any two particles at a distance less than R_1 such that their "available kinetic energy"*[18] *is larger than $\varepsilon^{-1}|\Lambda|^{\frac{1}{2}}V_0$.*

The justification of the name lies in the following:

Lemma 4.8. *If a configuration $\xi \in \widehat{\Omega}_{\Lambda,\,\varepsilon}$ contains a free two particles cluster, then, in a neighborhood of ξ μ is Microcanonical.*

Proof.

Since $\xi \in \widehat{\Omega}_{\Lambda,\,\varepsilon}$ it follows that a free two particles cluster has enough available energy to be moved anywhere in the box Λ. Accordingly we can produce allowed deformations that bring the free two particle cluster in contact with any given particle in Λ and the result follows from the repeated use of Proposition 4.4. $\square$

Consider a sequence of concentrical boxes Λ_k, $k \geq 1$, such that $\Lambda_{k+1} \supset \Lambda_k$ and $BR_1 < \mathrm{dist}(\partial\Lambda_k, \partial\Lambda_{k+1}) < (B+1)R_1$ for some fixed B large enough. Without loss of generality, we can assume that there exists a constant C such that $\Lambda_{CL} = \Lambda$. Given a partition $\mathcal{P}$, we will call an element "large" if it contains at least one particle belonging to $\Lambda_{CL} \backslash \Lambda_{CL-1}$.

As a result of our previous construction we are now in a situation in which all the particles belonging to Λ_1 (in the configuration ξ_0) must either be part of a large element of the final partition $\mathcal{P}_i$ or constitute a one particle cluster. To see this suppose that there is an element, not large, containing more than two particles, with one particle in Λ_1. Since it

[18] *By "available energy" of a group of m particles, with total momentum Π, we mean the maximal amount of kinetic energy that can be liberated and transferred to two particles α, β, i.e. $\sum_\gamma \phi(p_\gamma) - m\phi(\frac{1}{m}\Pi)$.*

is not large it does not touch the boundary so it is possible to extract a two particle cluster from its convex hull. Since we have already played this game it must be that there are no particles outside the convex hull of the element, otherwise one could obtain a coarser partition by using the above mentioned strategy. Accordingly, any other particle must be contain in the convex hull of such element. Suppose now that there exists another element containing more than one particle, consider the cluster belonging to such an element closer to the element under consideration, such cluster can clearly be translated to touch our element, forming in such a way a new coarser partition with respect to which ξ_0 will be certainly complete. The only alternative is that all the other clusters consist of only one particle, but our bound on the density would imply that our element contains at least $a(\frac{L}{2})^3$ particles, and therefore enough available kinetic energy to form a free two particle cluster (see lemma 3.2).

Therefore, all the elements containing more than one particle of whose at least one in Λ_1 must be large in the final $\mathcal{P}_i$ obtained with the previous construction. Hence, at least $a(\frac{L}{2})^3$ of the particles belong to large elements (the other $\rho_*(\frac{L}{2})^3$ of Λ_1 may be isolated one particles clusters).

Another interesting property of large elements is that they must contain at least one particle in each region $\Delta_k \equiv \Lambda_k \backslash \Lambda_{k-1}$. Otherwise, one can extract a two particle cluster from the element at move the two particles into the region Δ_k, since the partition is already the coarser obtainable with such deformations it is not possible that the two particles can be brought in contact with a new particle, i.e. no particle can be contained in Δ_k. This implies immediately that no other cluster with more than two particles can be present in Λ_{k_1} (again, if this would not be the case one could extract two particles from the other element and move them to the element into consideration), so the element would have enough energy to create a free two particle cluster.

It is not obvious that one can continue to extract particles form the cluster to join different elements once the partition has no longer clusters that do not extend to the boundary. In the following we will show that something can still be done.

Definition 4.9. *We call a large element "fat" if, for each region $\Delta_k = \Lambda_k \backslash \Lambda_{k-1}$, $k \in \{1, ..., CL\}$, the cluster has at least $L^{\frac{3}{4}}$ particles in Δ_k. A non-fat large element will be called "thin."*

If at least $a(\frac{L}{2})^3$ belong to fat elements then they will have at their disposal at least $\varepsilon a(\frac{L}{2})^3$ available energy. On the other hand a fat element must contain at least $\frac{(\frac{L}{2})^{\frac{7}{4}}}{10R_1}$ particles, so there can be at most $10R_1(\frac{L}{2})^{\frac{5}{4}}(\rho_* + a)$ fat elements. Hence, at least one fat element will enjoy $\frac{a\varepsilon}{16R_1(\rho_*+a)}(\frac{L}{2})^{\frac{7}{4}}$ kinetic energy. Taking L large enough ($L > \varepsilon^{-8}$) we can produce a free two particle cluster.

