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NONCOMMUTATIVE LATTICES AND THEIR CONTINUUM LIMITS

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Abstract

We consider finite approximations of a topological space M by noncommutative lattices of points. These lattices are structure spaces of noncommutative C^*-algebras which in turn approximate the algebra $\mathcal{C}(M)$ of continuous functions on M. We show how to recover the space M and the algebra $\mathcal{C}(M)$ from a projective system of noncommutative lattices and an inductive system of noncommutative C^*-algebras, respectively.

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1 Introduction

Physical models are usually given by a suitable carrier (configuration or phase) space together with a dynamics on it. Nontrivial topological properties of the carrier space may have deep consequences independently of any particular dynamics. Topological solitons in semiclassical physics and the existence of inequivalent quantizations (θ-states) are two familiar examples.

Realistic physical models, however, are often too complicated to be solved exactly. One is obliged to use approximation methods. In this context, lattice theories have become a standard tool in the study of non perturbative aspects of physical models, especially of gauge theories. An important development would consist in trying to accommodate in the lattice picture also the topological aspects of the models. However, usual lattice theories [1] generally mix aspects of the carrier space and of the dynamics. Moreover, it is not obvious how topological information can be encoded. One can ask, for example, how the underlying continuum space (-time) can be recovered from the discrete data. In typical models, the only topological information refers to the underlying space (-time) and is that of nearest neighbors as encoded in the Hamiltonian. This does not define per se a notion of limit from which the space (-time) itself is recovered. A more substantial problem is that this incomplete topological information has no bearings on the configuration space of fields which is topologically trivial.

In a previous paper [2] we made an attempt to clarify some of these questions. In particular, we have shown that the notions of inductive and projective limit allow to recover a topological space M from a sequence of lattice approximations. We found that although the lattices had a trivial topology, namely were sets of isolated points, from the limit Q^∞ one could still recover the space being approximated, and its algebra of continuous functions. However, due to the too simple topological properties of the model, the construction could not be complete. On one side, at a fixed level of approximation, all topological information on M was lost. On the other side, Q^∞ had a universal character, in the sense that it was the same for all spaces M. In order to recover the latter it was necessary the additional input of a projection from Q^∞ onto M. This projection, though provided by the specific system of lattices approximating M, was not definable directly from the knowledge of Q^∞ . In this paper we will show how these drawbacks can be overcome with the use of noncommutative lattices, i.e., T_0 topological spaces with a finite (or countable) number of points [3, 4] as opposed to ordinary Hausdorff lattices.

This paper is part of the program initiated in [3, 4] where we investigate these lattices, and their geometry from an algebraic point of view. The starting point is a paper of Sorkin [5], where an approximation scheme for topological spaces is proposed. The idea is to approximate the space with a finite (or countable) topological space which is however not Hausdorff, but only T_0. Those space are Partially Ordered Sets (posets) and the topology is determined by the partial order. This approximation greatly improves the traditional (Hausdorff) finite approximation. With a finite number of points it is possible to reproduce correctly some relevant topological properties of the space being approximated. In the same paper Sorkin pointed out that the notion of projective system gives a well defined notion of continuum limit, from which the original space can be reconstructed.

In [3, 4] we developed the essential tools for doing quantum physics on posets. It was observed that posets are genuine noncommutative spaces (noncommutative lattices). Indeed, one can associate with any poset P a noncommutative algebra $\mathcal{A}$ of 'operator valued' continuous functions, from which P can be reconstructed as a topological spaces much in the same way in which the Gel'fand-Naimark procedure allows to reconstruct a Hausdorff topological space from the commutative algebra of continuous functions defined on it. Thus the term noncommutative lattice, which we use interchangeably with the term poset. The algebraic framework provides the way to discuss quantum mechanical and field theoretical models, in the spirit of Connes' noncommutative geometry [6]. It is important to notice that these noncommutative spaces, although have a poorer geometry than their continuum counterpart (but not a trivial one), present an extremely rich algebraic structure. In this way, topological information enter non trivially at all stages of the construction. In [4] we have also explicitly shown how not trivial topological effects are captured by these topological lattices and their algebras, by constructing algebraically the θ-quantizations of a particle on the noncommutative lattice approximation of a circle.

In this paper we analyze the continuum limit of a sequence of noncommutative lattices and of their related noncommutative algebras. The picture which emerges is the following. Given a topological space M with a sequence of finer and finer cellular decompositions, we construct a sequence of noncommutative lattices by associating any such a lattice P^n to each decomposition. The natural projections $\pi^{(n,m)}$ among the lattices in the sequence define a projective system. The projective limit P^∞ is a quasi fiber bundle[1] on M, and M itself is homeomorphic to the quotient of P^∞ by the equivalence relation defined by the projection from P^∞ to M. As we said above, this quotient is naturally defined in P^∞ which is then 'not too different' from M itself. This construction is then dualized. To each noncommutative lattice P^n we associate a noncommutative algebra $\mathcal{A}_n$ whose structure space, $\widehat{\mathcal{A}}_n$, is P^n. By pull-back, the projections $\pi^{(n,m)}$ define immersions $\Phi_{(m,n)}$ among the algebras, giving rise to a structure of inductive system. The inductive limit $\mathcal{A}_\infty$ results to be the dual of P^∞, namely $\widehat{\mathcal{A}}_\infty = P^\infty$. The algebra of continuous functions on M is the center of $\mathcal{A}_\infty$. Finally, we consider, with an analogous construction, an inductive system of Hilbert spaces $\mathcal{H}^n$ on which the algebras $\mathcal{A}_n$ act. The Hilbert space $L^2(M)$ is recovered as a suitable subspace of the inductive limit $\mathcal{H}^\infty$.

[1] The fibers above two distinct points do not need to be homeomorphic and not even of the same dimension.

2 Continuum Limit of Noncommutative Lattices

In this section we introduce the notion of continuum limit of a sequence of finite non Hausdorff topological approximations of a topological space M. In [4] we used noncommutative lattices to approximate topological spaces. These lattices were constructed starting from open coverings through a quotienting procedure. In this section we will modify this construction starting instead from a cellular decomposition. This point of view is more suitable for the analysis of the continuum limit of a sequence of such lattices and their related algebras.

Consider a topological space M of dimension d which admits a locally finite cellular decomposition, $\Sigma = \{S_{m,\alpha}\ \alpha \in I \subset N,\ 0 \leq m \leq d\}$. For convenience we will use cubic cells and $S_{m,\alpha}$ will then be closed cubes of dimension m. The lattice $P^{\Sigma}(M)$ is constructed by associating a point $p_{m\alpha}$ to each cube $S_{m\alpha}$. In the following, if convenient for clarity, we will use interchangeably $p_{m\alpha}$ and $S_{m\alpha}$ to indicate either a point of the lattice or a cube. We introduce now in $P^{\Sigma}(M)$ the partial order relation, $\preceq$, given by the inclusion of cubes

$$p_{m\alpha} \preceq p_{n\beta} \Leftrightarrow S_{m\alpha} \subseteq S_{n\beta}. \tag{2.1}$$

This partial order relation defines naturally a T_0 topology on $P^{\Sigma}(M)$ [2]. The topology is generated by the open sets $\mathcal{O}(S_{m\alpha})$ defined as

$$\mathcal{O}(S_{m\alpha}) = \{S_{m'\alpha'} : \quad S_{m'\alpha'} \subseteq S_{m\alpha}\} \tag{2.2}$$

Any such topological space is also called poset (partially ordered set) and is usually represented by a Hasse diagram [5]. We will use a different diagrammatic representation which is more transparent in some of our examples and which is described in fig. 2.1.

Example 2.1. As a simple example consider the interval $[0, 1]$ and the cubic decomposition

$$\Sigma_2 = \{S_{01} = 0,\ S_{02} = \frac{1}{2},\ S_{03} = 1,\ S_{11} = [0, \frac{1}{2}], S_{12} = [\frac{1}{2}, 1]\}. \tag{2.3}$$

The corresponding diagram is shown in fig. 2.1.

The corresponding noncommutative lattice is $P^2(I) = (p_{01}, p_{02}, p_{03}, p_{11}, p_{12},)$, the partial order reads

$$p_{01} \preceq p_{11}\ ,\quad p_{02} \preceq p_{11},\ ,\quad p_{02} \preceq p_{12},\ ,\quad p_{03} \preceq p_{12},\ , \tag{2.4}$$

where we have omitted writing the relations $p_{ij} \preceq p_{ij}$.

