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Asymmetric conservative processes with random rates

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Summary. We study a one dimensional nearest neighbor simple exclusion process for which the rates of jump are chosen randomly at time zero and fixed for the rest of the evolution. The i-th particle's right and left jump rates are denoted p_i and q_i respectively; $p_i + q_i = 1$. We fix $c \in (1/2, 1)$ and assume that $p_i \in [c, 1]$ is a stationary ergodic process. We show that there exists a critical density ρ^* depending only on the distribution of $\{p_i\}$ such that for almost all choices of the rates and a (fixed) density $\rho^* < \rho \le 1$ there exists an invariant distribution for the process as seen from a tagged particle with asymptotic density ρ. Under this measure, the distribution of the distances between particles are independent random variables. We also show that under the invariant distribution, the position X_t of the tagged particle at time t can be sharply approximated by a Poisson process. Finally, we prove the hydrodynamical limit for zero range processes with random rate jumps.

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1. Introduction.

Let $c \in (1/2, 1)$ and $p = \{p_i \in [c, 1] : i \in \mathbb{Z}\}$ be a stationary ergodic process on $[c, 1]^{\mathbb{Z}}$. Let m be their joint distribution. For each random choice of p we consider an asymmetric simple exclusion process whose particles move with rates p. In fact we consider the simple exclusion process as seen from a tagged particle. The tagged particle is labeled zero and the other particles labeled in an increasing way according to their position. In words, the i-th particle attempts to jump to the nearest site to the right at rate p_i and performs the jump if the destination site is empty, otherwise it stays still. At rate $q_i = 1 - p_i$, the i-th particle does the same to the left. Let $x_i(t)$ be the position of the i-th particle at time t. Let η_t be the set of occupied sites at time t; hence $\eta_t = \{x_i(t) : i \in \mathbb{Z}\}$. The configuration η can be interpreted as a function from $\mathbb{Z}$ in $\{0,1\}$ given by $\eta(x) = 1\{x = x_i, \text{ for some } i\}$. The generator of the process seen from the tagged particle depends on the choice of p and is given by

$$
\begin{aligned}
L_p f(\eta) = \sum_{i\neq 0} \big(& 1\{x_{i+1} > x_i + 1\} p_i [f(\eta^{x_i, x_i+1}) - f(\eta)] \\
& + 1\{x_{i-1} < x_i - 1\} q_i [f(\eta^{x_i, x_i-1}) - f(\eta)]\big) \\
& + 1\{x_1 > 1\} p_0 [f(\tau_1 \eta^{0,1}) - f(\eta)] \\
& + 1\{x_{-1} < -1\} q_0 [f(\tau_{-1} \eta^{0,-1}) - f(\eta)]
\end{aligned}
\tag{1.1}
$$

where τ_x is the translation by x: $\tau_x \eta(z) = \eta(z - x)$, f is a local function and

$$
\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y \\ \eta(x) & \text{if } z = y \\ \eta(y) & \text{if } z = x \end{cases}
$$

is the configuration where the values at the sites x and y have been interchanged.

The existence of this process follows either from the Hille-Yosida Theorem as in Liggett (1985) and Andjel (1982) or using the graphical construction of Harris. See Ferrari (1992) for instance.

For a measure μ on $\{0,1\}^{\mathbb{Z}}$ let $\lim_{n\to\infty}(2n+1)^{-1} \int d\mu(\eta) \sum_{x=-n}^{n} \eta(x)$ be its global density (if the limit exists). In Section 2 we show that there exists a critical density ρ^* such that for any density $\rho \in (\rho^*, 1]$ and almost all choices of the rates p, there exists an invariant measure with global density ρ. ρ^* might be strictly positive as is shown by an example and there are no invariant product measure for densities in $[0, \rho^*)$. We then show that for any given ρ and for almost all choice of the rates, the position X_t of the particle originally at the origin is given by a Poisson process of rate $v(\rho, p)$ plus an error of order one. The "effective velocity" $v(\rho, p)$ is given by an explicit equation.

In the simple exclusion process as seen from a tagged particle with $p_i \equiv p_0$, the only extremal invariant measure with global density zero is the measure concentrating mass in the configuration with a single particle at the origin. On the other hand, if one starts the system with an ergodic measure such that the inter particle distances have a translation invariant distribution and the mean distance between two successive particles is infinite, then the process converges weakly to the configuration with a single particle at the origin (Andjel, Cocozza-Thivent and Roussignol (1986)). Based in this result we conjecture that in our case there are no extremal invariant measures with global density zero concentrating mass in the set of configurations with infinitely many particles. On the other hand, we show that there are measures concentrating mass in sets of configurations with n particles, for all n. We also study the process with initial configuration $\dots 111000 \dots$. In this case assuming that $m([c, c+\varepsilon]) > 0$ for all $\varepsilon > 0$, we show that the i-th particle to the left has an asymptotic velocity that converges as $i \to \infty$ to $2c - 1$.

When the rates are associated to the sites instead of the particles our result is weaker: we show that starting with the homogeneous product distribution ν_ρ, for almost all choices of the environment any tagged particle has a velocity at most $(2c-1)(1-\rho)$ almost surely.

In the last two sections we turn to the study of the hydrodynamical behavior of the process with random rates in infinite volume. We first change coordinates and denote by $\xi_t(k)$ the number of holes at time t between particle k and particle $k+1$. This transform the asymmetric simple exclusion process to a so–called zero range process. It is easy to check that ξ_t evolves as follows. At rate $\mathbf{1}\{\xi(k) \geq 1\} q_{k+1}$ one particle jumps from site k to site $k+1$ and at rate $\mathbf{1}\{\xi(k) \geq 1\} p_k$ one particle jumps from site k to site $k-1$. Under some restrictions on the initial measure, we prove that m almost surely the macroscopic behavior of this process is described by the entropy solution of a first order hyperbolic equation $\partial_t \rho + \partial_x v(\rho) = 0$ where $v(\rho)$ denotes the mean velocity of a particle for a process in equilibrium with density ρ.

In the last section we consider the hydrodynamical behavior of another type of zero range processes in random environment. This system corresponds to asymmetric simple exclusion processes where the rate at which the k-th particle jumps to the right and the rate at which the $k+1$-th particle jumps to the left is decelerated by a factor p_k picked from an ergodic stationary process.

2. Asymmetric exclusion with random rates associated to particles.

Without loss of generality we shall asume throughout this section that c is such that

$$m([c, c+\epsilon)) \; > \; 0 \quad \text{for all} \quad \epsilon \; > \; 0 \; . \tag{2.1}$$

For an integer i, denote by Z_i the (eventually infinite) function of p:

$$Z_i(p) \;=\; \frac{1}{q_{i+1}} \sum_{j=0}^{\infty} \prod_{k=i-j}^{i} \frac{q_{k+1}}{p_k} \,.$$

Let v^* the maximum value of v for which $Z_i < (1/v)$ almost surely for all $i \in \mathbb{Z}$:

$$v^* \;=\; \sup\{v > 0 : m(vZ_i < 1) = 1\}$$

Notice that $v^* \geq 2c - 1$.

Define the function $R : [0, v^*) \to \mathbb{R}_+$, by

$$R(w) = \left(m\left[(1 - wZ_i)^{-1}\right]\right)^{-1} \tag{2.2}$$

that does not depend on i as p is a stationary process. Notice that by the definition of v^*, the function R is well defined and positive in $[0, v^*)$. Moreover, $R(0) = 1$ and R is clearly decreasing in w. In particular it is invertible. Let v denote the inverse function: $v(R(w)) = w$. Denote by $R(v^*) \geq 0$ the limit of $R(w)$ as $w \uparrow v^*$.

Theorem 2.3. *For any $\rho \in [R(v^*), 1]$ and for almost all choice of p: (i) There exists a unique vector $\alpha(= \alpha(p, \rho)) = \{\alpha_i : i \in \mathbb{Z}\}$, such that for all i,*

$$\begin{aligned} &\alpha_i p_i - \alpha_{i-1} q_i = v(\rho) \\ &\alpha_i \in [0, 1). \end{aligned} \tag{2.4}$$

(ii) The measure ν_α defined by

$$\nu_\alpha(x_{i+1} - x_i = k_i : i \in A) = \prod_{i \in A} \alpha_i^{k_i - 1}(1 - \alpha_i), \quad A \subset \mathbb{Z}, \quad k_i \geq 1, \text{ finite} \tag{2.5}$$

with $\nu_\alpha(x_0 = 0) = 1$ is invariant for the process with generator L_p. (iii) The global density of this measure is ρ:

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{x=-n}^{n} \eta(x) = \rho \tag{2.6}$$

ν_α almost surely.
(iv) If $\rho < R(v^*)$, there are no invariant measures for L_p with global density ρ for which $\{x_i - x_{i-1}\}_i$ are independent.