If this is not the case then the total number of particles belonging to thin elements is bigger than $a(\frac{L}{2})^3$, then the available energy at their disposal is, at least, $\varepsilon a(\frac{L}{2})^3$. But a thin element is large, hence it contains at least $\frac{L}{BR_1}$ particles. This implies that there are, at most, $\frac{nBR_1}{L}$ thin elements. Hence, at least one of them will have more than $\frac{\varepsilon a L^4}{2nBR_1} > \frac{a}{2(\rho+a)BR_1}L\varepsilon$ available energy at its disposal. Accordingly, we have enough available kinetic energy so that one can construct a deformation that extracts two particles from the element in the region Δ_k containing less than $L^{\frac{3}{4}}$ particles. In fact, the element in $\Lambda_k \backslash \Lambda_{k_1}$ can invade at most a volume $\frac{4\pi}{3} R_1^3 L^{\frac{3}{4}}$ while the available volume is at least $30 R_1 L^2$, so there are both room and energy to extract two particles. We proceed to such an extraction in any direction and we call ξ_{i+1} the configuration so obtained. It is important to notice that ξ_{i+1} has a lover local density that ξ_0 and more available energy, moreover the densities both in Λ_1 and Λ are not changed, so ξ_{i+1} is still in $\hat{\Omega}_{\varepsilon,\Lambda}$. It is easy to convince oneself that ξ_{i+1} is still $\mathcal{P}_i$ complete. If such two particles cannot touch any other element while moving in the region Δ_k, this means that the element under consideration is the only large element, then the energy at its disposal is $\varepsilon a(\frac{L}{2})^3$, sufficient to create a free two particle cluster. Otherwise we obtain a new partition $\mathcal{P}_{i+1}$ where ξ_{i+1} is complete (see appendix II again). This shows that it is possible to eliminate progressively the thin elements until we reach a configuration in which their density is sufficiently low and the fat elements contain more than $a(L/2)^3$ particles.

And this conclude the proof of proposition 4.1. It suffices to apply the the previous discussion to each point in $\Xi \cap \widehat{\Omega}^{\Lambda,\varepsilon}$, accordingly in the neighborhood of each point μ is proportional to the Microcanonical measure. This implies also that conditioning on the positions and applying the argument illustrated at the beginning of section 4, follows that the conditional measure is constant on the surfaces of constant total momenta and energy.

5. PROOF OF THEOREM 1

The conclusion of the previous section is summarized by the following lemma:

Lemma 5.1. *For almost every configuration of the positions ω_q and any Λ_0, the conditional measure Q on p_{Λ_0} given ω_q and*

$$\sum_{q_j \in \Lambda} \phi(p_j) = const$$

$$\sum_{q_j \in \Lambda} p_j = const$$

is the Microcanonical measure on the corresponding surface.

At this point we are in the same situation as in [OVY] (after lemma (4.5) there). In fact, as a consequence of the previous lemma, the distribution of the momentum conditioned on the positions is given by a convex combination of measures of the form

$$\pi(dp \,|\, \Lambda) = \exp\Big[\sum_{i=1}^{3} \sum_{\alpha} \lambda_i p_\alpha^i - \lambda_4 \sum_{\alpha} \phi(p_\alpha)\Big] \,/\, \text{Normalization.}$$

Lemma 5.2. *For any configuration $\omega = \{(q_\alpha, p_\alpha)\}$, let $\vec{z}(\omega)$ be the density, momenta and* *kinetic energy associated with the configuration defined by*

$$z^0(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{\alpha=1}^N \chi(\delta q_\alpha)$$

$$z^\mu(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{\alpha=1}^N \chi(\delta q_\alpha) p^\mu_\alpha(\omega) \ , \ \ \mu = 1,2,3$$

$$z^4(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{\alpha=1}^N \chi(\delta q_\alpha)\, \phi(p_\alpha) \ .$$

Here χ is a cutoff function of total integral one, $\vec{z}^\mu(\omega)$ exist almost everywhere and are independent of the cutoff χ. Furthermore, $\vec{z}(\omega)$ are constants of the motion for L in the sense that

$$\int h(\vec{z}(\omega)\,)\, LF(\omega)\, dQ = 0,$$

for all local smooth functions F and all smooth functions h with compact support.