In an analogous way, one can construct the noncommutative lattice $P^n(I)$ corresponding to the cubic decomposition Σ_n in which the interval I is divided in 2^{n-1} subintervals

[2]A T_0 topology on a space M is a topology such that, given any two points of M, there exists an open set containing one point but not the other.

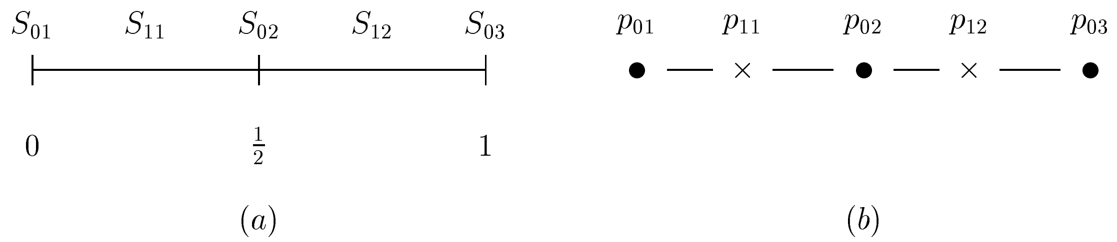

Fig. 2.1 Figure (a) shows the cell decomposition Σ_2 of the interval. Figure (b) represents the corresponding noncommutative lattice P^2. Any two points immediately related are joined by a line. The ordering relation is given by the convention that points denoted by $\times$ are preceded by points denoted by $\bullet$.

of equal length. In the following, when there is no ambiguity we shall write simply P^n instead of $P^n(I)$.

In the language of partially ordered sets, the smallest open set O_p containing a point $p \in P(M)$ consists of all q preceding p: $O_p = \{q \in P(M) \ : \ q \preceq p\}$. For instance, in fig. 2.1, this rule gives $\{p_{01}, p_{02}, p_{11}\}$ as the smallest open set containing p_{11}. These open sets generate the topology.

In order to recover M through a limiting procedure, a natural framework is the one of projective systems which we now briefly recall. A *projective* (or *inverse*) *system* of topological spaces is a family of topological spaces $Y^n, n \in \mathbf{N}$ [3] together with a family of continuous projections $\pi^{(m,n)} : Y^m \to Y^n$, $n \leq m$, with the requirements that $\pi^{(n,n)} = \mathbb{I}$, $\pi^{(n,m)} = \pi^{(n,p)}\pi^{(p,m)}$. The projective limit Y^∞ is defined as the set of coherent sequences, that is the set of sequences $\{x^n \in Y^n\}$ with $x^n = \pi^{(m,n)}(x^m)$. There is a natural projection $\pi^n : Y^\infty \to Y^n$ defined as:

$$\pi^n(\{x^m \in Y^m\}) = x^n \ . \tag{2.5}$$

The space Y^∞ is given a topology, by declaring that a set $\mathcal{O}^\infty \subset Y^\infty$ is open iff it is the inverse image of an open set belonging to some Y^n or a union (finite or infinite) of such sets.

Let us consider then a sequence $\Sigma_n = \{S^n_{m\alpha}, \ \alpha \in I^n\}$ of finer and finer cubic decompositions of M, obtained in the following way: Σ_{n+1} is obtained from Σ_n by an even subdivision of its cubes, the precise meaning of "even" is that for any point $x \in M$ and any open set $\mathcal{O}_x$ containing x, there must exist a level of approximation such that all cubes containing x will be contained in $\mathcal{O}_x$ from that level on [4]

$$\forall \ x \text{ and } \ \forall \ \mathcal{O}_x \ni x, \ \exists \ m \ \text{ such that } \ \forall \ n \geq m \ , {S^n}_\alpha \ni x \ \Rightarrow \ {S^n}_\alpha \subset \mathcal{O}_x \ . \tag{2.6}$$

[3] More generally, the index n could be taken in any directed set.
[4] We are really using decompositions which are 'fat' in the sense of [8].

The structure of projective system is given by the projections

$$\pi^{(m,n)} : P^m \to P^n \quad m > n \tag{2.7}$$

which associate to a cube of Σ_m the corresponding cube of Σ_n from which it comes, namely the lowest dimensional cube of Σ_n to which it belongs. Fig. 2.2 shows the case of the interval.

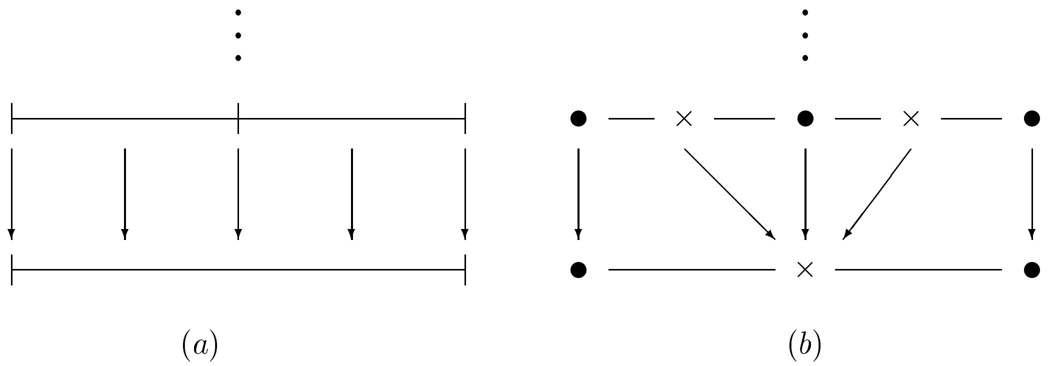

Fig. 2.2. The subdivision of a cell is illustrated in (a). The respective projection for the noncommutative lattices are indicated by arrows in (b)

We call $P^\infty(M)$ the projective limit of this projective system. A point in $P^\infty(M)$ is nothing but a coherent sequence $\{p^n\}$ of cubes, namely a sequence such that $p^{n+1} \subseteq p^n$. There exists a natural projection $\pi : P^\infty(M) \to M$. It is defined as follows:

$$\pi(\{p^n\}) = \bigcap_n p^n. \tag{2.8}$$

In this manner, we get a unique point of M. That this point is unique is a consequence of condition (2.6). Due to this projection, P^∞ results to be a quasi fiber bundle on M, namely a fiber bundle such that the fibers can change from point to point. A further characterization of P^∞ is given by the following observations. Let us introduce the set $M_0 \subset M$ which is the union of all closed $d-1$ cubes of all levels of approximations, namely

$$M_0 = \bigcup_n S^n_{d-1,\alpha}. \tag{2.9}$$

Then P^∞ is a quasi fiber bundle on M with the following properties:

1. Only the fibers above points of M_0 have more than one point. Suppose in fact that there are two distinct points, $p = \{p^n\}$ and $p' = \{p'^n\}$, of $P^\infty(M)$, projecting on the same point x of M. Since p and p' are different points of P^∞, there must be a level n such that $p^n \neq p'^n$. However, since p and p' project on the same point, the

intersection of p^n and p'^n must be not empty. But then the intersection can only belong to a lower dimensional cube of the boundary of both p^n and p'^n and thus x belongs to M_0.

2. Every fiber is a connected partially ordered set with a minimal point, namely a point p such that $p \preceq q$ for all the elements q in that fiber. This will be clear from the proof of Proposition 2. later on.

3. As it can be seen in simple examples, in general the nontrivial fibers are not all isomorphic.

Example 2.2. In the example of the interval the fibers of P^∞ are characterized as follows. I_0 is the set of all points of the interval of coordinates $m/2^n$, with m and n arbitrary non-negative integers. The fibers above points of $I - I_0$ have only one point. The fibers over the points of I_0 have three points and the structure of the poset V (except the boundary points of the interval whose fibers have only two points). The poset V is the poset shown in fig. 2.3 above the point x_3.