Remarks. If the vector p is constant, that is $p_i \equiv c$, then $v(\rho) = (2c-1)(1-\rho)$ and $\rho^* = 0$. On the other hand, this theorem implies that there are product invariant measures with global densities ρ *only* for $\rho > \rho^* = R(v^*)$. The critical density ρ^* may be strictly positive as is shown by an example below. It is not discarded that it could be invariant measures that are not product with global densities below ρ^*. It is an open problem to describe in some sense the evolution of the process starting from the Bernouilli product measure with density below the critical density.

Proof. Fix $0 \le v < 2c-1$. We consider the (non homogeneous) dynamical system a_i with state space $[0,1]$ defined by

$$a_i = \frac{v + a_{i-1}(1-p_i)}{p_i} := T_i(a_{i-1}) \ .$$

Of course T_i depends on p_i. In order to argue that the dynamical system a_i, $i \in \mathbb{Z}$ has a unique solution satisfying (2.4) we consider a coupling (a_i, b_i) between two realizations of the process starting from some j with different initial conditions a_j and b_j. To realize the coupling we use the same p_i for both processes and obtain for $i \ge j$,

$$b_i - a_i = \frac{(b_{i-1} - a_{i-1})(1-p_i)}{p_i} \le (b_j - a_j)\left(\frac{1-c}{c}\right)^{i-j} \tag{2.7}$$

From (2.7) we learn two things. First, that the process is attractive: $b \ge a$ implies $T_i b \ge T_i a$ for any i. This implies that if we consider $a_j = 0$ and $b_j = 1$, then for any $c_0 \in [0,1]$, it holds $a_i \le c_i \le b_i$ for $i \ge 0$. Second, for any initial condition $a_j < b_j$, the difference $b_i - a_i$ converges as $i \to \infty$ exponentially fast to zero. The rate of convergence is bounded above by $(1-c)/c < 1$ because $c > 1/2$. On the other hand, (2.7) implies that if we fix the sequence p and call $\alpha_i(n)$ the $(n+i)$-th iterate of the chain when $\alpha_{-n} = 1$:

$$\alpha_i(n) = T_{i-1}T_{i-2}\dots T_{-n}(1)$$

for $i > -n$. Then $\alpha_i(n)$ is a non decreasing sequence in n and it converges exponentially fast, as $n \to \infty$, to a limit α_i. We have proved that the double infinite sequence α is the unique solution of (2.4). This unique solution is given by

$$\alpha_i(p,v) = \alpha_i(p) = \frac{v}{q_{i+1}} \sum_{j=0}^{\infty} \prod_{k=i-j}^{i} \frac{q_{k+1}}{p_k} \ . \tag{2.8}$$

Recall the definition of v^*. From this explicit formula for α_i, we see that we may extend the domain of v for which there exist a solution of (2.4) to the set $[0, v^*)$.

Stationarity of $\{p_i\}$ implies that the process $\{\alpha_i\}$ is stationary.

The easiest way to show (ii) is to show

$$\nu_\alpha(gL_pf) = \nu_\alpha(fL_p^*g)$$

for f, g in a core of L, where the (guessed) adjoint generator L_p^* is defined by $L_p^* = L_{p^*}$, with

$$p_i^* = \frac{q_i\alpha_{i-1}}{\alpha_i}, \quad q_i^* = \frac{p_i\alpha_i}{\alpha_{i-1}}$$

It is sufficient to take f and g of the form $1\{x_{i+1} - x_i = k_i : i \in A\}$, with $k_i \ge 1$ and A a finite subset of $\mathbb{Z}$. Since the measure is product when applied to this kind of functions, the problem is reduced to show that for all i and all $k_i, k_{i-1} \ge 0$,

$$p_i\nu_\alpha(x_{i+1}-x_i = k_i, x_i-x_{i-1} = k_{i-1}+1) = q_i^*\nu_\alpha(x_{i+1}-x_i = k_i+1, x_i-x_{i-1} = k_{i-1})$$

and

$$p_i^* + q_{i+1}^* = 1$$

that immediately hold using (2.4). The invariance of ν_α has been proven by Jackson (1963) for finite systems and by Andjel (1982) for infinite systems. The proof using the adjoint can be found in Kipnis (1986) for the exclusion process with constant rates and (for a more complicate model) in Ferrari and Fontes (1994a).

To show (iii) we first look for the solution of (2.4). For a fixed realization p the (unique) solution of the system (2.4) is given by (2.8). Since p is ergodic, so must be $\{a_i\}$ and calling $mh(a_i) = \int m(dp)h(a_i(p))$,

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \frac{1}{1-a_k} = m(1/(1-a_i)), \quad m \text{ almost surely.} \tag{2.9}$$

that by stationarity does not depend on i. Call $m\nu_\alpha$ the measure on $\{0,1\}^{\mathbb{Z}}$ obtained by first choosing a p according to m and then for $i \in \mathbb{Z}$ choosing $x_{i+1} - x_i$ with geometric distributions as in (2.5) with $\alpha = \alpha(p)$. It is immediate that $m\nu_\alpha((x_{i+1} - x_i)^2) - (m\nu_\alpha(x_{i+1} - x_i))^2 <$ constant and $m\nu_\alpha(x_{i+1} - x_i) = m(1/(1-\alpha_i))$. Hence,

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} (x_{i+1} - x_i - m(1/(1-\alpha_i)) = 0 \quad m\nu_\alpha, \text{ almost surely.}$$

(See exercice 11 of page 314 of Grimmett and Stirzaker (1992) for instance.) which in turn implies again by ergodicity,

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{x=-n}^{n} \eta(x) = \frac{1}{m(1/(1-\alpha_i))}. \quad m\nu_\alpha, \text{ almost surely}$$

To show (iv) we first observe that invariant measures for which $\{x_i - x_{i-1}\}$ are independent must be product of geometric as in (2.5). This follows from direct computation with the generator. On the other hand, if the measure concentrates in configurations with infinitely many particles, the probability α_i that particles i and $i+1$ are at distance bigger than one must satisfy equation (2.4) for some $v \in [0, v^*)$. The global density of this measure is $R(v)$ that is necessarily bigger than ρ^*. □

We proved in Theorem 2.3 that there exist product invariant measures for all densities ρ greater than a critical density $\rho^* = R(v^*)$. We present below an example showing that the critical density ρ^* may be strictly positive.

Example. The independent case. Assume that p_i are i.i.d. random variables taking only two values c and 1 : $m[p_0 = c] = \theta = 1 - m[p_0 = 1]$. For each integer i, denote by T_i the distance from i to the first integer j on the left of i with $p_j = 1$:

$$T_i \;=\; \min\Big\{ j \geq 0;\; p_{i-j} = 1 \Big\}\;.$$

From formula (2.4) it is easy to see that α_i may be expressed as a simple function of T_i :

$$\alpha_i \;=\; \frac{v}{2c-1}\Big\{ 1 - 2(1-c)\left(\frac{1-c}{c}\right)^{T_i} \Big\}\;.$$

Since $c > 1/2$ and T_i may assume with positive probability large values, $v^* = 2c-1$. Furthermore, since T_i has a geometric distribution ($i.e.$, $m(T_i \geq k) = \theta^k$, $k \geq 0$), $[R(v)]^{-1}$ can be computed explicitely:

$$\frac{1}{R(v)} = m\left(\frac{1}{1-\alpha_i}\right) = \sum_{k\geq 0} (1-\theta)\theta^k \left[1 - \frac{v}{2c-1}\left(1 - 2(1-c)\left(\frac{1-c}{c}\right)^k \right) \right]^{-1}\;.$$

This is positive for all $v \leq 2c-1$. In the limit as $v \to 2c-1$ we get

$$\lim_{v\to 2c-1} [R(v)]^{-1} = \frac{1-\theta}{2(1-c)} \sum_{k\geq 0} \left[\frac{\theta c}{1-c}\right]^k \;<\; \infty$$

provided $\theta < (1-c)/c$. Hence, if $\theta < \frac{1-c}{c}$ we can compute the sum and obtain

$$\rho^* = \frac{2(1-c(1+\theta))}{1-\theta}$$

In this example therefore there are invariant product measures with global density ρ, only for $\rho \geq R(2c-1) = \dfrac{2(1-c(1+\theta))}{1-\theta} > 0$ for $\theta < \frac{1-c}{c}$. Figures 1 and 2 ilustrate this point. In particular we can think that if we perturb an homogeneous system with constant rates $p_i \equiv 1$, in such a way that a small fraction θ of the p_i's are modified to be c, then this perturbed system will not have product invariant measures with small global densities.

