Approximation of invariant surfaces by periodic orbits in high-dimensional maps. Some rigorous results.

Stathis Tompaidis [1] [2]

IRMAR

Université de Rennes I

35042 Rennes Cedex

France

Abstract. The existence of an invariant surface in high-dimensional systems greatly influences the behavior in a neighborhood of the invariant surface. We prove theorems that explain the behavior of periodic orbits in the vicinity of an invariant surface for symplectic maps and quasi-periodic perturbations of symplectic maps. Our results allow for efficient numerical algorithms that can serve as an indication for the breakdown of invariant surfaces.

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[2] e-mail: `stathis@lie.univ-rennes1.fr`

1. Introduction

Periodic orbits have long served as tools to study the long term behavior of dynamical systems (as witnessed, for example, by Poincaré, see [Po93]). In 1979, Greene proposed a numerical criterion, based on the behavior of periodic orbits, to determine the parameter values at which breakdown of certain invariant circles of twist maps of the annulus occurs. The criterion (henceforth called "Greene's criterion", see [Gr79] for a precise formulation) is remarkably accurate and has provided valuable intuition that led to the formulation of a renormalization group theory for the breakdown of invariant circles for twist maps of the annulus (see [McK82]).

Determining the parameter values at which breakdown of invariant surfaces occurs has significant practical importance, as invariant surfaces present barriers to phase-space diffusion. Part of Greene's criterion (initially conjectured in [Gr79] and later proved in [McK92, FL92]) asserts that twist maps of the annulus admit an invariant circle with diophantine rotation number as long as a certain limit, taken along periodic orbits in the neighborhood of the invariant circle and based on their stability, is equal to zero. Moreover, if the invariant circle is analytic, the limit is reached exponentially fast. Such behavior can, and has been, efficiently investigated numerically.

We present a similar result for certain high-dimensional symplectic and volume-preserving maps, satisfying non-degeneracy assumptions. If an invariant surface Γ exists and is analytic, or sufficiently differentiable, and motion on Γ is conjugate to rigid rotation with a diophantine rotation vector, we show that all the eigenvalues of the derivative of the map along periodic orbits in a neighborhood of Γ tend to 1 (exponentially, if the invariant surface is analytic) as the periodic orbit approaches Γ. A precise statement is given in section 2.

Our results are of a local nature and involve only a neighborhood of the invariant surface. Existence of an invariant surface imposes severe restrictions for the map in a neighborhood of the surface. We show that in an appropriate neighborhood of the invariant surface the map is close to integrable and using a perturbative argument we can control the behavior of periodic orbits. One can use the distance from the invariant surface as a smallness parameter and deduce that periodic orbits with rotation vectors close to the rotation vector of the invariant surface exist close to the surface. In [PW94] similar ideas were used to deduce long-term stability for orbits that come close to an invariant surface.

2. Notation and statement of results

We will study two distinct cases: (a) symplectic maps and (b) volume-preserving maps that are quasiperiodic perturbations of symplectic maps.

In the first case we consider C^r and analytic maps f from the space $\mathbf{T}^d \times \mathbb{R}^d$ to itself, satisfying

(i) they preserve the natural symplectic 2-form $\omega = \sum_{i=1}^d d\phi_i \wedge dA_i$

(ii) $\partial\phi'/\partial A$ is a non-singular matrix (of dimension d)

where ϕ' the first coordinate of $\tilde{f}(\phi, A)$ for $\tilde{f}$ a lift of f. We will call a function f satisfying (i) and (ii) a $2d$-dimensional non-singular symplectic map. Examples of C^r maps satisfying $(i), (ii)$ for $d = 1$ are called (positive or negative) twist maps of the annulus.

In the second case we consider maps $f : \mathbf{T}^{d+e} \times \mathbb{R}^d \to \mathbf{T}^{d+e} \times \mathbb{R}^d$ that are periodic or quasi-periodic skew-products on $\mathbf{T}^e$ where $f|_{\mathbf{T}^d \times \mathbb{R}^d} : \mathbf{T}^d \times \mathbb{R}^d \to \mathbf{T}^d \times \mathbb{R}^d$ satisfies $(i), (ii)$.