Proof.

This was proven in [OVY] for bounded ϕ'. For completeness, we present here the proof for unbounded ϕ'.

By the same argument used immediately after lemma (1.1) these limits clearly exist and are independent of the cutoff χ.

By condition (iii) in theorem (1.1)

$$\begin{aligned} 0 &= \int L(Fh(z^\mu_{\chi,\delta}(\omega)\,)\,)\, dQ \\ &= \int (LF)\, h(z^\mu_{\chi,\delta}(\omega)\,)\, dQ + \int F\, L\, h(z^\mu_{\chi,\delta}(\omega)\,)\, dQ \ . \end{aligned}$$

The first term converges to $\int h(z^\mu_\chi(\omega)) LF\, dQ$ as $\delta \to 0$. We only have to show that the second term converges to zero as $\delta \to 0$. Clearly, it suffices to show that as $\delta \to 0$

$$\int |L z^\mu_{\chi,\delta}|\, dQ \to 0 \ , \qquad \mu = 0, \ldots, 4 \ . \tag{5.1}$$

This is easy to show for $\mu = 0,1,2,3$ (as in [OVY] pag. 544).

For $\mu = 4$ we have

$$\begin{aligned} I\!\!E^Q\left(|Lz^4_{\chi,\delta}|\right) =& I\!\!E^Q\left(|\delta\, \delta^3 \sum_{i,\alpha} \chi_i(\delta q_\alpha)\, \phi_i(p_\alpha)\, \phi(p_\alpha)| \right) \\ &+ I\!\!E^Q\left(|\delta^3 \sum_{\alpha\neq\beta} \sum_i \chi(\delta q_\alpha)\, \phi_i(p_\alpha)\, V_i(q_\alpha - q_\beta)| \right) . \end{aligned}$$

Only the second term of the right end side present difficulties. Let $w_i(\vec{z})$ and $\sigma_i(\vec{z})$ denote the expectation and variance of $\phi_i(p_\alpha)$ with respect to Q conditioned on $\vec{z}$. These can be computed explicitly by using the characterization of the conditional measure given ω_q and $\vec{z}$. We can bound the second term of the RHS of the above expression by

$$
\begin{aligned}
&\mathbb{E}^{Q}\left(|\delta^3 \sum_i \sum_{\alpha\neq\beta} \chi(\delta q_\alpha)\, \phi_i(p_\alpha)\, V_i(q_\alpha - q_\beta)| \right) \\
&= \mathbb{E}^{Q}\left(|\delta^3 \sum_i \sum_{\alpha} \chi(\delta q_\alpha) \left[\phi_i(p_\alpha) - w_i\right] \sum_{\beta\neq\alpha} V_i(q_\alpha - q_\beta)| \right) \\
&\quad + \mathbb{E}^{Q}\left(|\delta^3 \sum_{\alpha\neq\beta} \sum_i \chi(\delta q_\alpha)\, V_i(q_\alpha - q_\beta)\, w_i| \right).
\end{aligned}
$$

The second term of the RHS (third line above) can be bounded as before. Using the Schwarz inequality the first term can be bounded by

$$
\begin{aligned}
&\sum_i \mathbb{E}^{Q}\left(\mathbb{E}^{Q}\left(\left(\left[\delta^3 \sum_\alpha \chi(\delta q_\alpha) \left[\phi_i(p_\alpha) - w_i\right] \sum_{\beta\neq\alpha} V_i(q_\alpha - q_\beta) \right]^2 \Big| \, \vec{z} \right) \right)^{1/2} \right) \\
&= \sum_i \mathbb{E}^{Q}\left(\sqrt{\sigma_i(\vec{z})}\, \delta^3\, \mathbb{E}^{Q}\left(\sum_\alpha \chi(\delta q_\alpha)^2 \left(\sum_\beta V_i(q_\alpha - q_\beta) \right)^2 \Big| \, \vec{z} \right)^{1/2} \right) \\
&\leq \sum_i \mathbb{E}^{Q}(\sigma_i(\vec{z}))^{1/2} \delta^3 \mathbb{E}^{Q}\left(\sum_\alpha \chi(\delta q_\alpha)^2 \left(\sum_{\beta\neq\alpha} V_i(q_\alpha - q_\beta) \right)^2 \right)^{1/2}.
\end{aligned}
$$