$units < .5in, .5in > xfrom-5.0to3.0, yfrom-1to2.5(.) < 6pt > [.2, .7]from-5.30to-20 < 6pt > [.2, .7]from-5-0.3to-52.5Figure1:$

$\theta \geq \frac{1-c}{c}$ at -4 -.7 0 at -5.15 -.15 $2c-1$ at -3 -.20 $R(v)$ at -5.4 2 1 at -5.15 1.6 v at -2.2 -.2 -5 1.6 -4 .5 -3 0 / -3 .05 -3 0 -3 -.05 / ¡6pt¿ [.2, .7] from -0.3 0 to 3 0 ¡6pt¿ [.2, .7] from 0 -0.3 to 0 2.5 *Figure 2:* $\theta < \frac{1-c}{c}$ at 1 -.7 0 at -.15 -.15 $2c-1$ at 2 -.20 ρ^* at -.18 .4 -.05 .4 0 .4 .05 .4 / $R(v)$ at -.4 2 1 at -.15 1.6 v at 2.8 -.2 0 1.6 1 .7 2 0.4 / 2 .05 2 0 2 -.05 /

Theorem 2.10. *Let $\rho \geq \rho^*$ and take p in the set of m–probability one for which there exists ν_α, the invariant measure defined in Theorem 2.3 with global density ρ and $\alpha = \alpha(p)$. Assume that the initial distribution of particles is given by ν_α and let X_t be the position at time t of the tagged particle originally at the origin. Then there exist Poisson processes N_t^0 and N_t^1 of rate $v = v(\rho)$, random variables R^0 and R^1 and a stationary process B_t such that the following inequalities hold*

$$N_t^0 + R^0 \geq X_t \geq N_t^1 + B_t^1 - B_0^1 - R^1$$

Furthermore B_t, R^0 and R^1 can be chosen such that they have an exponential moment independent of p (and t): there exist positive θ and C independent of p and t such that

$$\mathbb{E}e^{\theta R^0} < C, \ \mathbb{E}e^{\theta R^1} < C, \ \mathbb{E}e^{\theta B_t} < C.$$

Proof. We adapt a proof of Ferrari and Fontes (1994b). We consider two semi-infinite processes η_t^0 and η_t^1 as seen from the tagged particle.

For η^0_t the tagged particle never jumps to the left as if there were always full of particles to its left. The jumps to the right of the tagged particle are realized with rate p_0 and the jumps of the other particles with rates p_i, q_i, $i \geq 1$. Let X^0_t be the position of the tagged particle for η^0_t. The invariant measure for the process is the product measure ν^0_α on $\{0,1\}^{\mathbb{Z}^+}$ with $\alpha^0 = \{\alpha^0_i : i \geq 0\}$ the solution of (2.4) with boundary conditions $\alpha^0_{-1} = 0$.

For η^1_t the tagged particle jumps to the left at rate q_0 as if there were no particles to its left. Let X^1_t be the position of the tagged particle for η^1_t. The invariant measure for the process is the product measure ν^1_α on $\{0,1\}^{\mathbb{Z}^+}$ with $\alpha^1 = \{\alpha^1_i : i \geq 0\}$ the solution of (2.4) with boundary conditions $\alpha^1_{-1} = 1$.

By (2.7), if α is the double infinite solution of (2.4), then for $i \geq 0$, $\alpha^1_i \geq \alpha_i \geq \alpha^0_i$ and both $\alpha_i - \alpha^0_i$ and $\alpha^1_i - \alpha_i$ converge exponentially fast to zero. Now we couple a double infinite configuration η picked from ν_α with two semi-infinite configurations η^0 and η^1 picked from ν^0_α and ν^1_α respectively in such a way that if we call

$$\begin{aligned} \xi(i) &= x(i+1) - x(i) - 1, \ \ i \in \mathbb{Z} \\ \xi^0(i) &= x^0(i+1) - x^0(i) - 1, \ \ i \geq 0 \\ \xi^1(i) &= x^1(i+1) - x^1(i) - 1, \ \ i \geq 0 \end{aligned} \tag{2.11}$$

then under the (three way) coupled measure $\bar{\nu}_\alpha$,

$$\xi^1(i) \geq \xi(i) \geq \xi^0(i) \tag{2.12}$$

almost surely for $i \geq 0$ and there exist positive θ and C such that for $j = 0, 1$,

$$\bar{\nu}_\alpha \left\{ \exp\left(\theta \sum_{i=0}^{\infty} |\xi(i) - \xi^j(i)| \right) \right\} < C < \infty. \ \ m\text{–a.s.} \tag{2.13}$$

Statements (2.12) and (2.13) are easy consequences of (a) the attractiveness of the solutions α, α^1 and α^0, (b) the (uniform in p) exponential convergence to zero of $(\alpha_i - \alpha^j_i)$, which in turn are consequence of (2.7) and (c) the fact that ν_α ν^1_α and ν^0_α are product of geometric distributions.

Assume that ξ_t, ξ^0_t and ξ^1_t start each with the corresponding invariant distribution coupled in such a way that (2.12) and (2.13) hold. Now we couple η_t, η^1_t and η^0_t in such a way that the i-th particles of the three systems use the same exponential clocks and jump together as much as possible. In the particle system jargon this is the basic coupling for the zero range process ξ_t. Under initial distribution $\bar{\nu}_\alpha$ and using the coupling, the marginal processes ξ_t, ξ_t^0 and ξ_t^1 are stationary and keep the initial order: for all $t \geq 0$,

$$\xi_t^1(i) \geq \xi_t(i) \geq \xi_t^0(i). \tag{2.14}$$

This implies that the rate of jump to the right are also ordered for all times and that

$$X_t^0 + R^0 \geq X_t \geq X_t^1 - R^1 \tag{2.15}$$

where

$$R^1 = \sum_{x \geq 0} (\xi_0^1(x) - \xi_0(x)) \text{ and } R^0 = \sum_{x \geq 0} (\xi_0(x) - \xi_0^0(x)).$$

To see that the first inequality in (2.15) holds observe that each jump to the right of X_t can either be accompanied by a jump to the right of X_t^0 or not. In the first case the inequality holds if it holds before the jump. In the second case, the jump to the right is either corresponding to one of the extra holes R^0 or due to a hole that entered to the right of X_t after time zero. The first case is covered by R^0 that appear in the left hand side of (2.15). The second one cancels because the net flux generated by this hole is zero. The same argument is used to show the second inequality. Both R^1 and R^0 have exponential moments by (2.13).

Now, in distribution $X_t^0 = N_t^0$, a Poisson process with parameter v. The fact that it is Poisson follows from Burke's theorem because it is just the departure process of a network of queues. See Kelly (1979) or Theorem 1 of Ferrari and Fontes (1994a). The rate is given by

$$p_0 \alpha_0^0 = p_n \alpha_n^0 - q_n \alpha_{n-1}^0.$$

for all n. Since $\lim_{n\to\infty}(\alpha_n - \alpha_n^0) = 0$ and $p_n \alpha_n - q_n \alpha_{n-1} \equiv v$ by (2.4).

Finally in Theorem 2 of Ferrari and Fontes (1994a) it is proven that $X_t^1 = N_t^1 + B_t^1 - B_0^1$, where N_t^1 is a Poisson process with parameter v and B_t^1 is a stationary process distributed as a sum of independent geometric distributions with parameters $b_i^1/(1 - a_i^1 + b_i^1)$, $i \geq 0$, where a_i^1 is the solution of (2.4) for $i \geq 0$ assuming boundary condition $a_{-1}^1 \equiv 1$ and b_i^1 is the solution of (2.4) for $i \geq 0$ assuming boundary condition $b_{-1}^1 \equiv 1$ and $v = 0$. See display (20) of Ferrari and

Fontes (1994a). Clearly $b_i^1 \le ((1-c)/c)^i$ and $a_i^1 \ge b_i^1$. Hence B_t^1 has an exponential moment independent of p and t. $\square$

In the finite case one looks for solutions α^n of (2.4) fixing boundary conditions

$$\alpha_{-1}^n = \alpha_n^n = 1 \tag{2.16}$$

The proof of the following theorem is immediate as it is just a finite system of queues.

Theorem 2.17. *If the vector $p^n = (p_1^n, \dots, p_n^n)$ is such that (2.4) with boundary conditions (2.16) has a solution $\alpha_i^n \in (0,1)$, $i \in \{0, \dots, n\}$ then the measure ν^n defined by*

$$\nu^n(x_{i+1} - x_i = k_i : i \in \{0, \dots, n\}) = \prod_{i \in \{0,\dots,n\}} (\alpha_i^n)^{k_i - 1}(1 - \alpha_i^n), \tag{2.18}$$

is invariant for the process with generator L_p.