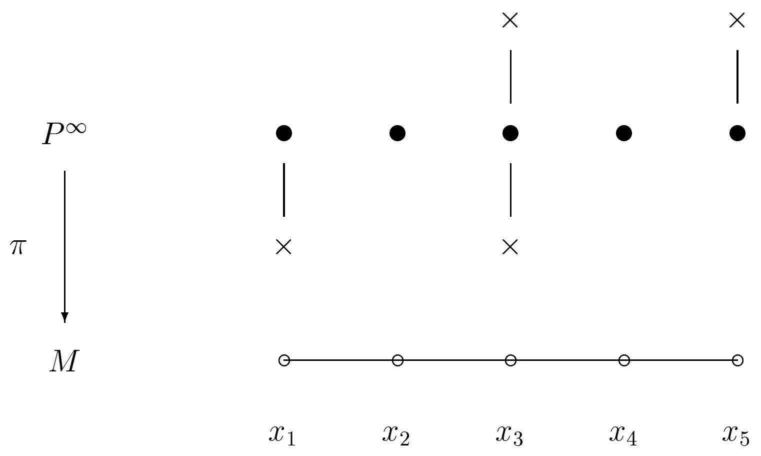

Fig. 2.3. The figure shows some of the fibers of P^∞. The fibers above points in I_0 can be identified with the poset V. For points in the complement of I_0, the fibers consist of only one point.

In [2], where we considered Hausdorff lattices, it was found that the projective limit, $Q^\infty(M)$, was bigger than the starting space M, and M was recovered from $Q^\infty(M)$ as a quotient defined by a projection from $Q^\infty(M)$ to M similar to the one in (2.8). Furthermore, $Q^\infty(M)$ was a universal space, namely it was a Cantor set independently on the starting space M and the information about M was contained in the projection. As in the case of Hausdorff lattices, P^∞ is bigger than M and M is recovered from P^∞

as a quotient [5]. However, unlike the Hausdorff case, P^∞ is not universal but is uniquely associated to M. In fact, as we will show in a moment, the quotient can now be defined also intrinsically from P^∞ without using the projection. We shall first prove the following.

Proposition 1. The space M is homeomorphic to $P^\infty/\sim$, where $\sim$ is the equivalence relation defined by the projection π in (2.8).

Proof.

We prove that: 1. π is continuous; 2. the quotient $P^\infty/\sim$ is actually homeomorphic to M.

1. π **is continuous**
We have to show that the inverse image of an open set B in M is open in $P^\infty(M)$. Let $p = \{S^n_{m,\alpha}(p)\}$ be a point belonging to $\pi^{-1}(B)$ and let $x = \pi(p)$. Because of condition (2.6) on the sequence of cubic decompositions there exists a $j \in N$ such that $n > j$ implies that all cubes $S^n_{m,\alpha}$ containing x are all contained in B. Consider then $\mathcal{O}^\infty_n = (\pi^n)^{-1}(\mathcal{O}(S^n_{m,\alpha}(p)))$ with $n > j$ which is an open set of $P^\infty(M)$ containing p. $\mathcal{O}^\infty_n$ is also entirely contained in $\pi^{-1}(B)$; all its points are coherent sequences whose representatives at level n are cells fully contained in $S^n_{m,\alpha}(p)$ and since the cube $S^n_{m,\alpha}(p)$ is fully contained in B they project to points in B.

2. $P^\infty/\sim$ **is homeomorphic to** M
To prove that the topology of M is equivalent to the quotient topology on $P^\infty/\sim$, it is then sufficient to show that the inverse image of a subset of M, which is *not* open, is not open in P^∞ as well. Consider then the set $\pi^{-1}(B) \subset P^\infty$, with $B \subset M$ *not* open. We will show that the assumption that $\pi^{-1}(B)$ is open leads to a contradiction. The statement that B is not open in the topology of M is equivalent to saying that there exists a sequence of points $\{x_i\}$ of M, not belonging to B, which converges to a point $x \in B$. From this sequence we will extract a particular subsequence $\{y_j\}$, still converging to x. We first introduce a countable basis of open neighborhoods for x, namely a countable family $\{\mathcal{O}_i\}$ of decreasing open sets containing x. Let us start with $\mathcal{O}_1$. Due to condition (2.6), there are one or more d-cubes $S^{n(1)}_{d,\alpha} \subset \mathcal{O}_1$, with $S^{n(1)}_{d,\alpha} \ni x$. At least one of these d cubes, call it $S^{n(1)}_{d,\alpha(1)}$, will contain an infinite number of elements of the sequence $\{x_i\}$. Then, choose y_1 to be any one of these elements. At the next level 2, there will again be at least one d-cube $S^{n(2)}_{d,\alpha(2)} \subset S^{n(1)}_{\alpha(1)}$, with $S^{n(2)}_{d,\alpha(2)}$ still containing an infinite number of elements of the sequence x_i. Again choose y_2 as any one of these elements. By iterating this procedure, we obtain the sequence $\{y_j\}$, which, being extracted from the original sequence, still converges to x. Moreover, $y_j \in S^{n(j)}_{d,\alpha(j)}$ and $\{S^{n(j)}_{d,\alpha(j)}\}$ is a coherent sequence [6] which thus defines a point $q \in Q^\infty$. By construction $\bigcap_j S^{n(j)}_{d,\alpha(j)} = x$, and consequently $\pi(q) = x$.

[5] This is the same result as the one found in [5]. The difference is that we consider simplicial decompositions instead of open coverings.

[6] A little care must be taken as $n(j)$ may not coincide with the level of the lattice.

Since $\pi^{-1}(B)$ is assumed to be open and recalling how the topology of Q^∞ is defined, there will be a $\overline{j}$ such that $\mathcal{O}_n^\infty = (\pi^{n(j)})^{-1}(\mathcal{O}(S^{n(j)}_{m,\alpha(j)}(p))) \subset \pi^{-1}(B)$ for $j \geq \overline{j}$. But then also $\pi^{-1}(y_j)$, with $j \geq \overline{j}$, must belong to $\pi^{-1}(B)$ and this implies, contrary to the hypothesis on the sequence $\{x_i\}$, that $y_j \in B$.

We show finally that the equivalence relation $\sim$ can be defined, without using any reference to the projection π, in the following way.

Proposition 2. For any $q_1, q_2 \in P^\infty$,

$$q_1 \sim q_2 \Leftrightarrow \exists q_3 \in P^\infty : q_3 \preceq q_1, q_3 \preceq q_2. \tag{2.10}$$

Proof.

The $\Rightarrow$ part can be proved by the explicit construction of q_3, for $q_1 \sim q_2$. The coherent sequence representing q_3 is in fact obtained by intersecting the elements corresponding to q_1 and q_2.

$$q_3{}^n = q_1{}^n \cap q_2{}^n \quad . \tag{2.11}$$

The $\Leftarrow$ part can be proven as follows. Suppose $q_3 \preceq q_1$, this implies that, at level n we have that

$$q^n{}_3 \subseteq q^n{}_1 \quad , \tag{2.12}$$

but then

$$\{\pi(q_3)\} = \bigcap_n q_3{}^n \subseteq \bigcap_n q_1{}^n = \{\pi(q_1)\} \quad . \tag{2.13}$$

From this equation it follows that $\pi(q_3) = \pi(q_1)$, hence $q_3 \sim q_1$. One can prove $q_3 \sim q_2$ in an analogous way.

So we now have a characterization of $P^\infty / \sim$ which is independent of the projection π, in other words $\sim$ is uniquely defined once P^∞ is given. With the use of noncommutative lattices we see then that, on one side, the lattice at finite level has a memory of the topology of the original manifold, on the other side, in the limit it is possible to recover unambiguously the original (Hausdorff) manifold M.