By the condition on ϕ and the entropy argument we have that $\mathbb{E}^{Q}(\sigma_i(\vec{z}))$ is finite. To bound the second expectation, let us divide the set $\left\{x \,|\, |x| \leq 2\delta^{-1}\right\}$ into boxes of size $2R_0$ (R_0 is the range of V). Let σ index the boxes and let N_σ be the number of particles in the σ box.

$$
\delta^3 \sum_i \mathbb{E}^{Q}\left(\sum_\alpha \chi(\delta q_\alpha)^2 \left[\sum_\beta V_i(q_\alpha - q_\beta) \right]^2 \right)^{1/2} \leq \text{const. } \delta^3\, \mathbb{E}^{Q}\left(\sum_\sigma N_\sigma^3 \right)^{1/2}
$$

By convexity and the inequality $(\sum_\sigma N_\sigma^3)^{1/3} \leq (\sum_\sigma N_\sigma^2)^{1/2}$ we see that the above expression is bounded by

$$
\text{const. } \delta^3 \mathbb{E}^{Q}\left(\left[\sum_\sigma N_\sigma^2 \right]^{3/4} \right) \leq \text{ const. } \delta^3 \left[\mathbb{E}^{Q}\left(\sum_\sigma N_\sigma^2 \right) \right]^{3/4}.
$$

By lemma (1.2)(i) and the translation invariance, $I\!\!E^{Q}(\sum_\sigma N_\sigma^2)$ is bounded by δ^{-3}; hence, the quantity under consideration is bounded by const. $\delta^{3/4}$. This concludes the proof of the lemma 4.2. □

By the previous lemma Q conditioned on $\vec{z}(\omega)$ is still invariant for L. Since we assume that Q is translation invariant, we can apply lemma 4.10 in [OVY] and obtain that these conditioned distributions are given by grancanonical Gibbs measures, concluding our proof.

APPENDIX 1.

To show that our condition on the noise and the kinetic energy in the non-Gaussian case are far from empty, we give here an example of stochastic perturbation that satisfies such condition. This is the only point where we use our requirement on the form of the kinetic energy ϕ.

Lemma A.1. *If $\{\eta^\theta_{\alpha\beta}\} = \{e_1 \wedge D_{\alpha\beta}E,\, e_2 \wedge D_{\alpha\beta}E,\, e_3 \wedge D_{\alpha\beta}E\}$ and $\phi(p_a) = \sum_{i=1}^3 \varphi(p^i_\alpha)$, with $(\varphi''')^2 + \varphi^{iv}\varphi'' = 0$ at most at finitely many points, then condition on the noise is satisfied.*

Proof.

A simple computation shows

$$\begin{aligned}
[X^i_{\alpha\beta},\, L]q_\alpha =& H_\alpha \eta^i_{\alpha\beta} \\
[X^i_{\alpha\beta},\, L]q_\beta =& - H_\beta \eta^i_{\alpha\beta} \\
[X^j_{\alpha\beta},\, [X^i_{\alpha\beta},\, L]]q^l_\alpha =& (\eta^j_{\alpha\beta})_l(\eta^i_{\alpha\beta})_l H'_{\alpha ll} + H_{\alpha ll}\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)_l \\
[X^i_{\alpha\beta},\, [X^j_{\alpha\beta},\, L]]q^l_\beta =& (\eta^j_{\alpha\beta})_l(\eta^i_{\alpha\beta})_l H'_{\beta ll} - H_{\beta ll}\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)_l
\end{aligned}$$

where $H_{\alpha ll}$ stand for the element ll of the diagonal matrix H_α. The matrix H'_α is the derivative of the matrix H_α; $(\cdot)_l$ stands for the l–th component of the corresponding vectors. Now, let us take the six vectors obtained by letting i, j vary only in $\{1, 2\}$. We define the vectors w^{ij} by

$$w^{ij}_l = (\eta^i_{\alpha\beta})_l(\eta^j_{\alpha\beta})_l.$$

Let us consider

$$\sum_{i=1}^2 \mu_i[X^i_{\alpha\beta},\, L] + \sum_{i,j=1}^2 \nu_{ij}[X^j_{\alpha\beta},\, [X^i_{\alpha\beta},\, L]] = 0$$