The tagged particle performes in this case a Poisson process in the sense of Theorem 2.10. See Ferrari and Fontes (1994b).

To close this section we consider the semi-infinite case. Let $\bar\eta$ be the configuration $\bar\eta(x) = \mathbf{1}\{x \le 0\}$. Assume that the measure m describing the random rates satisfies

$$\begin{aligned} &m \text{ is a product measure and} \\ &m[c,1] = 1 \end{aligned} \tag{2.19}$$

For fixed p let

$$v_i(p) = \lim_{t\to\infty} \frac{x_i(t) - x_i(0)}{t} \tag{2.20}$$

The limit exists by an application of the sub-additive ergodic theorem.

Theorem 2.21. *Under conditions (2.1) and (2.19) it holds that $v^* = 2c - 1$ and*

$$\lim_{i\to-\infty} v_i = 2c - 1 \qquad m \textit{ almost surely.}$$

We conjecture that the distribution around X_i converges to the measure $\nu_{\alpha(p,v^*)}$.

Proof. By assumption (2.19), $v_i \ge (2c-1)$. To show the other inequality consider a finite system with $n+1$ particles: $y_0(t) < \cdots < y_n(t)$ such that $y_0(t)$ jumps only to the right at rate 1 and $y_1(t), \dots, y_n(t)$ jump with rates b and $1-b$ to the right and left respectively. The process $(z_1(t),\dots,z_n(t)) = (y_1(t)-y_0(t)-1,\dots,y_n(t)-y_0(t)-1)$ is an ergodic positive recurrent continuous time Markov process in a countable state space and it has invariant distribution μ^n given by

$$\mu^n(z_1 \ge k_1,\dots,z_n \ge k_n) = \prod_{j=1}^{n} (\alpha_j^n)^{k_j}[1-\alpha_j^n], \quad k_j \ge 0$$

where

$$\alpha_j^n = \frac{1-\left(\dfrac{1-b}{b}\right)^j}{1-\left(\dfrac{1-b}{b}\right)^{n+1}}$$

Under this invariant measure the average velocity of the j-th particle is given by

$$v^n = b\mu^n(z_j > 0) - (1-b)\mu^n(z_{j+1} > 0) = \frac{2b-1}{1-\left(\dfrac{1-b}{b}\right)^{n+1}} \tag{2.22}$$

that does not depend on j. Furthermore we have that if the system starts with the configuration $z_j \equiv 0$ (that has positive μ^n measure), it holds the following law of large numbers

$$\lim_{t\to\infty} \frac{y_j(t)-y_j(0)}{t} = v^n.$$

Turning to the semi-infinite system, fix p. It is clear that $v_i \le v_{i+1}$. On the other hand, let $b(i,n,p) = \max\{p_j : i \le j \le i+n\}$. For this b consider the finite system $(y_0,\dots,y_n)$ defined above with initial positions $(x_i,\dots,x_{i+n})$. Hence one can couple in such a way that

$$x_i(t) - x_i(0) \le y_0(t) - y_0(0)$$

and

$$\limsup_{t\to\infty} \frac{x_i(t)-x_i(0)}{t} \le \lim_{t\to\infty} \frac{y_0(t)-y_0(0)}{t} = \frac{2b(i,n,p)-1}{1-\left(\frac{1-b(i,n,p)}{b(i,n,p)}\right)^{n+1}},$$

by (2.22) and a law of large numbers for a positive recurrent Markov chain. Take $\varepsilon > 0$. Take b and n such that $v^n < 2c-1+\varepsilon$, where v^n is given by (2.22). Let $I(p) = \max\{i : b(i,n,p) < b\}$ the first time that n successive p_j are smaller than b.

By assumption (2.1) I is finite with m-probability one. Hence we have that for all $\varepsilon > 0$ for i sufficiently large,

$$2b(i, n, p) - 1 \leq 2c - 1 + \varepsilon$$

This proves the Theorem. $\square$

When the rates are attached to the sites instead of the particles, there is no description of the invariant measure available. Presumably in this case the invariant measure is not product and it might have long correlations. A simple coupling argument shows that if m concentrates in $(c, 1)$ and the particles are distributed initially according to the translation invariant product measure ν_ρ, then the tagged particle must move at velocity at least $(2c-1)(1-\rho)$. This is the contents of the next theorem.

Theorem 2.23. *Assume that the rates p are attached at the sites and that m concentrates in $[c, 1]$. Let η_t be the simple exclusion process with these rates distributed initially with the (non invariant) distribution ν_ρ. Let X_t be the position of the tagged particle initially located at the origin. Then*

$$\liminf_{t\to\infty} \frac{X_t}{t} \geq (2c-1)(1-\rho) \quad \textit{almost surely.}$$

Proof. We couple η_t with a simple exclusion process ζ_t with (constant) rates c and $1-c$ for right and left jumps respectively. The product measure ν_ρ conditioned to have a particle at the origin is invariant for this process as seen from the tagged particle. Let Z_t be the position of the tagged particle initially at the origin. Under initial distribution ν_ρ, Kipnis (1986) proved that

$$\lim_{t\to\infty} \frac{Z_t}{t} = (2c-1)(1-\rho). \tag{2.24}$$

Since $p_x \geq c$ for all x we can couple η_t and ζ_t in such a way that each time a jump from x to $x+1$ is attempted for the system ζ_t, the same jump is attempted for the system η_t. Analogously, since $1-c \geq q_i$, each time a jump from x to $x-1$ is attempted for the system η_t, the same jump is attempted for the system ζ_t. Since the rates are different, under this coupling the two systems will differ. Let $x_i(t)$ and $z_i(t)$ the positions of the particles of the η_t and ζ_t systems respectively at time

t. Here $x_0(t) = X_t$ and $z_0(t) = Z_t$. At time zero $x_i(0) \equiv z_i(0)$. We claim that for all $t \geq 0$

$$x_i(t) \geq z_i(t) \tag{2.25}$$

To show (2.25) it suffices to see that each jump involving positions $x_i(t)$ and $z_i(t)$ keep the inequality unaltered. Assume that at time t (2.25) holds and that there is an event involving $x_i(t)$ and/or $z_i(t)$ in the time interval $(t, t+dt)$. If $x_i(t) > z_i(t)$ then, after the event, $x_i(t) \geq z_i(t)$ because at most a jump of length 1 occurred. If $x_i(t) = z_i(t)$, there are four possibilities: (a) an attempt for both particles to jump to the right; (b) an attempt for the $x_i(t)$ particle to jump to the right; (c) an attempt for both particles to jump to the left and (d) an attempt for the $z_i(t)$ particle to jump to the left. Since by (2.25), $x_{i+1}(t) \geq z_{i+1}(t)$, in case (a) after the jump $x_i(t+dt) \geq z_i(t+dt)$. Case (c) is similar and cases (b) and (d) are immediate because the jumps are in the good sense. This proves (2.25) for all t. In particular $X_t \geq Z_t$. This and (2.24) show the theorem. $\square$

3. Hydrodynamical limit of one dimensional asymmetric zero range processes with random jumps.

In this section we study the hydrodynamical behaviour of the asymmetric process with random rates introduced in last section. In order to do it, we first change coordinates.

For an integer k, define $\xi_t(k)$ to be the number of holes between the $(k+1)$-th and the k-th particles at time t :

$$\xi_t(k) \;=\; x_{k+1}(t) \;-\; x_k(t) \;-\; 1\ .$$

It is then easy to check that for each realization $\{p_k;\ k \in \mathbb{Z}\}$ ξ_t is a Markov process on $\mathbb{N}^{\mathbb{Z}}$ whose generator acts on cylinder function as

$$(L_p' f)(\xi) \;=\; \sum_{\substack{j,k\in\mathbb{Z}\\ |k-j|=1}} p(k,j)\mathbf{1}\{\xi(k) \geq 1\}[f(T_{k,j}\xi) - f(\xi)]\ . \tag{3.1}$$

In this formula, for fixed integers j and k and for configurations ξ with at least one particle at k, $T_{k,j}\xi$ stands for the configuration obtained from ξ letting one particle jump from site k to site j :

$$\Big(T_{k,j}\xi\Big)(i) = \begin{cases} \xi(i) & \text{if } i \neq k, j \\ \xi(k) - 1 & \text{if } i = k \\ \xi(j) + 1 & \text{if } i = j. \end{cases}$$

Moreover, with notation introduced in last section, $p(k, k+1) = q_{k+1}$ and $p(k, k-1) = p_k$ so that $p(k, k+1) + p(k+1, k) = 1$. This is the so called zero range process with jump rate $g(n) = \mathbf{1}\{n \geq 1\}$ and random transition probabilities $p(j, k)$.