Before going to the algebraic description of the noncommutative lattices and their continuum limit let us consider a construction which will be useful in the following. A simple way to obtain the noncommutative lattice for a topological space, which is the cartesian product of lower dimensional spaces, is through the cartesian product of the corresponding noncommutative lattices. Suppose that P is the cartesian product of the noncommutative lattices P^i's. A relation on P is given by declaring that any two elements of P are related iff each component of one element is related to the corresponding component of the other. As an example, consider the cartesian product of a noncommutative lattice $P = \{p_1, \cdots, p_n\}$ with a noncommutative lattice $Q = \{q_1, \cdots, q_m\}$. The cartesian product

$$P \times Q = \{(p_j, q_k) \;\; p_j \in P \;, q_k \in Q\} \;, \tag{2.14}$$

is given the order relation

$$(p_i, q_j) \preceq (p_k, q_l) \Leftrightarrow p_i \preceq p_k \ \text{ and } \ q_j \preceq q_l \ . \tag{2.15}$$

A simple example is shown in fig. 2.4

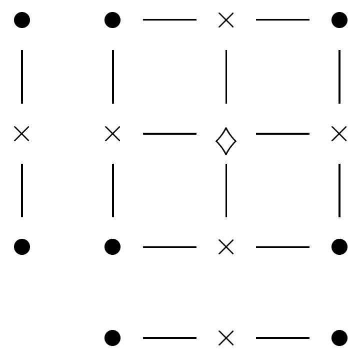

Fig. 2.4. The figure shows the cartesian product $P^1 \times P^1$. Points connected by links are ordered according to the convention $\bullet \preceq \times \preceq \diamondsuit$.

With this construction one can obtain the noncommutative lattice for any d-dimensional cubic decomposition of a space M. One takes the cartesian product of d copies of the noncommutative lattice $P^1(I)$ (like the 2-dimensional example of fig. 2.4) for each elementary d-dimensional cube in the decomposition. One then joins the cubes by identifying some of the faces. The projective system $\{P^n(M), \pi^{(n,m)}(M)\}$ and the projective limit $P^\infty(M)$ space can also be obtained in this way from the corresponding ones of the interval. If M is a d-dimensional cube C_d, a projective system is given by $\{P^n(C_d), \pi^{(n,m)}(C_d)\}$ where

$$P^n(C_d) = \underbrace{P^n \times P^n \times \ldots \times P^n}_{d} \ , \tag{2.16}$$

where P^n are the noncommutative lattices of the interval, and

$$\pi^{(m,n)}(C_d)(p_1, \cdots, p_d) \equiv (\pi^{(m,n)} p_1, \cdots, \pi^{(m,n)} p_d). \tag{2.17}$$

The projective limit is

$$P^\infty(C_d) = \underbrace{P^\infty \times P^\infty \times \ldots \times P^\infty}_{d} . \tag{2.18}$$

Finally the cube itself is obtained from $P^\infty(C_d)$ as the quotient defined by the projection $\pi(C_d) = (\pi, \cdots, \pi)$ from $P^\infty(C_d)$ onto the cube.

This construction has a straightforward algebraic translation in terms of tensor products.

3 The Continuum Limit of Algebras

3.1 The Algebras of Noncommutative Lattices

We shall now associate with each noncommutative lattice P an algebra of operator valued functions which contain the same topological information, in the sense that P can be reconstructed as a topological space from the algebra. For a Hausdorff space M, the natural choice would be the algebra of continuous functions $\mathcal{C}(M)$ on M. As it is known [9], from $\mathcal{C}(M)$ the space M itself can be reconstructed as the set $\widehat{\mathcal{C}(M)}$ of irreducible representations (IR's) of $\mathcal{C}(M)$ or, equivalently, as the set of complex homomorphisms of $\mathcal{C}(M)$. The topology on $\widehat{\mathcal{C}(M)}$ is then defined by pointwise convergence,

$$p_n \to p \iff p_n(f) \to p(f) \quad \forall f \in \mathcal{C}(M), \quad p_n, p \in \widehat{\mathcal{C}(M)}. \tag{3.1}$$

Actually there is a complete correspondence between abelian C^*-algebras (with unit) and (compact) Hausdorff spaces [9].

However, since our noncommutative lattice P is not Hausdorff, the usual algebra of $\mathbb{C}$-valued continuous functions on P is not able to capture the topological structure of P. It is easy to see, indeed, that the only continuous complex valued functions on P are the constant ones and $\mathcal{C}(P)$ is then equal to $\mathbb{C}$. Since $\widehat{\mathbb{C}} = \widehat{\mathcal{C}(P)}$ consists of a single point, we see that the structure space of $\mathcal{C}(P)$ is not P and is actually trivial. The algebra $\mathcal{C}(P)$, in other words, because of the continuity property of its elements, identifies those points of P which cannot be separated by the topology and thus gives rise to a space with a single point.

It is however possible to reconstruct the noncommutative lattice P and its topology from a *noncommutative* C*-algebra of operator-valued functions. Given a noncommutative C^*-algebra, $\mathcal{A}$, one can construct a topological space $\widehat{\mathcal{A}}$, not Hausdorff in general, by considering again the set of all IR's (in general not anymore one dimensional) of $\mathcal{A}$. One gives a topology to $\widehat{\mathcal{A}}$, called regional topology, by giving a notion of closeness among IR's of different dimensions [9]. For a particular class of algebras, called postliminal algebras [9], which includes the ones we will be interested in, there is an equivalent but easier, construction of $\widehat{\mathcal{A}}$ in terms of a particular class of ideals of $\mathcal{A}$. The kernels of IR's of an algebra $\mathcal{A}$ are bilateral ideals, called primitive ideals. Generally, different IR's may have the same kernel, however for postliminal algebras there is a one to one correspondence between IR's and primitive ideals. For these algebras then, the space $\widehat{\mathcal{A}}$ can be equivalently regarded as the set of primitive ideals and the topology can be equivalently given as the Hull kernel topology. This topology is defined by giving the procedure to construct the closure, $\overline{S}$, of an arbitrary set, S, of primitive ideals. If $S = \{\mathcal{I}_\lambda, \ \lambda \in \Lambda\}$ where $\mathcal{I}_\lambda$ is a family of primitive ideals parametrized by λ, then

$$kerS \equiv \{\bigcap_{\lambda\in\Lambda} \mathcal{I}_\lambda\} \tag{3.2}$$

and

$$\overline{S} \equiv Hull\ kerS \equiv \{\mathcal{I} : kerS \subset \mathcal{I},\ \mathcal{I} \in \hat{\mathcal{A}}\}. \tag{3.3}$$

Since what we have said about primitive ideals holds also in the commutative case, we give a simple abelian example to clarify this definition. Consider the algebra $C(I)$ of complex continuous functions on an interval I, and for $a < b \in I$ let S be

$$S = \{\mathcal{I}_\lambda,\ \lambda \in]a,b[\} \tag{3.4}$$

where $\mathcal{I}_\lambda$ is the primitive ideal of $C(I)$ given by the kernel of the homomorphism

$$\lambda : f \to \lambda(f) \equiv f(\lambda) \quad f \in C(I) \tag{3.5}$$

that is the set of functions vanishing at λ

$$\mathcal{I}_\lambda = \{f : f(\lambda) = 0\}. \tag{3.6}$$

$KerS$ is then

$$kerS = \bigcap_{\lambda \in]a,b[} \mathcal{I}_\lambda = \{f : f(\lambda) = 0\ \forall \lambda \in]a,b[\}. \tag{3.7}$$

Now, since the functions f are continuous, we also have

$$\begin{aligned} \overline{S} = Hull\ kerS &= \{\mathcal{I} : kerS \subset \mathcal{I}\} & (3.8)\\ &= S \bigcup \{\mathcal{I}_a, \mathcal{I}_b\} & (3.9)\\ &= \{\mathcal{I}_\lambda,\ \lambda \in [a,b]\}. & (3.10) \end{aligned}$$

which IS the closure of the open interval $]a,b[$.

¿From this definition of topology we see that if an ideal $\mathcal{I}_{\lambda_1}$ is included in an ideal $\mathcal{I}_{\lambda_2}$, then the point $\mathcal{I}_{\lambda_2}$, being in the closure of $\mathcal{I}_{\lambda_1}$, is a limit point of $\mathcal{I}_{\lambda_1}$. The topology of a noncommutative lattice or, which is the same, the order relation of the poset, is then simply encoded in the inclusion relation among the primitive ideals of the corresponding algebra.

As it is proven in [9], each finite T_0 topological space is the structure space of a noncommutative C^*-algebra.

Before giving few examples of the reconstruction theorem we recall few facts about compact operators which play a crucial role in our algebras. An operator in a Hilbert space $\mathcal{H}$ is said to be of finite rank if the orthogonal complement of its null space is finite dimensional. An operator k in $\mathcal{H}$ said to be compact if it can be approximated arbitrarily closely in norm by finite rank operators. If $\lambda_1, \lambda_2, ...$ are the eigenvalues of k^*k for such a k, with $\lambda_{i+1} \leq \lambda_i$ and an eigenvalue of multiplicity n occurring n times in this sequence, (here and in what follows, $*$ denotes the adjoint for an operator) then $\lambda_n \to 0$ as $n \to \infty$. It follows that the operator $\mathbb{1}$ in an infinite dimensional Hilbert space is not compact.