We say that $\mathbf{x}$ is a periodic orbit of type (P/N), $P \in \mathbb{Z}^c$, where $c = d$ for the case of the symplectic maps or $c = d + e$ for the case of volume-preserving maps, $N \in \mathbb{N}^*(\equiv \mathbb{N} - \{0\})$, if $f^N(\mathbf{x}) = \mathbf{x}$ and $\tilde{f}^N(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}} + (P, 0)$, where $\tilde{f}, \tilde{\mathbf{x}}$ a (fixed) lift of $f, \mathbf{x}$ to the universal cover of $\mathbf{T}^c \times \mathbb{R}^d$. We will call N the period of the orbit. Notice that only periodic skew-products can have periodic orbits. For c-vectors we will use the norm $\|\omega\|_c = \sum_{i=1}^c |\omega_i|$.

We define the *rotation vector* of an orbit of $\tilde{f}$ as the c-dimensional vector

$$\omega = \lim_{i\to\infty} \frac{\pi_1(\tilde{f}^i(x, y)) - x}{i}$$

if the limit exists, where π_1 the projection on the first c (angle) coordinates $\pi_1(x, y) = x$. For a periodic orbit of type (P/N) the rotation vector is $\omega = P/N$.

We will consider sets with rotation vectors that are not well approximated by rational vectors. We define a c-dimensional vector to be of type (K, τ) if

$$|P \cdot \omega| \geq \frac{K}{\|P\|_c^\tau}, P \in \mathbb{Z}^c, P \neq 0, K > 0 \tag{2.1}$$

It is known (see [Ar88]) that if $\tau > c - 1$ the set of vectors of type (K, τ) has positive Lebesgue measure in the unit c-dimensional cube.

We now state our results for periodic orbits that approach invariant sets of f.

Theorem 2.1. *Let $f \in C^r(\mathbf{T}^d \times \mathbb{R}^d), r > 1$ satisfy (i), (ii) and admit a C^r invariant surface Γ, homotopic to $\mathbf{T}^d$, on which the motion is C^r conjugate to rigid rotation with rotation vector ω of type (K, τ). Moreover assume that in a neighborhood of Γ there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_d$ small enough.*

Then, for $k \in \mathbb{N}, k < \frac{r-1}{\tau}$ we can find $D_k > 0$, such that the eigenvalues $\lambda_1, \ldots, \lambda_{2d}$ of the derivative $Df^N(x_{(P/N)})$ satisfy

$$|\lambda_i - 1| \leq D_k \|N\omega - P\|_d^{k/2} N, \ i = 1, \ldots, 2d$$

In the case where the map f and the invariant surface are analytic in a poly-strip I_δ around the invariant surface Γ and analytically conjugate to rigid rotation, we can compute the coefficients D_k and choose the k that gives the best bound.

Theorem 2.2. *Let $f : \mathbf{T}^d \times \mathbb{R}^d \to \mathbf{T}^d \times \mathbb{R}^d$ analytic satisfy (i), (ii) and admit an analytic invariant surface Γ, homotopic to $\mathbf{T}^d$, on which the motion is analytically conjugate, with conjugacy γ, to rigid rotation with rotation vector ω of type (K, τ). Moreover assume that in a neighborhood of Γ there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_d$ small enough. If f, γ are bounded in a neighborhood of the invariant surface then the eigenvalues $\lambda_1, \ldots, \lambda_{2d}$ of the derivative $Df^N(x_{(P/N)})$ satisfy*

$$|\lambda_i - 1| \leq \tilde{D}_1 N \exp(-\tilde{D}_2 \|N\omega - P\|_d^{\frac{-1}{2(1+\tau)}})$$

where $\tilde{D}_2$ depends on the width of the domain of analyticity of f, γ.

In the case $d = 1$, the behavior of the eigenvalues is completely determined by the trace of the derivative along the periodic orbit. In analogy with that case, we define the residue of a periodic orbit with period N, as

$$R(\mathbf{x}) = \frac{1}{4d}\left[2d - \mathrm{Tr}(Df^N(\mathbf{x}))\right] \tag{2.2}$$

Our definition is an extension of the one used by Greene in [Gr79] for two-dimensional twist maps of the annulus. The factor $(4d)^{-1}$ assures that the residue of elliptic periodic orbits (i.e. orbits for which the eigenvalues of $D\tilde{f}^N$ lie on the unit circle) is between zero and one.

In [Gr79] Greene formulated a criterion for the breakdown of invariant curves of twist maps based on the behavior of the residue of periodic orbits. As indicated by Theorem 2.2, an analog of the criterion in higher dimensions should consider the behavior of *additional* quantities, other than the residue, such as the *eigenvalues* of Df^N along periodic orbits.