We already obtained in last section the invariant measures of this process. Translated to our context they write as follows. For each v in $[0, v^*)$, consider $\alpha_i(p) = \alpha_i(p, v)$ the solution of (2.4) and denote by $\mu_{p,v}$ the product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals given by

$$\mu_{p,v}\{\xi; \xi(i) = k\} \;=\; (\alpha_i)^k (1 - \alpha_i) \qquad \text{for } k \in \mathbb{N} \text{ and } i \in \mathbb{Z}\,.$$

Define $M\colon [0, v^*) \to \mathbb{R}_+$ as the expected value of particles at site 0 for the measure $\mu_{p,v}$:

$$M(v) \;=\; \mu_{p,v}\big[\eta(0)\big] \;=\; \frac{\alpha_0}{1 - \alpha_0}\,.$$

We have thus that M is a smooth and strictly increasing positive function with $M(0) = 0$.

Define by $\rho\colon [0, v^*) \to \mathbb{R}_+$ the expected value of $M(v)$:

$$\rho(v) \;=\; m\Big[M(v)\Big]\,. \tag{3.2}$$

From its definition and from the properties of M, it follows that ρ vanishes at 0 and that ρ is continuous and strictly increasing. Moreover, with the very same arguments presented in the proof of Theorem 2.2(iii), one obtains that m almost surely,

$$\lim_{n\to\infty} |\Lambda_n|^{-1} \sum_{k\in\Lambda_n} \xi(k) \;=\; \rho(v) \qquad \mu_{p,v} \quad \text{almost surely}\,.$$

Here, for a positive integer n, Λ_n represents a box of length $2n+1$ centered at the origin :

$$\Lambda_n \;=\; \{-n, \ldots, n\}$$

and $|\Lambda_n|$ its volume.

We now introduce the initial measures considered in this section. For a bounded continuous function $\lambda_0 : \mathbb{R} \to \mathbb{R}_+$ denote by $\{\mu^N_{\lambda_0};\ N \geq 1\}$ the sequence of product measures on $\mathbb{N}^{\mathbb{Z}}$ with marginals given by

$$\mu^N_{\lambda_0}\{\xi ; \xi(k) = n\} \;=\; \Big[1 + \lambda_0(k/N)\Big]^{-1} \left(\frac{\lambda_0(k/N)}{1 + \lambda_0(k/N)}\right)^n$$

for $k \in \mathbb{Z}$ and $n \geq 0$. $\{\mu^N_{\lambda_0};\ N \geq 1\}$ is defined so that expected number of particles at k is equal to $\lambda_0(k/N)$.

To ensure that, for m almost all environments p, μ^N is bounded above (for the natural partial order on the space of all probability measures of $\mathbb{N}^{\mathbb{Z}}$) by some invariant measure $\mu_{p,v}$, we will have to assume that the initial profile λ_0 is bounded above by $\sup_{v<v^*} \min_i \{\alpha_i(p,v)/(1-\alpha_i(p,v))\}$. This value corresponds to the minimum density at a site for the measure $\mu_{p,v}$. It is therefore the minimum density at a site for the largest product invariant measure.

(**H1**) The continuous initial profile λ_0 is bounded above :

$$\sup_{x\in\mathbb{R}} \lambda_0(x) \;<\; \sup_{v<v^*}\ \min_{i\in\mathbb{Z}} \frac{\alpha_i(p,v)}{1-\alpha_i(p,v)}\ \cdot$$

We believe however that it should not be too difficult to remove this technical assumption. We also assume that p takes at most a finite number of values denoted by $c = c_1 < \cdots < c_b \leq 1$:

(**H2**) There exists $b \in \mathbb{N}$ and $c = c_1 < \cdots < c_b \leq 1$ such that

$$m\Big\{p;\ p_0 = c_i \quad \text{for some} \quad 1 \leq i \leq \alpha\Big\} \;=\; 1\ .$$

We are now ready to state the main theorem of this section. For each realization p of the environment and for each probability measure μ on $\mathbb{N}^{\mathbb{Z}}$ denote by $P^N_{p,\mu}$ the probability measure on the path space $D([0,\infty), \mathbb{N}^{\mathbb{Z}^d})$ corresponding to the Markov process with generator L'_p, defined in (3.1), accelerated by N and starting from μ. Denote by $E^N_{p,\mu}$ expectation with respect to $P^N_{p,\mu}$.

Theorem 3.3 . *Under the assumptions (***H1***) and (***H2***) stated above, for every compact supported continuous function H, every $t \geq 0$ and every strictly positive δ,*

$$\lim_{N\to\infty} P^N_{p,\mu^N_{\lambda_0}} \left[\left| N^{-d} \sum_k H(k/N)\eta_t(k) \; - \; \int H(x)\lambda(t,x)\, dx \right| > \delta \right] \; = \; 0$$

m almost surely. In this last formula λ is the unique weak entropy solution of the first order quasilinear hyperbolic equation

$$\begin{cases} \partial_t \lambda \; + \; \partial_x a(\lambda) \; = \; 0 \\ \lambda(0,\cdot) \; = \; \lambda_0(\cdot) \end{cases}$$

and a is the inverse of the function ρ defined in (3.2).

The proof of this theorem is similar to the one of Theorem 4.2 presented in next section and thus omitted.

4. Asymmetric zero range processes with random rates.

In section 3 we studied the hydrodynamical behaviour of one dimensional zero range processes with random transition probabilities. In this section, instead, we shall study zero range processes with random rate jumps and fixed deterministic transition probabilities.

As in section 1 consider a family of random variables $\{p_k;\ k \in \mathbb{Z}^d\}$ with values in an interval $[c,1]$ for some strictly positive constant c. Denote by m the joint distribution and assume that is is stationary and ergodic. We will call p the environment and shall assume that $m([c, c+\epsilon)) > 0$ for all $\epsilon > 0$. For each realization of p, let η_t be the Markov process on $\mathbb{N}^{\mathbb{Z}^d}$ whose generator acts on cylinder functions as

$$(L^2_p f)(\eta) = \sum_{k,j\in\mathbb{Z}^d} r(j) p_k g(\eta(k)) [f(T_{k,k+j}\eta) - f(\eta)] \; , \tag{4.1}$$

where $r(\cdot)$ is a finite range transition probability on $\mathbb{Z}^d$ and g a positive non decreasing bounded function vanishing at 0 : there exists $\Gamma < \infty$ such that $r(j) = 0$ for $|j| > \Gamma$, $0 = g(0) < g(1)$ and $\lim_{k\to\infty} g(k) = g(\infty) < \infty$. We further assume that $r(k) + r^*(k)$ is irreducible. This process can be interpreted as follows. For a fixed realization of the environment p, at each site k a particle jumps to site $k+j$ at rate $r(j)p_k g(\eta(k))$. In this sense the rate at which particles leave a site is randomly decelerated by the value of the environment at that site. In the one dimensional nearest neighbor case with $g(n) = \mathbf{1}\{n \geq 1\}$ this model corresponds to the simple exclusion process studied in sections 1 and 2 where we fixed the probability at which each particle jumps to the right or to the left and where we decelerate the rate at which the k-th particle jumps to the right and the $k+1$-th particle jumps to the left by a factor p_k picked from a stationary ergodic process.

For further use we denote by $\gamma = (\gamma_1, \dots, \gamma_d)$ the mean drift of each elementary particle :

$$\gamma_i \;=\; \sum_k k_i \, r(k)$$

for $1 \leq i \leq d$ if $k = (k_1, \dots, k_d)$.

The first question is naturally to find invariant measures for the process. Fortunately there are product invariant measures. For each $0 \leq \varphi < g(\infty)$, define the partition function $Z(\cdot)$ on $\mathbb{R}_+$ by

$$Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(1)\cdots g(k)} \; .$$

We used here the convention that $g(1)\cdots g(0) = 1$. It is clear that $Z(\cdot)$ is an increasing function which converges to ∞ as φ approaches $g(\infty)$. Recall from Andjel (1982) that in the case where m is a Dirac measure concentrated on the set $\{p; p_k = 1, k \in \mathbb{Z}^d\}$, that is, in the case with a deterministic translation invariant environment, the translation invariant product measures $\{\mu_\varphi;\; 0 \leq \varphi < g(\infty)\}$, with marginals given by

$$\mu_\varphi\{\eta; \eta(k) = n\} = = \frac{1}{Z(\varphi)} \frac{\varphi^n}{g(1)\cdots g(n)}$$

for $n \geq 0$, are invariant. From this result product invariant measures for the random environment case are easily guessed.