Applying the above vector fields to q_α, q_β, we have

$$\begin{aligned}
0 =& \sum_{i=1}^2 \mu_i H_\alpha \eta^i_{\alpha\beta} + \sum_{i,j=1}^2 \nu_{ij}\left[H'_\alpha w^{ij} + H_\alpha\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)\right] \\
0 =& - \sum_{i=1}^2 \mu_i H_\beta \eta^i_{\alpha\beta} + \sum_{i,j=1}^2 \nu_{ij}\left[H'_\beta w^{ij} - H_\beta\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)\right]
\end{aligned}$$

If we multiply the first by $(H_\alpha)^{-1}$, the second by $(H_\beta)^{-1}$, and add one to the other, then we get

$$0 = \sum_{i,j=1}^{2} \nu_{ij} A w^{ij}$$

where $A = 1/2\{H'_\alpha H_\alpha^{-1} + H'_\beta H_\beta^{-1}\}$. Notice that A is invertible out of a set of codimension 1 (see later for more details), consequently

$$\begin{aligned} 0 &= \sum_{i,j=1}^{2} \nu_{ij} w^{ij} \\ 0 &= \sum_{i=1}^{2} \mu_i \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} \nu_{ij} e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta} \end{aligned}$$

To conclude we need an explicit representations of the vectors involved in the previous equations. Let $D^i_{\alpha\beta}E = \zeta_i$, $h_i = H_{\alpha ii} + H_{\beta ii}$, then a direct computation yields

$$\begin{aligned} \eta^1 &= (0,\, -\zeta_3,\, \zeta_2) \\ \eta^2 &= (\zeta_3,\, 0,\, -\zeta_1) \\ w^{11} &= (0,\, \zeta_3^2,\, \zeta_2^2) \\ w^{12} = w^{21} &= (0,\, 0,\, -\zeta_1\zeta_2) \\ w^{22} &= (\zeta_3^2,\, 0,\, \zeta_1^2) \\ e^1 \wedge (H_\alpha + H_\beta)\eta^1_{\alpha\beta} &= (0,\, -\zeta_2 h_3,\, -\zeta_3 h_2) \\ e^1 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} &= (0,\, \zeta_1 h_3,\, 0) \\ e^2 \wedge (H_\alpha + H_\beta)\eta^1_{\alpha\beta} &= (\zeta_2 h_3,\, 0,\, 0) \\ e^2 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} &= (-\zeta_1 h_3,\, 0,\, -\zeta_3 h_1). \end{aligned}$$

Immediately follows $\nu_{22} = \nu_{11} = 0$ and $\nu_{12} = -\nu_{21}$, which, substituted in the remaining equations, yields

$$\Omega \begin{pmatrix} \mu_1 \\ \mu_2 \\ \nu_{12} \end{pmatrix} = 0$$

For some matrix Ω with $\det(\Omega) = \zeta_1\zeta_2\zeta_3(h_2 + h_3)$. Since the determinant is equal zero on a set of codimension one, we have that the vector are linearly independent, out of a set of codimension one.

This set of codimension one consists of $\cup_i\{p \mid \varphi'''(p^i_\alpha)\varphi''(p^i_\alpha) = -\varphi'''(p^i_\beta)\varphi''(p^i_\beta)\}$, where the matrix A is not invertible[19], and $\cup_i\{p^i_\alpha = p^i_\beta\}$, where the matrix Ω is not

[19] The condition of the hypothesis insure that such set is a smooth codimension one manifold unless $\varphi'''(p^i_\alpha)^2 + \varphi^{iv}(p^i_\alpha)\varphi''(p^i_\alpha) = \varphi'''(p^i_\beta)^2 + \varphi^{iv}(p^i_\beta)\varphi''(p^i_\beta) = 0$, which can happen only on a set of codimension two.