Notice that, due to invariance under cyclic permutations, the residue of a periodic orbit is the same for all the points of the orbit. Also, since the definition only involves derivatives, the residue is invariant under C^1 changes of variables. For integrable maps (i.e. a map conjugate to $\tilde{g}(x,y) = (x + h(y), y)$ for $h : \mathbb{R}^d \to \mathbb{R}^d$) the residue of all periodic orbits is zero. From Theorem 2.1, Theorem 2.2 we have the following corollary:

Corollary 2.3. *Let $f \in C^r(\mathbf{T}^d \times \mathbb{R}^d), r > 1$ (analytic) satisfy (i), (ii) and admit a C^r (analytic) invariant surface Γ, homotopic to $\mathbf{T}^d$, on which the motion is C^r (analytically) conjugate to rigid rotation with rotation vector ω of type (K, τ). Moreover assume that in a neighborhood of Γ there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_d$ small enough.*

Then, for $k \in \mathbb{N}, k < \frac{r-1}{\tau}$ we can find $C_k > 0$, such that

$$|R(x)| \le C_k \|N\omega - P\|_d^{k/2} N$$

$$(|R(x)| \le \tilde{C}_1 N \exp(-\tilde{C}_2 \|N\omega - P\|_d^{\frac{-1}{2(1+\tau)}}))$$

Remark. In the case $d = 1$ the continued-fraction convergents to ω provide a series of numbers $\{M_i/N_i\}_{i=0}^{\infty}$ such that

$$|\omega - M_i/N_i| \le K N_i^{-2}, \text{ for all } i, \omega \tag{2.3}$$

In that case it is possible to show that if an analytic invariant curve exists

$$\lim_{i\to\infty} \sup |R(x_i)|^{1/N_i} \le 1$$

where the limit is taken along continued fraction convergents. Unfortunately, in higher dimensions, we are not aware of an approximation scheme that can produce convergents to an arbitrary rotation vector with d components that satisfy an inequality similar to (2.3) (such schemes exist for certain classes of rotation vectors though – e.g. golden vectors of the Jacobi-Perron algorithm for $d = 2$, see [Kos91]).

Remark. Theorem 2.1 and Theorem 2.2 are local results that apply in a neighborhood of the invariant surface. Thus, assumptions (i), (ii) can be relaxed to assumptions (i) and (ii) holding only in a neighborhood of the invariant surface Γ.

For the case of volume-preserving maps that are quasi-periodic skew-products of symplectic maps over $\mathbf{T}^e$ we have

Theorem 2.4. *Let $f : \mathbf{T}^{d+e} \times \mathbb{R}^d \to \mathbf{T}^{d+e} \times \mathbb{R}^d \in C^r, r > 1$ (analytic) be a quasi-periodic skew-product of a symplectic map satisfying (i), (ii) over $\mathbf{T}^e$ such that $f|_{\mathbf{T}_e}$ is rigid rotation with a diophantine rotation vector. Assume that f admits a C^r (analytic) invariant surface Γ, homotopic to $\mathbf{T}^{d+e}$, on which the motion is C^r (analytically) conjugate to rigid rotation with rotation vector ω of type (K, τ). Moreover assume that in a neighborhood of Γ, f there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_{d+e}$ small enough.*

Then, for $k \in \mathbb{N}, k < \frac{r-1}{\tau}$ we can find $D_k > 0$, such that $2d$ of the eigenvalues $\lambda_1, \ldots, \lambda_{2d}$ of the derivative $Df^N(x_{(P/N)})$ satisfy

$$|\lambda_i - 1| \le D_k \|N\omega - P\|_d^{k/2} N,\ \ i = 1, \ldots, 2d$$

$$(|\lambda_i - 1| \le \tilde{D}_1 N \exp(-\tilde{D}_2 \|N\omega - P\|_d^{\frac{-1}{2(1+\tau)}}),\ \ i = 1, \ldots, 2d)$$

The remaining e eigenvalues are identically 1.

Remark. In Theorem 2.4 we consider periodic orbits not of f itself but a closeby f^* that is a periodic skew-product over $\mathbf{T}^e$ with rational rigid rotation ω_2. The neighborhood of the invariant surface can be thought of as a neighborhood of the invariant surface in action variables of an extension of the original map $\tilde{f} : \mathbf{T}^{d+e} \times \mathbb{R}^{d+e} \to \mathbf{T}^{d+e} \times \mathbb{R}^{d+e}$ such that $\tilde{f}|_{\mathbf{T}^{d+e}\times\mathbb{R}^d} = f$ and $\tilde{f}|_{\mathbb{R}^e} = \mathrm{Id}$. A particular example of such an extension for the map

$$f(\phi_1, \phi_2, A_1) = (\phi_1 + A_1 + g(A_1, \phi_1, \phi_2), \phi_2 + \omega_2, A_1 + g(A_1, \phi_1, \phi_2))$$

is given by

$$\tilde{f}(\phi_1, \phi_2, A_1, A_2) = (\phi_1 + A_1 + g(A_1, \phi_1, \phi_2), \phi_2 + A_2, A_1 + g(A_1, \phi_1, \phi_2), A_2)$$

at the value of $A_2 = \omega_2$.