The set $\mathcal{K}$ of all compact operators k in a Hilbert space is a C*-algebra . It is a two-sided ideal in the C*-algebra $\mathcal{B}$ of all bounded operators [9]. The construction of $\mathcal{A}$

for a noncommutative lattice rests on the following result from the representation theory of $\mathcal{K}$. The representation of $\mathcal{K}$ by itself is irreducible [9] and it is the <u>only</u> IR of $\mathcal{K}$ up to equivalence.

Example 3.1. The simplest nontrivial noncommutative lattice is $\widetilde{P} = \{p_1, p_2\}$ with $p_1 \prec p_2$. It is shown in fig. 3.1 (a).

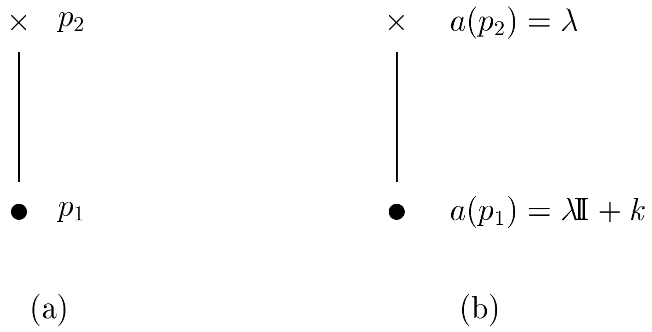

Fig. 3.1. (a) is the poset $\widetilde{P}$ with two points. (b) shows the values of a generic element $\lambda\mathbb{I} + k$ of its algebra A at its two points p_1 and p_2.

We associate an infinite dimensional Hilbert space $\mathcal{H}$ to this poset. The corresponding algebra is then the subalgebra of the bounded operators $\mathcal{B}(\mathcal{H})$ given by

$$\mathcal{A} = \mathbb{C}\mathbb{1} + \mathcal{K}(\mathcal{H}) = \{\lambda\mathbb{1} + k \, : \, \lambda \in \mathbb{C}, k \in \mathcal{K}(\mathcal{H})\} \, . \tag{3.11}$$

In order to construct the IR's of $\mathcal{A}$ we recall a known theorem [10] which states that the IR's of a $C^* - subalgebra$ of $\mathcal{B}(\mathcal{H})$ which includes the algebra of compact operators $\mathcal{K}(\mathcal{H})$ are of two kinds: either they vanish on the compact operators or they are unitary equivalent to the defining representation. In our case then we will have only two IR's (up to unitary equivalence), the first one, π_1, which is one dimensional and vanishes on $\mathcal{K}(\mathcal{H})$, is given by

$$\pi_1(\lambda\mathbb{I}_{\mathcal{H}} + k) = \lambda. \tag{3.12}$$

The second one, π_2, is just the defining representation:

$$\pi_2(a) = a, \;\; a \in \mathcal{A}. \tag{3.13}$$

The kernels $\mathcal{I}_1$, $\mathcal{I}_2$ of π_1 and π_2 are then the only two primitive ideals of $\mathcal{A}$, and are

$$\mathcal{I}_1 \;=\; \mathcal{K}(\mathcal{H}) \tag{3.14}$$
$$\mathcal{I}_2 \;=\; \{0\}. \tag{3.15}$$

Since the second ideal is included in the first one we see that $\widehat{\mathcal{A}}$ is actually the noncommutative lattice $\widetilde{P}$ of fig. 3.1 (a) if we identify $\mathcal{I}_1$ and $\mathcal{I}_2$ with p_1 and p_2 respectively. An arbitrary element $\lambda\mathbb{1} + k$ of $\mathcal{A}$ can be regarded as a "function" on it if, in analogy to the commutative case, we set

$$\begin{aligned} (\lambda\mathbb{1} + k)(p_2) &:= \lambda \\ (\lambda\mathbb{1} + k)(p_1) &:= \lambda\mathbb{1} + k \ . \end{aligned} \tag{3.16}$$

Notice that in this case the function $\lambda\mathbb{1} + k$ is not valued in $\mathbb{C}$ at all points. Indeed, at different points it is valued in different spaces, $\mathbb{C}$ at p_2 and a subset of bounded operators on an infinite Hilbert space at p_1.

Example 3.2. We next consider the noncommutative lattice V consisting of three points p_1, p_2, p_3 as in fig. 3.2.

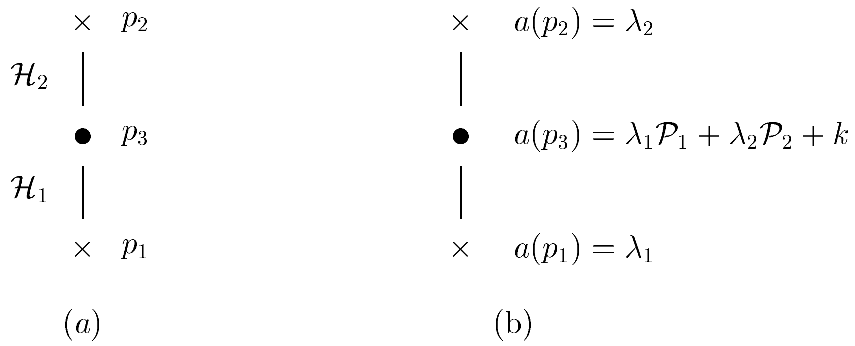

Fig. 3.2. (a) shows the noncommutative lattice V and the association of an infinite dimensional Hilbert space $\mathcal{H}_i$ to each of its "arms". (b) shows the values of a typical element $a = \lambda_1\mathcal{P}_1 + \lambda_2\mathcal{P}_2 + k$ of its algebra at its three points.

The noncommutative lattice V has two arms 1 and 2. In order to construct the associated algebra, we attach an infinite-dimensional Hilbert space $\mathcal{H}_i$ to each arm i as shown in fig. 3.2. Let $\mathcal{P}_i$ be the orthogonal projector on $\mathcal{H}_i$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K}_{12} = \{k_{12}\}$ be the set of all compact operators in $\mathcal{H}_1 \oplus \mathcal{H}_2$. The associated algebra is then

$$\mathcal{A} = \mathbb{C}\mathcal{P}_1 + \mathbb{C}\mathcal{P}_2 + \mathcal{K}_{12} \ . \tag{3.17}$$

We will indicate an element a of this algebra either as $a = \lambda_1\mathcal{P}_1 + \lambda_2\mathcal{P}_2 + k$ or, with an obvious notation, $a = (\lambda_1, \lambda_2) + k$. Using the same characterization of IR's as in the previous example we can see that this algebra has three IR's π_1, π_2, π_3. The representations π_1 and π_2, which are the ones vanishing on the compact operators, are one-dimensional and are given by

$$\pi_1(\lambda_1\mathcal{P}_1 + \lambda_2\mathcal{P}_2 + k) = \lambda_1 \tag{3.18}$$
$$\pi_2(\lambda_1\mathcal{P}_1 + \lambda_2\mathcal{P}_2 + k) = \lambda_2. \tag{3.19}$$

The representation π_3 is the defining one of $\mathcal{A}$:

$$\pi_3(a) = a \quad a \in \mathcal{A}. \tag{3.20}$$

The corresponding kernels, $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ are

$$\begin{aligned} \mathcal{I}_1 &= ker\pi_1 = \{a \in \mathcal{A} : a = \lambda\mathcal{P}_2 + k\} & (3.21)\\ \mathcal{I}_2 &= ker\pi_2 = \{a \in \mathcal{A} : a = \lambda\mathcal{P}_1 + k\} & (3.22)\\ \mathcal{I}_3 &= ker\pi_3 = \{0\}. & (3.23) \end{aligned}$$

Since the inclusion relations among the primitive ideals are $\mathcal{I}_3 \subset \mathcal{I}_1$ and $\mathcal{I}_3 \subset \mathcal{I}_2$, we see that, with the identifications $p_i = \pi_i$, $\widehat{\mathcal{A}}$ is equal to P.