invertible. To get codimension two we have to analyze all the different cases one by one, since they are treated all in the same way we will consider only the points on the set $\{p^1_\alpha = p^1_\beta\}$, and we will leave the rest to the skeptical reader. We can clearly ignore points of the above set that also belong to some other singular set: they belong to a set of codimension two. For points in the set under consider we will have $\zeta_1 = 0$, while all the other components will be different from zero. This implies that $w^{12} = w^{21} = e^1 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} = 0$, we need then to produce more vectors, i.e., compute more commutators. It turns out to be sufficient to compute

$$
\begin{aligned}
&[X^2,\, [X^1_{\alpha\beta},\, [X^2_{\alpha\beta},\, L]]]q_\alpha = H_\alpha v_1 + H'_\alpha v_2\\
&[X^2,\, [X^1_{\alpha\beta},\, [X^2_{\alpha\beta},\, L]]]q_\beta = -H_\beta v_1 + H'_\beta v_2\\
&[X^2,\, [X^2_{\alpha\beta},\, [X^1_{\alpha\beta},\, L]]]q_\alpha = H'_\alpha v_2\\
&[X^2,\, [X^2_{\alpha\beta},\, [X^1_{\alpha\beta},\, L]]]q_\beta = H'_\beta v_2
\end{aligned}
$$

where $v_1 = (0,\, \zeta_3 h_1 h_3,\, 0)$, and $v_2 = (0,\, 0,\, -\zeta_2\zeta_1 h_1)$. We have then to study the linear combination

$$
\sum_{i=1}^{2}\mu_i[X^i_{\alpha\beta},\, L] + \sum_{i,j=1}^{2}\nu_{ij}[X^j_{\alpha\beta},\, [X^i_{\alpha\beta},\, L]] + \varepsilon_1[X^2,\, [X^1_{\alpha\beta},\, [X^2_{\alpha\beta},\, L]]] \\
+ \varepsilon_2[X^2,\, [X^2_{\alpha\beta},\, [X^1_{\alpha\beta},\, L]]] = 0
$$

where ν_{12}, ν_{21} are taken to be zero since the corresponding commutators, when restricted to the q_α, q_β space, would not contribute anything of interest. As before, we apply the vectors to the coordinates q_α, q_β, we multiply by H_α^{-1} and H_β^{-1} and add the corresponding equations, in so doing we obtain

$$
\sum_{i,j=1}^{2}\nu_{ij}w^{ij} + (\varepsilon_1 + \varepsilon_2)v_2 = 0
$$

from this follows immediately $\nu_{11} = \nu_{22} = 0$, $\varepsilon_1 = -\varepsilon_2$. Substituting in the original equation we get

$$
0 = \sum_{i=1}^{2}\mu_i H_\alpha \eta^i_{\alpha\beta} + \varepsilon_1 H_\alpha v_1
$$

which implies $\mu_i = \varepsilon_i = 0$ on a set of codimension two. □

APPENDIX II

We will prove here that if ξ is $\mathcal{P}$–complete, and two particle can be extracted from an element P_1 to join P_2 (or viceversa), then ξ is complete for the partition $\mathcal{P}_*$ obtained from $\mathcal{P}$ joining P_1 and P_2.

Choose $\eta \in \Pi(\xi, \mathcal{P}_*)$. Call α, β the two particles that are allowed to move along γ.

The rough idea is to transfer energy and momentum between the elements[20] P_1 and P_2 by using the particles α, β. Unfortunately, there are limits to how much momentum or energy we can transfer to a particles, due to the necessity to conserve the total energy and momentum of the clusters. To overcome this we will show that each $\eta \in \Pi(\xi, \mathcal{P}_*)$ can be deformed into the special configuration $\zeta \in \Pi(\xi, \mathcal{P}_*)$ defined by,[21]

$$
\begin{aligned}
p_\sigma &= \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} \quad \forall \sigma \notin \{\alpha, \beta\} \\
p_\alpha &= \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} + \lambda v \\
p_\beta &= \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} - \lambda v,
\end{aligned}
$$

with some fixed $v \in I\!R^3$, $\|v\| = 1$, and λ determined by^{2}0

$$K(\xi, P_1 \cup P_2) = [\#(P_1 \cup P_2) - 2]\phi\left(\frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)}\right) + \phi(p_\alpha) + \phi(p_\beta).$$

The desired allowed transformation will then be obtained by deforming ξ into ζ and then by running backward the allowed transformation that connects η to ζ (since the reverse of an allowed transformation it is still an allowed transformation).