Our results cover the case that f admits an invariant surface on which motion is conjugate to rotation. In [FL92] it was shown that if f admits an invariant set on which motion is semi-conjugate to rotation then there are periodic orbits approaching the invariant set under certain conditions on the Lyapunov exponents of f on the invariant set. We include the statements of the theorems in [FL92] for completeness.

Theorem 2.5. *(Theorem 2.3 in* [FL92]*)*
Assume Γ is a hyperbolic set of rotation vector ω and that $\{x_n\}$ is a sequence of periodic points of type (M_n/N_n) such that orbit(x_n) *converges to Γ. Then, for sufficiently large n, $|R(x_n)|^{1/N_n} > \lambda > 1$. Actually, if the hyperbolic set has (maximum) Lyapunov exponent γ, then $\lim_n R(x_n)^{1/N_n} = e^{\gamma}$.*

Theorem 2.6. *(Theorem 4.3 in* [FL92]*)*
Let $f : M \to M$ be a C^2 diffeomorphism leaving invariant the ergodic measure μ. Assume that, with respect to this measure, f has no zero Lyapunov exponents. Then, for almost every point x_0 in the support of μ, it is possible to find a sequence $\{x_n\}_{n=0}^{\infty}$ of periodic points that converge to x_0. Moreover, the sequence of orbits can be chosen in such a way that the Lyapunov exponents of x_n converge to the Lyapunov exponents of x_0.

3. Proof of the results

3.1. The C^r case for symplectic maps

In this section we will consider the case of symplectic maps f, satisfying conditions $(i),(ii)$. The proof consists of three parts. In the first part we will construct a normal form in the neighborhood of the invariant surface and approximate the map in that neighborhood with an integrable mapping. The distance between our map and the integrable map can be made $O(\|H\|_d^k)$ where H are the actions in an appropriate coordinate system, for k depending on the smoothness of the invariant surface and the type of the rotation vector.

In the second part we will show that in a small enough neighborhood of the invariant surface the rotation vector of periodic orbits that stay in the neighborhood cannot differ from the rotation vector of the invariant surface more than the size of the neighborhood.

The last part is a perturbation argument, that allows us to estimate the eigenvalues of the derivative along periodic orbits that stay close to the invariant surface.

We begin the proof by making a change of variables to a new system of coordinates, more convenient for studying a neighborhood of the invariant surface.

Proposition 3.1. *Let f as above, Γ a C^r invariant surface (which is a graph of a C^r function $\gamma : \mathbf{T}^d \to \mathbb{R}^d$) and $f|_\Gamma$ C^r conjugate to rigid rotation with rotation vector ω. Then we can find a symplectic, C^{r-1} mapping h defined in a neighborhood of Γ, with a C^{r-1} inverse in a neighborhood of Γ such that*

$$h \circ f \circ h^{-1}(\phi, A) = (\phi + \omega + Av(\phi, A), A + A^2 u(\phi, A)) \tag{3.1}$$

where A^2 implies all quadratic combinations of the various A's.

Proof. The proof consists of two steps. We first shift the action coordinates so that $(\phi, 0)$ becomes the invariant surface. Then we use the conjugacy to rigid rotation to deduce (3.1).

Define $h_1 : \mathbf{T}^d \times \mathbb{R}^d \hookleftarrow$ by

$$h_1(\phi, A) = (\phi, A + \gamma(\phi))$$

Then h_1 is in C^r, symplectic and sends $\mathbf{T}^d \times \{0\}^d$ to the graph of γ. Thus $h_1 \circ f \circ h_1^{-1}$ leaves the surface $\mathbf{T}^d \times \{0\}^d$ invariant.

$$h_1 \circ f \circ h_1^{-1}(\phi, A) = (v_1(\phi, A), Au_1(\phi, A))$$

Since the motion on the surface is C^r conjugate to rigid rotation, there is a C^r function $\delta : \mathbf{T}^d \to \mathbf{T}^d$ with a C^r inverse (hence $[D\delta]^{-1}$ exists) such that $v_1(\delta(\phi), 0) = \delta(\phi + \omega)$.