For $0 \leq \varphi < cg(\infty)$ denote by ν_φ^p the product measure on $\mathbb{N}^{\mathbb{Z}^d}$ with marginals given by :

$$\nu_\varphi^p\{\eta; \eta(k) = n\} \;=\; \frac{1}{Z(\varphi p_k^{-1})} \frac{\left[p_k^{-1}\varphi\right]^n}{g(1)\cdots g(n)}$$

for $n \geq 0$. To keep notation as simple as possible we will often omit the superlabel p of ν_φ^p. A direct computation shows that ν_φ is an invariant measure for the process for each $0 \leq \varphi < cg(\infty)$. Moreover if we denote by $L_p^{2,*}$ the generator defined in (4.1) with $r^*(j) = r(-j)$ replacing r, we have that L_p^* is the adjoint of L_p^2 in $L^2(\nu_\varphi)$.

We now turn to the study of the mean density of particles under each measure ν_φ. Define $M\colon [0, g(\infty)) \to \mathbb{R}_+$ as the expected value of the mean density of particles under the measure μ_φ :

$$M(\varphi) \;=\; \mu_\varphi\big[\eta(0)\big] \;.$$

A straightforward computation shows that $M(0) = 0$ and that M is a smooth strictly increasing bijection since it is given by

$$M(\varphi) \;=\; \varphi Z'(\varphi) Z(\varphi)^{-1} \;=\; \varphi \partial_\varphi \log Z(\varphi) \;.$$

Denote by $A\colon \mathbb{R}_+ \to [0, g(\infty))$ the inverse function of M, which is therefore smooth and strictly increasing. Define $\rho\colon [0, cg(\infty)) \to \mathbb{R}_+$ by

$$\rho(\varphi) \;=\; m\Big[M(\varphi p_0^{-1})\Big] \;.$$

From its definition and from the properties of M, it follows that ρ vanishes at 0 and that ρ is continuous and strictly increasing. Moreover, with the very same arguments presented in the proof of Theorem 2.2(iii), one obtains that m almost surely,

$$\lim_{n\to\infty} |\Lambda_n|^{-1} \sum_{k\in\Lambda_n} \eta(k) \;=\; \rho(\varphi) \qquad \nu_\varphi^p \quad \text{almost surely} \;.$$

Denote by $a\colon \mathbb{R}_+ \to [0, cg(\infty))$ the inverse function of ρ which is continuous, strictly increasing and vanishing at 0.

A natural question for zero range processes in random environment is the following. We fix a density ρ_0 and start at time 0 with the translation invariant measure associated to the density ρ_0 invariant for the usual zero range process : $\mu_{A(\rho_0)}$. We let then the process evolve with a random environment. We would like to prove that m–almost surely the state of the process as time increases converges to the invariant measure associated to the density ρ_0: $\nu^p_{a(\rho_0)}$.

One approach to solve this problem would be to prove a strong form of hydrodynamical limit and then use the attractiveness of the process (cf. Landim (1993a)). This approach however relies strongly on the translation invariance of the process.

We present here the first step in this program: a proof of the hydrodynamical behaviour of zero range processes in random environment, that is, we describe the macroscopic behaviour of the process starting from a local equilibrium state. The first step is therefore to define the initial states considered here. For a bounded continuous function $\lambda_0 : \mathbb{R}^d \to \mathbb{R}_+$ denote by $\{\mu^N_{\lambda_0};\ N \geq 1\}$ the sequence of product measures on $\mathbb{N}^{\mathbb{Z}^d}$ with marginals given by

$$\mu^N_{\lambda_0}\{\eta; \eta(k) = n\} = \ \mu_{A(\lambda_0(k/N))}\{\eta; \eta(0) = n\}$$

for $k \in \mathbb{Z}^d$ and $n \geq 0$.

To ensure that for m almost all environments p μ^N is bounded above (for the natural partial order on the space of all probability measures of $\mathbb{N}^{\mathbb{Z}^d}$) by some invariant measure ν^p_φ, we will have to assume that the initial profile λ_0 is bounded above by $M(g(\infty)c')$ for some $c' < c$:

(**H1**) There exists $c' < c$ such that the continuous initial profile λ_0 is bounded above by $M(g(\infty)c')$: $\sup_{x\in\mathbb{R}^d} \lambda_0(x) \leq M(g(\infty)c')$.

We believe however that it should not be too difficult to remove this technical assumption. We also assume that p takes at most a finite number of values denoted by $c = c_1 < \cdots < c_\alpha \leq 1$:

(**H2**) There exists $\alpha \in \mathbb{N}$ and $c = c_1 < \cdots < c_\alpha \leq 1$ such that

$$m\Big\{p;\ p_0 = c_i \quad \text{for some} \quad 1 \leq i \leq \alpha\Big\} \ = \ 1\ .$$

We are now ready to state the main theorem of this section. For each realization p of the environment and for each probability measure μ on $\mathbb{N}^{\mathbb{Z}^d}$ denote by $P^N_{p,\mu}$ the probability measure on the path space $D([0,\infty), \mathbb{N}^{\mathbb{Z}^d})$ corresponding to the Markov process with generator L^2_p accelerated by N and starting from μ. Denote by $E^N_{p,\mu}$ expectation with respect to $P^N_{p,\mu}$.

Theorem 4.2 . *Under the assumptions (*__H1__*) and (*__H2__*) stated above, for every compact supported continuous function H, every $t \geq 0$ and every strictly positive δ,*

$$\lim_{N\to\infty} P^N_{p,\mu^N_{\lambda_0}} \left[\ \left| N^{-d}\sum_k H(k/N)\eta_t(k) \ - \ \int H(x)\lambda(t,x)\, dx \right| > \delta \right] \ = \ 0$$

m almost surely. In this last formula λ is the unique weak entropy solution of the first order quasilinear hyperbolic equation

$$\begin{cases} \partial_t\lambda \ + \ \displaystyle\sum_{i=1}^d \gamma_i\partial_{x_i} a(\lambda) \ = \ 0 \\ \lambda(0,\cdot) \ = \ \lambda_0(\cdot)\ . \end{cases}$$

The proof of this theorem is analogous to the one of Theorem 1 presented in sections 5 and 6 of Landim (1993b). We therefore concentrated only on the main differences. Also, to keep notation simple, we present the proof in dimension 1 and assume that the environment takes only two values $c = c_1 < c_2 \le 1$.

Following Rezakhanlou (1990), the first step in order to prove hydrodynamical behaviour of asymmetric attractive interacting particle systems is to prove an entropy inequality at the microscopic level.

Lemma 4.3 . *Under the assumptions of Theorem 4.2, for every smooth positive function H with compact support in $(0,\infty)\times\mathbb{R}$, for every nonnegative constant w and every positive ϵ,*

$$\lim_{\ell\to\infty}\lim_{N\to\infty} P^N_{p,\mu^N_{\lambda_0}}\Bigg[\int_0^\infty dt\, N^{-1}\sum_k \Big\{\partial_t H(t,k/N)\Big|\eta^\ell_t(k)-w\Big| + \gamma\partial_x H(t,k/N)\Big|a\big(\eta^\ell_t(k)\big)-a\big(w\big)\Big|\Big\} \ \ge -\epsilon\Bigg] \;=\; 1$$

m almost surely.

Here and below, for an integer k and a positive integer ℓ, $\eta^\ell(k)$ represents the mean density of particles in a box of length $2\ell+1$ centered at k :

$$\eta^\ell(k) \;=\; (2\ell+1)^{-1}\sum_{|j-k|\le\ell}\eta(j)\ .$$

Lemma 4.3 is the main step in the proof of Theorem 4.2. In possession of this microscopic entropy inequality to prove the theorem we just have to follow the arguments in Rezakhanlou (1990). The slight modifications needed are left to the reader and we concentrate on the proof of Lemma 4.3. The proof of this lemma is analogous to the one of Proposition 5.1 and Corollary 5.7 in Landim (1993b). The unique difference being the proof of the one block estimate. In order to state this result, for a positive integer ℓ and a realization p of the random environment, denote by $V_\ell(p,\eta)$ the cylinder function

$$V_\ell(p,\eta) \;=\; \Big|(2\ell+1)^{-1}\sum_{|k|\le\ell} p_k g(\eta(k)) \;-\; a\big(\eta^\ell(0)\big)\Big|\ .$$

Lemma 4.4(one block estimate). *Let μ^N be a sequence of product measure on $\mathbb{N}^{\mathbb{Z}}$ associated to a profile bounded by $M(c'g(\infty))$ for some $c' < c$. For every positive continuous function $H: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with compact support,*

$$\lim_{\ell\to\infty} \lim_{N\to\infty} E^N_{p,\mu^N} \left[\int_0^\infty dt\, N^{-1} \sum_k H(t, k/N) V_\ell(\theta_k p, \tau_k \eta_t) \right] \;=\; 0$$

m almost surely. In this formula, for an integer k, $\theta_k p$ denotes the translation by k unities of the environment p ($(\theta_k p)_j = p_{k+j}$) and $\tau_k \eta$ the translation by k unities of a configuration η ($(\tau_k)\eta(j) = \eta(k+j)$).