Example 3.3. In general, consider a noncommutative lattice P which is obtained by joining two other noncommutative lattices P_1 and P_2 by identifying some closed set of points. This operation has a simple algebraic translation. The algebra $\mathcal{A}$ of P is obtained in two steps; first consider the direct sum $\overline{\mathcal{A}} = \mathcal{A}_1 \oplus \mathcal{A}_2$ of the two algebras associated to P_1 and P_2. The algebra $\mathcal{A}$ of P is then equal to the subalgebra of $\overline{\mathcal{A}}$ consisting of those elements which have the same value at the points (seen as IR's) which are identified.

Consider the construction of the algebra $\mathcal{A}_n$ of a generic noncommutative lattice P^n for the interval. Such a noncommutative lattice is composed of a number $N = 2^{n-1} - 1$ of V's plus the elementary noncommutative lattice $\widetilde{P}$ with two points, of example 3.1, at each end. Number the arms and attach an infinite dimensional Hilbert space $\mathcal{H}_i$ to each arm i. To a V with arms $i, i+1$, attach the algebra $\mathcal{A}_i$ with elements $\lambda_i\mathcal{P}_i + \lambda_{i+1}\mathcal{P}_{i+1} + k_{i,i+1}$. Here λ_i, λ_{i+1} are any two complex numbers, $\mathcal{P}_i, \mathcal{P}_{i+1}$ are orthogonal projectors on $\mathcal{H}_i$, $\mathcal{H}_{i+1}$ in the Hilbert space $\mathcal{H}_i \oplus \mathcal{H}_{i+1}$ and $k_{i,i+1}$ is any compact operator in $\mathcal{H}_i \oplus \mathcal{H}_{i+1}$. This is as before. But now, for gluing the various algebras together, we also impose the condition $\lambda_j = \lambda_k$ if the lines j and k meet at a top point. Taking into account also the algebras $\mathcal{A}_0$ and $\mathcal{A}_{N+1}$ for the noncommutative lattice $\widetilde{P}$ at the beginning and at the end, we can express the algebra $\mathcal{A}_n$ as a direct sum plus this condition. A generic element a_n of $\mathcal{A}_n$ has then the form

$$a = (\lambda_1\mathcal{P}_1 + k_1) \bigoplus_{i=1}^{N} a_i \oplus (\lambda_{N+1}\mathcal{P}_{2N+2} + k_{2N+2}), \qquad a_i = \lambda_i\mathcal{P}_{2i} + \lambda_{i+1}\mathcal{P}_{2i+1} + k_{2i,2i+1} \tag{3.24}$$

There is a systematic and simple construction of $\mathcal{A}$ for any noncommutative lattice (that is, any "finite T_0 topological space") which is given in [11].

It should be remarked that actually the noncommutative lattice does not uniquely fix its algebra as there are in general many non-isomorphic (noncommutative) C*-algebras with the same poset as structure space [11]. This is to be contrasted with the Gel'fand-Naimark result asserting that the (commutative) C*-algebra associated to a Hausdorff topological space (such as a manifold) is unique.

Fig. 3.3. A noncommutative lattice for the segment constructed by gluing several V posets and a pair of two point posets at the ends.

3.2 Direct Systems of Algebras and their Limits

In this section we shall consider the question of the continuum limit of the system of algebras $\mathcal{A}_n$ associated with the noncommutative lattices P^n which are obtained from the cubic decompositions of a topological space M. We shall construct an inductive system of algebras whose inductive limit $\mathcal{A}_\infty$ will have the following two properties:

a) The structure space of $\mathcal{A}_\infty$ is P^∞.

b) The algebra of continuous functions on M coincides with the center of $\mathcal{A}_\infty$.

We prove these statements for the interval I. We shall then show how to extend the results to any topological space M admitting cubic decompositions.

A natural framework to define the continuum limit of the algebras $\mathcal{A}_n$ is that of inductive system of algebras which dualizes the projective system of noncommutative lattices. Let us then recall the notion of inductive system and inductive limit of C^*-algebras. An *inductive system* of C^*-algebras is a sequence of C^*-algebras $\mathcal{A}_n$, together with norm non-increasing immersions $\Phi_{(n,m)} \ : \ \mathcal{A}_n \to \mathcal{A}_m, \ n < m$, such that the composition law $\Phi_{(n,m)}\Phi_{(m,p)} = \Phi_{(n,p)}, \ n < m < p$, holds.

The inductive limit $\mathcal{A}_\infty$ is the C*-algebra consisting of equivalence classes of "Cauchy sequences" $\{a_n\}, \ \ a_n \in \mathcal{A}_n$. Here by Cauchy sequence we mean that $||\Phi_{(n,m)}(a_n) - a_m||_m$ goes to zero as n and m go to infinity. Two sequences $\{a_n\}$ and $\{b_n\}$ are equivalent if $||a_n - b_n||_n$ goes to zero. The norm in $\mathcal{A}_\infty$ is defined by

$$||a||_\infty = \lim_{n\to\infty} ||a_n||_n \tag{3.25}$$

where $\{a_n\}$ is any of the representatives of a. In case the algebras $\mathcal{A}_n$ are realized as an increasing sequence of C^*-subalgebras of a given C^*-algebra $\mathcal{A}$, the inductive limit is just the closure of the union of the algebras $\mathcal{A}_n$. [7].

[7]For a more detailed account of the definition see for example [9] or [10].

Let us now consider the interval $I = [0, 1]$. Its projective system of noncommutative lattices P^n has been described in section 2 and is illustrated in fig. 2.2. The algebra $\mathcal{A}_n$ of P^n has been introduced in section 3.1 and the form of the generic element is given in eq. (3.24).

In this case the immersions $\Phi_{(n,m)}$ are obtained through a suitable definition of pull-back of the projections $\pi^{(m,n)}$. The usual definition of pull-back would be

$$p^m_{l\alpha}(\Phi_{(n,m)}(a^n)) = p^n_{l'\alpha'}(a_n) \tag{3.26}$$

where

$$\pi^{(m,n)}(p^m_{l\alpha}) = p^n_{l'\alpha'}, \qquad a^n \in \mathcal{A}_n, \quad p^m_{l\alpha} \in P^m, \quad p^n_{l'\alpha'} \in P^n. \tag{3.27}$$

(Here we are thinking of the elements $p^n_{l\alpha} \in P^n$ as IR's of $\mathcal{A}_n$.) Eq. (3.26) however, makes sense only when the two representations $p^m_{l\alpha}$ and $p^n_{l'\alpha'}$ have the same dimension but this is not always the case. Due to the refinement procedure of the decompositions and to the properties of the projections $\pi^{(m,n)}$, the representations have different dimensions when $l = 0$ and $l' = 1$. Then, $p^m_{0\alpha}$ is a point of P^m corresponding to a zero dimensional cube and thus is an infinite dimensional representation

$$p^m_{0\alpha}(a^m) = \mu_\alpha \mathbb{I}_{\mathcal{H}_\alpha} + \mu_{\alpha+1}\mathbb{I}_{\mathcal{H}_{\alpha+1}} + k_\alpha \ . \tag{3.28}$$

Also, $p^n_{1'\alpha'}$ is a point of P^n corresponding to a one dimensional cube and is the one dimensional representation

$$p^n_{1\alpha'}(a^n) = \lambda_{\alpha'}. \tag{3.29}$$

Eq. (3.26) is replaced in this case by

$$p^m_{0\alpha}(\Phi_{(n,m)}(a^n)) = p^n_{1\alpha'}(a_n) \otimes \mathbb{I}_{\mathcal{H}_\alpha \oplus \mathcal{H}_{\alpha+1}}. \tag{3.30}$$

This equation is now meaningful and can be solved for the value of $\Phi_{(n,m)}(a^n)$ at the point $p^m_{0\alpha}$. More explicitly, it gives

$$\mu_\alpha \mathbb{I}_{\mathcal{H}_\alpha} + \mu_{\alpha+1}\mathbb{I}_{\mathcal{H}_{\alpha+1}} + k_\alpha = \lambda_{\alpha'} \otimes \mathbb{I}_{\mathcal{H}_\alpha \oplus \mathcal{H}_{\alpha+1}} \tag{3.31}$$

whose solution is

$$\mu_\alpha = \mu_{\alpha+1} = \lambda_{\alpha'}, \qquad k_\alpha = 0 \ . \tag{3.32}$$