We introduce (for $r > 1$) the C^{r-1} symplectic transformation

$$h_2(\phi, A) = (\delta(\phi), [D\delta]^{-1} A)$$

with

$$h_2^{-1} \circ h_1 \circ f \circ h_1^{-1} \circ h_2(\phi, A) = (\phi + \omega + Av_2(\phi, A), Au_2(\phi, A)). \tag{3.2}$$

We have

$$\frac{\partial A'_i}{\partial A_j}\Big|_{A=0} = \frac{\partial \phi'_i}{\partial \phi_j}\Big|_{A=0} = 0, \quad i \neq j$$

$$\frac{\partial \phi'_i}{\partial \phi_j}\Big|_{A=0} = 1, \quad \forall\, i$$

$$\frac{\partial A'_i}{\partial \phi_j}\Big|_{A=0} = 0, \quad \forall\, i, j$$

so, since the map is symplectic

$$\frac{\partial A'_i}{\partial A_i}\Big|_{A=0} = 1$$

Moreover, from condition (ii) $\partial \phi'_i / \partial A_i \neq 0$ or $v_2(\phi, 0) \neq 0$. This concludes the proof of Proposition 3.1.

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Remark. In the case $d = 1$, Birkhoff's theorem guarantees that an invariant curve of a non-singular symplectic map, with irrational rotation number is a graph. Birkhoff's theorem fails in the case that the twist condition (condition (ii) for $d = 1$) is violated. Also for higher dimensions we are not aware of an analog of Birkhoff's theorem (in the case $d = 2$ there is an analog of Birkhoff's theorem for a class of maps that can be expressed as a finite number of compositions of one-dimensional twist maps – see [Ma91]). In the general case, the condition that the invariant curve is a graph over $\mathbf{T}^d$ can be substituted by a more local condition (weaker in the case of the maps we have been studying and also applying for singular symplectic maps – i.e. maps with zero torsion). If Γ is homotopic to $\mathbf{T}^d$ there are coordinates, in a neighborhood of Γ for which the invariant surface reduces to a graph. Then, condition (ii) needs only be satisfied in a neighborhood of the invariant surface, in the transformed coordinates (3.2) (i.e. $v_2(\phi, 0) \neq 0$) for the conclusions of Theorem 2.1 to be valid.

We introduce some further notation. In the following we use $\{i_m\}$ as a multi-index. The statement $\{i_m\}$ will denote all possible combinations of indices $i_{1,m}, \ldots, i_{j,m}$ such that $\sum_l i_{l,m} = m$. Moreover, the expression $A^{\{i_m\}}$ will mean all possible combinations of the different A's raised to all possible indices allowed from the condition $\sum_l i_{l,m} = m$. Also, a symbol $Q_{\{i_m\}}$ "multiplying" $A^{\{i_m\}}$ will denote a multitude of functions, one for each combination of the A's allowed (e.g. $Q_{\{i_1\}}$ corresponds to d functions, $Q_{\{i_2\}}$ corresponds to $d(d-1)/2$ functions, etc.)

We can now construct a normal form for f in a neighborhood of the invariant surface. We first construct d approximate integrals in a small neighborhood of the invariant surface.

Lemma 3.2. *Let $f \in C^r$ as above and ω a rotation vector of type (K, τ). Then, given any $k \in \mathbb{N}$, $k < \frac{r-1}{\tau}$, we can find functions $H_{\{i_0\}}, H_{\{i_1\}}, \ldots, H_{\{i_k\}} : \mathbf{T}^d \to \mathbb{R}^d$ and constants C_k such that $H : \mathbf{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$*

$$H = \sum_{m=0}^{k} A^{\{i_m\}} H_{\{i_m\}}(\phi)$$

satisfies

$$\|H \circ f - H\| \leq C_{k+1} \|A\|_d^{k+1}$$

Proof. Expanding in A we have

$$H \circ f(\phi, A) = \sum_m (A + A^{\{i_2\}} u(\phi, A))^{\{i_m\}} H_{\{i_m\}}(\phi + \omega + Av(\phi, A))$$

$$= \sum_m (A + A^{\{i_2\}} u(\phi, A))^{\{i_m\}} \left[\sum_{l=0}^{k} c_{\{j_l\}} \frac{\partial H_{\{i_m\}}}{\partial A^{\{j_l\}}} (\phi + \omega + Av(\phi, A))|_{A=0} + O(A^{\{m+k+1\}}) \right]$$