Since, by convexity, $K(\xi, P_1 \cup P_2) \geq \#(P_1 \cup P_2)\phi\left(\frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)}\right)$, if $\lambda = 0$ then $\Pi(\xi, \mathcal{P}_*)$, restricted to the particles in $P_1 \cup P_2$ consists of only the point ξ and we have nothing to prove. Otherwise we proceed as follows: we make an allowed deformation that set all the moments in $P_1 \backslash \{\alpha, \beta\}$ equal to $\frac{1}{\#P_1}\pi(\xi, P_1)$ while $p_\alpha = \frac{1}{\#P_1}\pi(\xi, P_1) + \nu_1 v$ and $p_\beta = \frac{1}{\#P_1}\pi(\xi, P_1) + \nu_1 v$, and ν_1 is determined by the conservation of $K(\xi, P_1)$. Then we move the coordinates of the particles α, β accordingly to γ but without changing their momenta. Once they get in touch with P_2 we change the momenta of the particles in P_2 to

$$p_* = \frac{1}{\#P_2 + 2}(\pi(\xi, P_2) + \frac{2}{\#P_1}\pi(\xi, P_1)),$$

apart from $p_\alpha = p_* + \nu_2 v$ and $p_\beta = p_* - \nu_2 v$, again ν_2 is determined by the conservation of the kinetic energy of the new cluster $P_2 \cup \{\alpha, \beta\}$. Finally, we move back the particles α, β to their original position in the configuration ξ and share again their momentum among all the particles in P_1 has we have done at the beginning. Let us call $\xi_{1,1}$ the configuration reached in such a way. Calling $\delta_0 = \frac{1}{\#P_1}\pi(\xi, P_1) - \frac{1}{\#P_2}\pi(\xi, P_2)$ and $\delta_1 = \frac{1}{\#P_1}\pi(\xi_{1,1}, P_1) - \frac{1}{\#P_2}\pi(\xi_{1,1}, P_2)$ a direct computation shows that

$$\delta_1 = \left(1 - \frac{2\#(P_1 \cup P_2)}{\#P_1(\#P_2 + 2)}\right)\delta_0.$$

[20] Note that $\{\alpha, \beta\} \subset P_1$ and that in the configuration ξ P_1 still form an element.

[21] For each $P \subset \{1, ..., n\}$, by $\pi(\xi, P)$ and $K(\xi, P)$ we mean, respectively, the total momentum and kinetic energy, in the configuration ξ, of the particles belonging to P; by $\#P$ we mean the cardinality of the set P.

If we iterate further the procedure just described we see that the difference between the average momentum in P_1 and P_2 goes to zero, this shows that we are getting closer and closer to the configuration ζ; unfortunately only asymptotically. Nevertheless, after a finite number of iterations we will get to a configuration ζ_0 for which

$$2\phi\left(\frac{\pi(\zeta_0,\, P_1)}{2} - \frac{(\#P_1-2)\pi(\xi,\, P_1\cup P_2)}{2\#(P_1\cup P_2)}\right) + (\#P_1-2)\phi\left(\frac{\pi(\xi,\, P_1\cup P_2)}{\#(P_1\cup P_2)}\right) < K(\zeta_0,\, P_1). \tag{A2.1}$$

Let $p_\sigma(\eta)$ be the momentum of the particle σ in the configuration η. We deform ζ_0 into ζ_1 defined by

$$\begin{aligned} p_\sigma(\zeta_1) =& \frac{\pi(\xi,\, P_1\cup P_2)}{\#(P_1\cup P_2)} \quad \text{for } \sigma\in P_1\backslash\{\alpha,\, \beta\} \\ p_\alpha(\zeta_1) =& \frac{\pi(\xi,\, P_1\cup P_2)}{\#(P_1\cup P_2)} + \frac{1}{2}\left[\pi(\zeta_0,\, P_1) - \#P_1\frac{\pi(\xi,\, P_1\cup P_2)}{\#(P_1\cup P_2)}\right] + \nu v \\ p_\beta(\zeta_1) =& p_\alpha(\zeta_1) - 2\nu v, \end{aligned}$$

where ν is defined by $K(\zeta_1,\, P_1) = K(\zeta_0,\, P_1)$. All this is possible provided (A2.1) is satisfied; in fact, (A2.1) express simply that there is sufficient energy to deform the momenta of the particles in P_1 to the above values. After achieving the configuration ζ_1, to obtain the configuration ζ it suffices to take the particles α, β to P_2, adjust the momenta of the particles of P_2 to the value $\frac{\pi(\xi,\, P_1\cup P_2)}{\#(P_1\cup P_2)}$, which will make all the momenta agree with the ones in the configuration ζ and take $\{\alpha,\, \beta\}$ back to their original position in ξ.

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