This is shown in fig. 3.4. In general, for $\Phi_{(n,n+1)} : \mathcal{A}_n \to \mathcal{A}_{n+1}$ we have

$$\begin{aligned} \Phi_{n,n+1}[(\lambda_1 \mathcal{P}_1 + k_1) \bigoplus_{i=1}^{N} (\lambda_i \mathcal{P}_{2i} + \lambda_{i+1}\mathcal{P}_{2i+1} + k_{2i,2i+1}) \oplus (\lambda_{N+1}\mathcal{P}_{2N+2} + k_{2N+2})] = \\ = (\lambda_1 \mathcal{P}_1 + k_1) \bigoplus_{i=1}^{N} \Big[(\lambda_i \mathcal{P}_{4i-2} + \lambda_i \mathcal{P}_{4i-1}) \oplus (\lambda_i \mathcal{P}_{4i} + \lambda_{i+1}\mathcal{P}_{4i+1} + k'_{4i,4i+1}) \oplus \\ \oplus (\lambda_{i+1}\mathcal{P}_{4i+2} + \lambda_{i+1}\mathcal{P}_{4i+3})] \oplus (\lambda_{N+1}\mathcal{P}_{4N+4} + k'_{4N+4}) \end{aligned} \tag{3.33}$$

where $k'_{4i,4i+1} = k_{2i,2i+1}$ and $k'_{4N+4} = k_{2N+2}$.

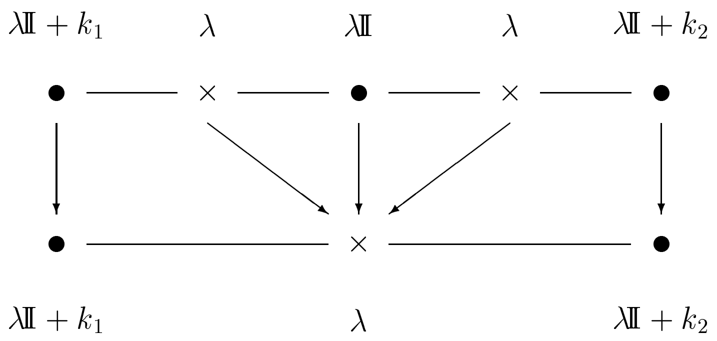

Fig. 3.4. The figure shows two noncommutative lattices corresponding to two consecutive levels of approximation. The arrows indicates the projection. The pullback of a generic function on the lower level to the upper level is indicated.

With these immersions $\Phi_{(n,m)}$, the set of algebras $\mathcal{A}_n$ is made into an inductive system. We want to find now the inductive limit $\mathcal{A}_\infty$, and prove that it is the dual of P^∞ and that its center is $C(I)$.

Instead of constructing abstractly the inductive limit $\mathcal{A}_\infty$ we will represent it as a C^*-algebra of operator valued functions on the interval. This construction is suggested by the following theorem due to Dauns and Hofmann [12]:

Dauns-Hofmann theorem: let $\mathcal{A}$ be a C^*-algebra with identity and M the space of maximal ideals of the center of $\mathcal{A}$ with the Hull-kernel topology. Then $\mathcal{A}$ is isometrically *-isomorphic to the C^*-algebra of all continuous sections $\Gamma(\pi)$ of a C^*- bundle $\xi = (\pi, B, M)$ over M. The fiber above $m \in M$ is the quotient C^*-algebra $\mathcal{A}/m\mathcal{A}$, the isometric *-isomorphism is the Gel'fand representation $x \to \hat{x}$, where $\hat{x}(m) = x + m\mathcal{A}$, and the norm of $\hat{x}$ is given by

$$||\hat{x}|| = sup\{||\hat{x}(m)|| : m \in M\}. \tag{3.34}$$

Further, the real valued map $m \to ||\hat{x}(m)||$ on M is upper semicontinuous [7] for each $x \in \mathcal{A}$.

Let us then construct the C^*-algebra $\mathcal{A}_\infty$ as an algebra of operator valued functions on I. Recall that P^∞ is a quasi fiber bundle on I whose fibers at the points of the set $I_0 = \{x = \bigcup_{n,\alpha} S^n_{0,\alpha}\}$ are the noncommutative lattice V (see example 2.2) while they are made of a single point for $I - I_0$. Now, the algebra corresponding to V is $\{\lambda_1 \mathbb{I}_{\mathcal{H}_1} + \lambda_2 \mathbb{I}_{\mathcal{H}_2} + k\}$ while the one corresponding to a point is $\mathbb{C}$. We are thus led to consider an element of $\mathcal{A}_\infty$ as an operator valued function on I of the kind

$$a(x) = \begin{cases} (\lambda_+(x), \lambda_-(x)) + k(x) & x \in I_0 \\ \lambda(x) & x \in I - I_0 \end{cases} \tag{3.35}$$

where :

a) the function $\lambda(x)$ is continuous and bounded in $I - I_0$ and $\lim_{x\to\overline{x}^\pm} \lambda(x) = \lambda_\pm(\overline{x})$ for all $\overline{x} \in I_0$.

b) $\forall\epsilon \geq 0$ there exist only a finite number of points $\overline{x} \in I_0$ such that $||(\lambda_+(\overline{x}), \lambda_-(\overline{x})) + k(\overline{x})|| > \epsilon$.

The norm of any $a(x) \in \mathcal{A}_\infty$ is defined as $||a|| = sup_{x\in I}||a(x)||$.

It is easy to see that any element a_n at finite level can be regarded as a function of this kind having as $\lambda(x)$ a stepwise function and $k(x)$ different from zero only at a finite number of points. This provides an embedding $\Phi_n : \mathcal{A}_n \to \mathcal{A}_\infty$. It can then be verified that, due to conditions a and b, the closure of the union of the algebras $\Phi_n(\mathcal{A}_n)$ is $\mathcal{A}_\infty$ so that $\mathcal{A}_\infty$ is the inductive limit of the system $\{\mathcal{A}_n, \pi^{(m,n)}\}$.

Although we have given the structure of the fibers and realized the algebra $\mathcal{A}_\infty$ as sections, we have not specified yet the topology of the bundle with respect to which $\mathcal{A}_\infty$ is the set of all continuous functions. The existence of such a topology, however, follows from the fact that our presentation of $\mathcal{A}_\infty$ coincides with the one given in the Dauns-Hofmann theorem. Indeed,

1. The center of $\mathcal{A}_\infty$ is the set of continuous functions of the interval, namely is the set of sections such that

$$\lambda_-(x) = \lambda_+(x) \ , \quad k(x) = 0 \ , \qquad \forall x \in I. \tag{3.36}$$

   Indeed, since the product is pointwise, even the characterization of the center can be done pointwise. Now, only at the points x in I_0 the algebras $\mathcal{A}_\infty(x)$ will be nonabelian, and their elements will be of the form $(\lambda_-(x), \lambda_+(x)) + k(x)$. It is then easy to check that for an element of this kind to be in the center, eq.(3.36) has to be satisfied at that point.

2. One easily proves that the fibers, $\mathcal{A}(x)$, over any $x \in I$, given by the Dauns-Hofmann theorem as the quotient $\mathcal{A}_\infty/x\mathcal{A}_\infty$, coincides with ours. They are either a copy of $\mathbb{C}$ if $x \in I - I_0$ or a copy of the algebra for a V if $x \in I_0$.

3. The sections that, according to the theorem, should be associated to elements of $\mathcal{A}_\infty$ are the sections that we used to define the elements themselves.

Finally, we want to show that the structure space of $\mathcal{A}_\infty$ is just P^∞. Now, under the conditions of the Dauns-Hofmann theorem, by a result due to Varela [13], each IR of $\mathcal{A}_\infty$ factors through the evaluation map, namely, given an IR T of $\mathcal{A}_\infty$ there exists a unique point $x \in I$ and a point $q \in \widehat{\mathcal{A}}(x)$ such that

$$T(a) = q(a(x)). \tag{3.37}$$

since $\widehat{\mathcal{A}}(x)$ is homeomorphic to the fiber $\pi^{-1}(x)$ of P^∞ we see then that there is a correspondence one to one between IR's of $\mathcal{A}_\infty$ and points of P^∞. That $\widehat{\mathcal{A}}_\infty$ is homeomorphic to P^∞ can be finally checked by observing that closed sets of P^∞ are also closed in the Hull-kernel topology of $\mathcal{A}_\infty$ and vice versa.