$$= \sum_{m=0}^{k} A^{\{i_m\}} [H_{\{i_m\}}(\phi + \omega) + H_{\{i_{m-1}\}}(\phi + \omega) u(\phi, 0) + L_{\{i_m\}}(\phi)] + O(A^{\{k+1\}})$$

where $c_{\{j_l\}}$ the coefficients of the Taylor expansion and $L_{\{i_m\}}$ depends on $H_{\{i_0\}}, H_{\{i_1\}}, \ldots, H_{\{i_{m-2}\}}$ and their derivatives up to order m, as well as on the derivatives of $H_{\{i_{m-1}\}}$. Notice that changes in $H_{\{i_{m-1}\}}$ by a constant do not affect $L_{\{i_m\}}$.

Matching terms by order we have

$$\begin{aligned} H_{\{i_0\}}(\phi) =& H_{\{i_0\}}(\phi + \omega) \\ H_{\{i_m\}}(\phi) =& H_{\{i_m\}}(\phi + \omega) + H_{\{i_{m-1}\}}(\phi + \omega) u(\phi, 0) + L_{\{i_m\}}(\phi) \end{aligned} \tag{3.3}$$

Equations (3.3) are of the form

$$g(\phi + \omega) - g(\phi) = f(\phi) \tag{3.4}$$

It is well known (see [SM71], [Ar88]) that for the case of ω of type (K, τ), given $f \in C^q$ with zero average over the d-torus, there exists $g \in C^{q-\tau}$ that satisfies (3.4) (for $q > \tau$). For $\{i_0\}$, the only possible continuous solution is $H_{\{i_0\}} =$ constant (from the condition $\int_{\mathbf{T}^d} L_{\{i_1\}} d\phi = 0$, if $\int_{\mathbf{T}^d} u(\phi, 0) d\phi \neq 0$ we get $H_{\{i_0\}} = 0$). If, for $\{i_m\}, m > 0$, $H_{\{i_0\}}, H_{\{i_1\}}, \ldots, H_{\{i_{m-2}\}}$ are uniquely determined and $H_{\{i_{m-1}\}}$ is determined up to a constant, then $L_{\{i_m\}}$ is completely determined. Moreover, $H_{\{i_m\}}$ can be determined up to a constant if and only if

$$\int_{\mathbf{T}^d} [L_{\{i_m\}}(\phi) + u(\phi, 0) H_{\{i_{m-1}\}}(\phi + \omega)] d\phi = 0 \tag{3.5}$$

which uniquely determines the average value of $H_{\{i_{m-1}\}}$ in the case $\int_{\mathbf{T}^d} u(\phi, 0) \neq 0$. In the case $\int_{\mathbf{T}^d} u(\phi, 0) = 0$ we can show that the choice $\int_{\mathbf{T}^d} H_{\{i_m\}}(\phi) = 0,\ m \geq 0$ satisfies (3.5).

Consider the truncation $H^{[\leq m-1]} = \sum_{m=0}^{m-1} A^{\{i_m\}} H_{\{i_m\}}(\phi)$, satisfying (3.3) up to order $m - 1$. Then we have

$$\int_{\mathbf{T}^d} \{H^{[\leq m-1]}(\phi, A) - H^{[\leq m-1]} \circ f(\phi, A)\} d\phi = 0$$

since f symplectic implies f preserves volume in phaser-space. We have

$$H^{[\leq m-1]}(\phi, A) - H^{[\leq m-1]} \circ f(\phi, A) = A^{\{i_m\}} \left(L_{\{i_m\}}(\phi) + u(\phi, 0) H_{\{i_{m-1}\}}(\phi + \omega) \right) + O(A^{\{i_{m+1}\}})$$

thus, condition (3.5) is satisfied.

The process can, inductively, be carried out as long as $L_{\{i_k\}}$ is smooth enough (at least $C^{\tau+\epsilon}$). Since in every step of the induction the smoothness of $L_{\{i_k\}}$ decreases by τ, we have the bound $k\tau > r-1$ or $k < \frac{r-1}{\tau}$. If f is C^∞ or analytic the induction can be carried out for all $k \in \mathbb{N}$.

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We have constructed d functions H that are approximate integrals in the vicinity of the invariant surface. Since $H_{\{i_0\}} = 0$, H is a small perturbation of A and the surface $H = h$, for $\|h\|_d$ small, is topologically nontrivial.

Defining

$$\bar{H}(h) = \int_{H=h} A d\phi$$

the function $\bar{H}$ is conserved under f up to $O(\|A\|_d^{k+1})$ in a neighborhood of $A = 0$.