Proof: First of all, since the measure μ^N is associated to a profile bounded by $M(c'g(\infty))$ it is bounded above by a product invariant measure ν^p_φ for some φ. Therefore it is enough to prove the lemma for the cylinder function V_ℓ replaced by

$$V_{\ell,b}(p,\eta) \;=\; V_\ell(p,\eta)\mathbf{1}\{\eta^\ell(0) \le b\}$$

for each fixed b.

On the other hand, by standard coupling arguments, we can show that it is enough to prove the lemma, for every fixed integer B, for the product measure μ^N replaced by a product measure with marginals equal to the marginals of μ^N in the interval $\{-BN, \dots, BN\}$ and equal to the marginals of an invariant measure ν^p_φ outside the interval. Denote by $\mu^N_B = \mu^N_{p,\varphi,B}$ the measure obtained in this way.

From the entropy inequality, we have that

$$\begin{aligned} & E^N_{p,\mu^N_B} \left[\int_0^{t_0} dt\, N^{-1} \sum_k H(k/N) V_{\ell,b}(\theta_k p, \tau_k \eta_t) \right] \\ & \quad \le \frac{H\Big(\mu^N_B \big| \nu^p_\varphi\Big)}{\beta N} \;+\; \frac{1}{\beta N} \log E^N_{p,\nu^p_\varphi} \left[\exp\left\{ \int_0^{t_0} dt\, \beta \sum_k H(k/N) V_{\ell,b}(\theta_k p, \tau_k \eta_t) \right\} \right]. \end{aligned}$$

Here H is a positive continuous function with compact support that bounds above $H(t,\cdot)$ for every $t \ge 0$, t_0 is large enough so that $[0,t_0] \times \mathbb{R}$ contains the support of H and $H\Big(\mu^N_B \big| \nu^p_\varphi\Big)$ stands for the entropy of μ^N_B with respect to ν^p_φ. A direct computation shows that for every $B \in \mathbb{N}$,

$$\limsup_{N\to\infty} N^{-1} H\Big(\mu^N_B \big| \nu^p_\varphi\Big) \;\le\; C(B)\,.$$

Since we may let β increases to ∞ after N and ℓ, to prove the lemma it is enough to show that for every positive β,

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log E^N_{p,\nu^p_\varphi}\left[\exp\left\{\int_0^{t_0}dt\,\beta\sum_k H(k/N)V_{\ell,b}(\theta_k p,\tau_k\eta_t)\right\}\right] \;\le\; 0\;.$$

By standard arguments, even though the measure ν^p_φ is not reversible (cf. Benois et al. (1993)), the expected value in the above formula is bounded by $\exp\{Ct_0 e_N\}$, where e_N is the largest eigenvalue of the operator

$$N(L^2_p + L^{2,*}_p) + \beta\sum_k H(k/N)V_{\ell,b}(\theta_k p,\tau_k\eta)\;.$$

By the variational formula for the largest eigenvalue of of a symmetric operator we obtain that the logarithm of the last expected value divided by N is bounded above by

$$\beta C(t_0)\sup_f\left\{N^{-1}\sum_k H(k/N)\int V_{\ell,b}(\theta_k p,\tau_k\eta)f(\eta)\nu^p_\varphi(d\eta)\;-\;\beta^{-1}D_{N,p}(f)\right\}\;.$$

In this formula $D_{N,p}(f)$ stands for the Dirichlet form given by

$$D_{N,p}(f)\;=\;2^{-1}\sum_{k,j}\int\big[r(j)+r(-j)\big]g(\eta(k))p_k\Big[\sqrt{f(T_{k,k+j}\eta)}-\sqrt{f(\eta)}\Big]^2\nu^p_\varphi(d\eta)$$

and the supremum is taken over all densities with respect to the measure ν^p_φ.

The first step in the proof that the last variational formula converges to 0 is to localize the integrals to microscopic blocks. We need some notation in order to do it. For an integer k, denote by $\Lambda_{k,\ell}$ the block $\{k-\ell,\dots,k+\ell\}$, by $X_{k,\ell}$ the configuration space $\mathbb{N}^{\Lambda_{k,\ell}}$, by $\nu^p_{\varphi,k,\ell}$ the marginal on $X_{k,\ell}$ of ν^p_φ, by $f_{k,\ell}$ the density with respect to $\nu^p_{\varphi,k,\ell}$ of the marginal of the measure $f(\eta)\nu^p_\varphi(d\eta)$ on $X_{k,\ell}$ and by $D_{k,\ell,p}$ the Dirichlet form on $X_{k,\ell}$ given by

$$D_{k,\ell,p}(f)\;=\;\sum_{i,j\in\Lambda_{k,\ell}}\int g(\eta(j))p_j\Big[\sqrt{f(T_{j,i}\eta)}-\sqrt{f(\eta)}\Big]^2\nu^p_{\varphi,k,\ell}(d\eta)\;.$$

Since $r(\cdot)+r^*(\cdot)$ is irreducible and the Dirichlet form is convex, by Schwarz inequality the last supremum is bounded above by

$$\beta C_1(t_0,H)\sup_f N^{-1}\sum_{|k|\le hN}\left\{\int V_{\ell,b}(\theta_k p,\eta)f_{k,\ell}(\eta)\nu^p_{\varphi,k,\ell}(d\eta)\right.$$
$$\left.\qquad\qquad -\;C_1(\ell)N\beta^{-1}D_{k,\ell,p}(f_{k,\ell})\right\}$$

for some constant $C_1(\ell)$ that depends only on ℓ and on the transition probability r. Here h is a positive constant related to the support of the function H. We now have to consider separately the integers k for which the environment p around k behaves badly.

For $i = 1, 2$, denote by m_i the probability of p_0 to be equal to p_i under m: $m_i = m\{p_0 = c_i\}$ and denote by $M^i_{k,\ell}(p)$ the mean number of sites in $\Lambda_{k,\ell}$ which takes value c_i:

$$M^i_{k,\ell}(p) \;=\; (2\ell+1)^{-1}\Big|\{j \in \Lambda_{k,\ell};\; p_j = c_i\}\Big| \;.$$

For a sufficiently small positive ϵ, let E_ϵ denote the interval $[m_1 - \epsilon, m_1 + \epsilon]$. Since g and a are bounded functions, the last supremum is bounded above by

$$\begin{aligned} &\beta C_1(t_0, H, g) N^{-1} \sum_{|k|\le hN} \mathbf{1}\{M^1_{k,\ell}(p) \notin E_\epsilon\} \\ &+ \beta C_2(t_0, H) \sup_{p; M^1_{0,\ell}(p)\in E_\epsilon} \sup_f \Big\{ \int V_{\ell,b}(p,\eta) f(\eta) \nu^p_{\varphi,0,\ell}(d\eta) - C_1(\ell) N \beta^{-1} D_{0,\ell,p}(f) \Big\} \end{aligned} \tag{4.5}$$

In the second line the first supremum is taken over all environments p with $M^i_{0,\ell}(p)$ in E_ϵ and the second supremum over all densities with respect to $\nu^p_{\varphi,0,\ell}$. Notice that the expression inside braces in the second line does not depend on the value of the environment outside Λ_ℓ. We may therefore think the environment to be just defined in Λ_ℓ. In particular the first supremum is taken over a finite set of possibles environments. Keep also in mind that the measure $\nu_{\varphi,0,\ell}$ that appears in the definition of the Dirichlet form $D_{0,\ell,p}$ is the same that integrates $V_{\ell,b} f$. On the other hand, since we assumed the environment to be ergodic and stationary the first line of this expression converges m almost surely, as $N \uparrow \infty$, to

$$2h\beta C(t_0, H, g) m\Big\{M^1_{0,\ell}(p) \notin E_\epsilon\Big\} \;.$$

Again by ergodicity, for every positive ϵ, this expression converges to 0 as $\ell \uparrow \infty$. It thus remains to prove that the second line converges to 0 as $N \uparrow \infty$, $\ell \uparrow \infty$ and then $\epsilon \downarrow 0$. In order to do it we need to introduce some notation.