The generic case of a topological space M can be treated by reducing it to the one of the interval. Given a space M, consider a sequence of finer and finer decompositions Σ_n which satisfies condition (2.6). The corresponding projective system of noncommutative lattices $P^n(M)$ and its limit $P^\infty(M)$ were described at the end of section 2. In order to construct the dual inductive system, consider again a cubic decomposition of M. The algebra $\mathcal{A}_n(C_d)$, of the noncommutative lattice $P^n(C_d)$ of a single cube given in eq. (2.16), is simply the tensor product

$$\mathcal{A}_n(C_d) = \underbrace{\mathcal{A}_n \otimes \mathcal{A}_n \cdots \otimes \mathcal{A}_n}_{d} \ , \tag{3.38}$$

where $\mathcal{A}_n$ is the algebra of the noncommutative lattice P^n. This fact is a consequence of properties of IR's of the algebra of compact operators and can be proven straightforwardly. The algebraic operation of joining cubes is a direct generalization of the one described in Example 3 of section 3.1. We thus obtain a sequence of algebras $\{\mathcal{A}_n(M)\}$ dual to $\{P^n(M)\}$. The inductive system is $\{\mathcal{A}_n(M), \Phi_{(n,m)}(M)\}$ where, as usual, the immersions $\Phi_{(n,m)}(M)$ are defined as the pull-backs of the projections $\pi^{(m,n)}(M)$. Since the immersions $\Phi_{(n,m)}(M)$ do not mix different factors in the direct sums of algebras, the gluing procedure and the limiting procedure commute. In order to find $\mathcal{A}_\infty(M)$ and to show that is the dual of $P^\infty(M)$ one can concentrate then on a single cube. For a single cube, the following properties of $\mathcal{A}_n(C_d)$ can be checked along the same lines as for the interval,

1 The inductive limit $\overline{\mathcal{A}}_\infty$ is the tensor product of the inductive limits $\mathcal{A}_\infty$

$$\overline{\mathcal{A}}_\infty = \underbrace{\mathcal{A}_\infty \otimes \mathcal{A}_\infty \cdots \otimes \mathcal{A}_\infty}_{d} \tag{3.39}$$

2 The structure space of the algebra $\mathcal{A}_\infty(x)$, generated by evaluating the elements of $\overline{\mathcal{A}}_\infty$ at the point x of the cube, is homeomorphic to the fiber of $\overline{P}^\infty$ at the point x.

3 The center of $\overline{\mathcal{A}}_\infty$ is the algebra of continuous functions of the cube.

4 The section associated by the Dauns-Hofmann theorem to an element of $\overline{\mathcal{A}}_\infty$ is the operator valued function which defines the element itself.

4 Direct Systems of Hilbert Spaces and their Limits

In the previous sections we have considered the approximation of a topological space M and its algebra of continuous functions. In this section we want to extend this analysis to the Hilbert space of square integrable functions $L^2(M)$. We saw that the inductive limit of algebras, $\mathcal{A}_\infty$, consisted of operator valued functions. We will see that the Hilbert space $\mathcal{H}^\infty$, on which $\mathcal{A}_\infty$ acts naturally, contains $L^2(M)$ as a subspace and is the inductive limit of a sequence of Hilbert spaces $\mathcal{H}^n$. On these spaces, $\mathcal{H}^n$, the algebras $\mathcal{A}_n$ act naturally as bounded operators.

As before we start considering the interval I. The algebra $\mathcal{A}_\infty$ acts naturally on the following Hilbert space $\mathcal{H}^\infty$

$$\mathcal{H}^\infty = L^2(I) \bigoplus_{\alpha \in I_0} \mathcal{H}_\alpha \tag{4.1}$$

where $\mathcal{H}_\alpha$ are infinite dimensional Hilbert spaces. An element $\psi = (f, \psi_\alpha)$ of $\mathcal{H}^\infty$ consists then of a function f of $L^2(I)$ and a countable collection of vectors ψ_α in $\mathcal{H}_\alpha$ such that

$$||\psi|| = \sum_\alpha < \psi_\alpha, \psi_\alpha > + ||f||^2 < \infty. \tag{4.2}$$

The space $\mathcal{H}^\infty$ is actually the inductive limit of the following sequence of Hilbert spaces $\mathcal{H}^n$ which is naturally associated to the noncommutative lattices P^n and their algebras $\mathcal{A}_n$. Each point $S^n_{m\alpha}$ of P^n corresponds to an IR of $\mathcal{A}_n$ on a Hilbert space $\mathcal{H}^n_{m\alpha}$. The space $\mathcal{H}^n$ is then the direct sum of these $\mathcal{H}^n_{m\alpha}$

$$\mathcal{H}_n = \bigoplus_{m\alpha} \mathcal{H}^n_{m\alpha}. \tag{4.3}$$

where $\mathcal{H}^n_{1\alpha} = \mathbf{C}$ and $\mathcal{H}^n_{0\alpha}$ is infinite dimensional. The inner product of two elements $\psi^n = \oplus_{m\alpha}\psi^n_{m\alpha}$ and $\phi^n = \oplus_{m\alpha}\phi^n_{m\alpha}$ of $\mathcal{H}_n$, is

$$(\psi^n, \phi^n)_n = \sum_\alpha \frac{1}{2^{n-1}} (\psi^n_{1\alpha})^* \phi^n_{1\alpha} + \sum_\alpha (\psi^n_{0\alpha}, \phi^n_{0\alpha})_{0\alpha}. \tag{4.4}$$

The algebra $\mathcal{A}_n$ can obviously be realized as bounded operators on $\mathcal{H}_n$. The isometric immersions $j^{n,n+1}$ of $\mathcal{H}^n$ into $\mathcal{H}^{n+1}$ defining the inductive limit are given by the pull back of the projections $\pi^{(n+1,n)}$ from P^{n+1} to P^n. Namely

$$j^{n,n+1}(\oplus_{m\alpha}\psi^n_{m\alpha}) = \oplus^{n+1}_{m\alpha}\psi^{n+1}_{m\alpha} \tag{4.5}$$

where

$$\psi^{n+1}_{0\alpha} = \begin{cases} \psi^n_{0\beta} & if \;\; \pi^{n+1,n}(S^{n+1}_{0\alpha}) = S^n_{0\beta} \\ 0 & if \;\; \pi^{n+1,n}(S^{n+1}_{0\alpha}) = S^n_{1\beta} \end{cases} \tag{4.6}$$

$$\psi^{n+1}_{1\alpha} = \psi^{n+1}_{1\alpha} \quad \text{where} \;\; \pi^{n+1,n}(S^{n+1}_{1\alpha}) = S^n_{1\beta} \tag{4.7}$$

The inductive limit of the inductive system of Hilbert spaces $\mathcal{H}_n$ is obviously $\mathcal{H}^\infty$. The space $L^2(I)$ is recovered as a subspace of $\mathcal{H}^\infty$.

The generalization to a generic space M can be done following the argument of the previous section. The construction is straightforward due to the fact that tensor products and direct sums of algebras act respectively on tensor products and direct sums of the corresponding Hilbert spaces.

5 Conclusions

In this paper we have discussed topological approximations of a topological space M by means of noncommutative lattices. The space M can be recovered from a projective limit $P^\infty(M)$ of a projective system associated with the noncommutative lattice approximations. In contrast to what happens for approximations with Hausdorff lattices [2] the projective limit $P^\infty(M)$ is not universal and it is not 'too different' from M itself. The latter can be naturally recovered from $P^\infty(M)$.

Any noncommutative lattice is the structure space of a noncommutative C*-algebra of operator valued functions on it. The collection of all such algebras is made into an inductive system of C^*-algebras, from whose limit $\mathcal{A}^\infty(M)$, the algebra $\mathcal{C}(M)$ of continuous functions on M can be obtained as a subalgebra. Again, in contrast to the Hausdorff approximations the algebra $\mathcal{A}^\infty(M)$ is not an universal object anymore.

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