We change coordinates, in such a way that $\bar{H}$ replaces A, using a generating function S

$$S(\Phi, A) = (A + \int_{\mathbf{T}^d} \sum_{m=2}^{k} A^{\{i_m\}} H_{\{i_m\}}(s) ds)\Phi \tag{3.6}$$

The function S generates the symplectic transformation

$$\begin{aligned} \bar{H} &= D_1 S(\Phi, A) = A + \int_{\mathbf{T}^d} \sum_{m=2}^{k} A^{\{i_m\}} H_{\{i_m\}}(s) ds \\ \phi &= D_2 S(\Phi, A) = \Phi(1 + \frac{\partial}{\partial A} \int_{\mathbf{T}^d} \sum_{m=2}^{k} A^{\{i_m\}} H_{\{i_m\}}(s) ds) \end{aligned} \tag{3.7}$$

In the new coordinates

$$f(\Phi, \bar{H}) = (\Phi + \omega + \bar{H}\Delta(\bar{H}), \bar{H}) + E(\Phi, \bar{H}) \tag{3.8}$$

where the remainder satisfies (in appropriate norms) $\|E\| \le C_k \|\bar{H}\|_d^{k+1}$ and $\Delta(0) \neq 0$.

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Remark. Another way to construct the normal form would be to perform successive canonical transformations (for example using the method of Lie transforms) and reduce f to an integrable map, up to $O(A^{\{k+1\}})$, in a neighborhood of the invariant surface. The method of successive canonical transformations has been used in the case $d = 1$ in [McK92], whereas the method of constructing an approximate integral in [FL92]. We favor the method of constructing approximate integrals, since it lends itself to efficient numerical implementations.

In the case that the map f is analytic, our estimates hold in a complex neighborhood of $\mathbf{T}^d \times \{0\}^d$ of the form $\{|\mathrm{Im}\,\Phi_i| < \xi, |\bar{H}_i| \le \xi, \quad i = 1, \ldots, d\}$ for some $\xi > 0$.

In the new $(\Phi, \bar{H})$ coordinates, we have $\|DE\| \le C_k \|\bar{H}\|_d^k$ and

$$Df(\Phi, \bar{H}) = \begin{pmatrix} 1 & F(\bar{H}) \\ 0 & 1 \end{pmatrix} + O(\|\bar{H}\|_d^k) \tag{3.9}$$

where $F(\bar{H}) = \Delta(\bar{H}) + \bar{H}\Delta'(\bar{H})$.

In a neighborhood of the invariant surface only periodic orbits with rotation vectors close to the rotation vector of the invariant surface are allowed. Since $F(0) \neq 0$, using the implicit function theorem, we conclude that the actions $\bar{H}_{\mathrm{per}}$ of a periodic orbit of period N in the vicinity of the invariant surface are bounded by

$$C_1 \|N\omega - P\|_d \le \|\bar{H}_{\mathrm{per}}\|_d \le C_2 \|N\omega - P\|_d$$

The existence of periodic orbits for maps that are close to integrable (such as map (3.8) in a neighborhood of the invariant surface) has been studied in the case where f has a generating function, in [BK87] and [LW93]. It was shown that some periodic orbits of the integrable system persist, for small enough perturbation, and their distance from the original periodic orbits can be bounded by the size of the perturbation. Although in [LW93] only Hamiltonian flows were considered (which correspond to maps with a generating function) the methods used could be easily extended to periodic orbits of symplectic maps that do not have a generating function.

The last part of the proof consists of a simple perturbative argument. Since we are interested in the eigenvalues of the derivative along periodic orbits, we estimate the norm of products of matrices close to the ones appearing in (3.9).

Lemma 3.3. *Let $\{A_i\}_{i=1}^N$ be a set of $2d \times 2d$ matrices of the form*

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$$

with

$$\max\left(1, \sup_{1\le i\le N} \left(\sup_{1\le l,k\le d} |(a_i)_{lk}|\right)\right) \le A$$

and $\{B_i\}_{i=1}^N$ satisfy

$$\sup_{\substack{1\le i\le N \\ 1\le j,k\le 2d}} |(B_i)_{jk} - |(A_i)_{jk}| \le \epsilon \quad \text{with } \epsilon < A\ .$$

Then, all the eigenvalues of $B = B_1 \dots B_N$, $\lambda_1, \dots, \lambda_{2d}$ satisfy

$$|1 - \lambda_i| \le 2\{(1 + 3d\sqrt{A}\sqrt{\epsilon})^N - 1\}$$

Proof. We introduce the following norms for vectors and matrices: for a vector in $\mathbb{R}^{2d}$ we define $\|v\|_\delta = \sum_{i=1}^d (|v_i|\delta + |v_{i+d}|)$ and for any $2d \times 2d$ matrix C, $\|C\|_\delta = \sup_{v\in\mathbb{R}^{2d}} \|Cv\|_\delta / \|v\|_\delta$. Then, if λ is an eigenvalue of C, we have $|\lambda| \le \|C\|_\delta$.