For $0 \le \ell_1 \le 2\ell+1$, $\ell_1 + \ell_2 = \ell$, let $\nu^{\ell_1,\ell_2}_\varphi$ be the product measure on $\mathbb{N}^{\{-\ell_1,\dots,\ell_2-1\}}$ with marginals given by

$$\nu^{\ell_1,\ell_2}_\varphi\{\eta; \eta(k) = n\} \;=\; \begin{cases} \mu_{\varphi p_1^{-1}}\{\eta; \eta(k) = n\} & \text{if } -\ell_1 \le k < 0, \\ \mu_{\varphi p_2^{-1}}\{\eta; \eta(k) = n\} & \text{if } 0 \le k < \ell_2. \end{cases}$$

Thus on the left of the origin $\nu_\varphi^{\ell_1,\ell_2}$ has the marginals of the translation invariant measure $\mu_{\varphi p_1^{-1}}$ and on the right of the origin it has the same marginals as $\mu_{\varphi p_2^{-1}}$. On the other hand, recall the explicit form of the cylinder function $V_{\ell,b}$ and notice that if p and p' are two environments with the same total number of sites in Λ_ℓ equal to p_1 ($M^1_{k,\ell}(p) = M^1_{k,\ell}(p')$) then the supremum over all densities in the second line of (4.5) is the same for p and p'. Therefore the second line of (4.5) is equal to

$$C_2(t_0, H) \max_{\ell_1/(2\ell+1)\in E_\epsilon} \sup_f \left\{ \int V_{\ell,b}(q,\eta) f(\eta) \nu_\varphi^{\ell_1,\ell_2}(d\eta) \; - \; C_1(\ell) N \beta^{-1} D_{\ell_1,\ell_2}(f) \right\} .$$

In this formula the maximum is taken over all positive integers ℓ_1 such that $\ell_1/2\ell{+}1$ belongs to E_ϵ, the supremum is taken over all densities f with respect to $\nu_\varphi^{\ell_1,\ell_2}$, q is the environment which is equal to p_1 at the left of the origin and to p_2 at the origin and at the right of the origin and D_{ℓ_1,ℓ_2} is the Dirichlet form on $\mathbb{N}^{\{-\ell_1,\dots,\ell_2-1\}}$ given by

$$D_{\ell_1,\ell_2}(f) \;=\; \sum_{-\ell_1\le i,j<\ell_2} \int g(\eta(j)) q(j) \Big[\sqrt{f(T_{j,i}\eta)} - \sqrt{f(\eta)} \Big]^2 \nu_\varphi^{\ell_1,\ell_2}(d\eta) \; .$$

Since $V_{\ell,b}(q,\eta)$ vanishes on the subset of configurations with total number of particles bigger that $(2\ell+1)b$, we may restrict the supremum over densities to densities concentrated on configurations with at most $(2\ell+1)b$ particles. Moreover since $V_{\ell,b}(q,\eta)$ is bounded, we may also restrict the supremum over densities to the set of densities with Dirichlet form bounded by $C(\ell,\beta)N^{-1}$. Since the space of densities concentrated on configurations with at most $(2\ell+1)b$ particles is compact for the weak topology and since the Dirichlet form is lower semicontinuous, the limit sup as $N \uparrow \infty$ of the last expression is bounded above by

$$\beta C_2(t_0, H) \max_{\ell_1} \sup_f \left\{ \int V_{\ell,b}(q,\eta) f(\eta) \nu_\varphi^{\ell_1,\ell_2}(d\eta) \right\}$$

where the supremum is now taken over all densities f concentrated on configurations with at most $(2\ell+1)b$ particles and with Dirichlet form equal to 0.

A density whose Dirichlet form equals to 0 is constant on hyperplanes with fixed total number of particles. Thus for $0 \le K \le b(2\ell{+}1)$, denote by $\nu_K^{\ell_1,\ell_2}$ the marginal of $\nu_\varphi^{\ell_1,\ell_2}$ on the hyperplane $\{\eta \in \mathbb{N}^{\{-\ell_1,\dots,\ell_2-1\}};\ \sum_{-\ell_1\le k<\ell_2} \eta(k) = K\}$:

$$\nu_K^{\ell_1,\ell_2}\Big(\, \cdot \, \Big) \;=\; \nu_\varphi^{\ell_1,\ell_2}\Big(\, \cdot \, \Big| \sum_{k=-\ell_1}^{\ell_2-1} \eta(k) = K \Big) \; .$$

Notice that the right hand side does not depend on φ. With this notation we may rewrite the last expression as

$$\beta C_2(t_0, H) \max_{\ell_1} \max_{0\le K\le b(2\ell+1)} \left\{ \int V_\ell(q,\eta) \nu_K^{\ell_1,\ell_2}(d\eta) \right\} .$$

To show that this expression converges to 0, we have to divide the box Λ_ℓ in boxes of some size n fixed. Let then $\ell \uparrow \infty$ and use a local central limit theorem to claim that the measure $\nu_K^{\ell_1,\ell_2}$ in $\{-n_1, \dots, n_2\}$ converges to some meaasure $\nu_{\varphi^*}^{n_1,n_2}$ if $K/2\ell$ converges to ρ and $\ell_1/2\ell$ to u_1. At this point we would have replaced the measures concentrated on fixed hyperplanes by product measures. To conclude the argument it would remain to use the law of large numbers.

We now turn to a detailed proof. Recall the explicit form of V_ℓ and fix a positive integer n. For $i = 1, 2$, denote by u_i the ratio $\ell_i/2\ell + 1$. Divide $\{-\ell_1, \dots, -1\}$ into $\Big[\ell_1/[nu_1]\Big]$ blocks of lenght $[nu_1]$ and eventually one of smaller size. Here for a real v, $[v]$ stands for the integer part of v. We do the same dividing the interval $\{0, \dots, \ell_2 - 1\}$ into blocks of lenght $[nu_2]$. Since g and a are bounded function, a simple computation shows that for n sufficiently large the last expression is bounded above by

$$C' \max_{\ell_1} \max_{0\le K\le b(2\ell+1)} \int \Big| \frac{1}{[nu_1]+[nu_2]} \sum_{k=-[nu_1]}^{[nu_2]-1} q_k g(\eta(k)) - a(K/2\ell+1) \Big| \nu_K^{\ell_1,\ell_2}(d\eta)$$
$$+\ O(n/\ell)\ +\ O(1/n)\ .$$

Here the constant C' depends on β and H. The local central limit theorem shows that for every fixed integer n_1, n_2, the marginal on $\mathbb{N}^{\{-n_1,\dots,n_2\}}$ of $\nu_K^{\ell_1,\ell_2}$ converges, as $\ell \uparrow \infty$, as $K/2\ell$ converges to ρ and as $\ell_1/2\ell$ converges to π_1 to $\nu_{\varphi(\rho,\pi_1)}^{n_1,n_2}$, where $\varphi(\rho, \pi_1)$ is implicitly given by

$$\pi_1 \nu_{\varphi(\rho,\pi_1)}^{n_1,n_2}[\eta(-1)]\ +\ (1-\pi_1)\nu_{\varphi(\rho,\pi_1)}^{n_1,n_2}[\eta(0)]\ =\ \rho$$

that is

$$\pi_1 M(\varphi p_1^{-1})\ +\ (1-\pi_1)M(\varphi p_2^{-1})\ =\ \rho\ .$$

This convergence is uniformly on compact intervals of $K/2\ell$ and ℓ_1/ℓ. Therefore the limit as $\ell \uparrow \infty$ of the last expression is bounded above by

$$C_3 \sup_{\pi_1\in E_\epsilon} \sup_{0\le\rho\le b} \left\{ \int \Big| \frac{1}{[nu_1]+[nu_2]} \sum_{k=-[nu_1]}^{[nu_2]-1} q_k g(\eta(k)) - a(\rho) \Big| \nu_{\varphi(\rho,\pi_1)}^{n_1,n_2}(d\eta) \right\}$$
$$+\ O(1/n)\ .$$

Since $\nu^{n_1,n_2}_{\varphi(\rho,\pi_1)}$ is a product measure $\{q_k g(\eta(k))\}$ are bounded independent random variables with expected value equal to $\varphi(\rho,\pi_1)$. Therefore, by the law of large numbers, the last expression converges, as $n\uparrow\infty$, to

$$C_3 \sup_{\pi_1\in E_\epsilon}\sup_{0\le\rho\le b}\Big|\varphi(\rho,\pi_1)-a(\rho)\Big|\ .$$

Notice that $\varphi(m_1,\rho)=a(\rho)$. Letting now $\epsilon\downarrow 0$ we conclude the proof of the one block estimate. $\square$

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