For the matrices A_i, B_i and for $\delta < 1$

$$\begin{aligned} \|A_i\|_\delta &\le 1 + d\max(1, |(a_i)_{jk}|)\delta \le 1 + dA\delta \\ \|A_i - B_i\|_\delta &\le \epsilon \max(d + d\delta, d + d\delta^{-1}) = \epsilon d(1 + \delta^{-1}) \end{aligned} \tag{3.10}$$

To prove the claim about the eigenvalues of B, notice that the eigenvalues of $B - I$, $\mu_1, \dots, \mu_{2d}$ satisfy

$$|\mu_i| \le \|B - A_1 \cdots A_N + A_1 \cdots A_N - I\|_\delta \le \|B - A_1 \cdots A_N\|_\delta + \|A_1 \cdots A_N - I\|_\delta$$

We write

$$B = B_1 \dots B_N = (A_1 + (B_1 - A_1))(A_2 + (B_2 - A_2)) \dots (A_N + (B_N - A_N))$$

Expanding, grouping terms, we get

$$
\begin{aligned}
B = A_1 \cdots A_N &+ \sum_i A_1 \cdots A_{i-1}(B_i - A_i)A_{i+1} \cdots A_N \\
&+ \sum_{i,j} A_1 \cdots A_{i-1}(B_i - A_i)A_{i+1} \cdots A_{j+1}(B_j - A_j)A_{j+1} \cdots A_N \\
&+ \cdots \\
&+ (B_1 - A_1) \cdots (B_N - A_N)
\end{aligned}
$$

or

$$
\begin{aligned}
\|B - A_1 \cdots A_N\|_\delta \le &\binom{N}{1} \max_i \|A_i\|_\delta^{N-1} \|B_i - A_i\|_\delta \\
&+ \binom{N}{2} \max_i \|A_i\|_\delta^{N-2} \|B_i - A_i\|_\delta^2 \\
&+ \cdots \\
&+ \binom{N}{N} \max_i \|B_i - A_i\|_\delta^N
\end{aligned}
$$

and using the estimates (3.10)

$$\|B - A_1 \ldots A_N\|_\delta \le [1 + dA\delta + d(1 + \delta^{-1})\epsilon]^N - [1 + d\delta A]^N$$

Choosing $\delta = (\epsilon/A)^{1/2} < 1$ we obtain

$$\|B - A_1 \ldots A_N\|_\delta \le [1 + 3d\sqrt{A}\sqrt{\epsilon}]^N - 1 \tag{3.11}$$

Similarly $\|A_1 \cdots A_N - I\|_\delta$ can be bounded, following the same steps as above by

$$\|A_1 \cdots A_N - I\|_\delta \le (1 + d\sqrt{A}\sqrt{\epsilon})^N - 1$$

Since $\mu_i = \lambda_i - 1$ we have

$$|\lambda_i - 1| \le (1 + 3d\sqrt{A}\sqrt{\epsilon})^N + (1 + d\sqrt{A}\sqrt{\epsilon})^N - 2 \le 2\{(1 + 3d\sqrt{A}\sqrt{\epsilon})^N - 1\}$$

∎

Putting all the estimates together, for N large enough, we can bound all the eigenvalues of $Df^N(\mathbf{x})$ for a (P/N) periodic orbit by

$$|\lambda_i - 1| \le D_k \|N\omega - P\|_d^{k/2} N$$

This concludes the proof of Theorem 2.1.

∎

3.2. The analytic case

To prove Theorem 2.2 we only need to compute the values of the constants C_k, D_k and choose the best value for k. The optimal bound depends on the diophantine properties of the rotation vector ω.

In this section we use the following norms for analytic functions over a complex neighborhood $\mathcal{T}_\delta = \{(\phi, A) | \mathrm{Re}\, \phi_i \in [0,1], |\mathrm{Im}\, \phi_i| \leq \delta, |A_i| \leq \delta\}$ of the invariant surface

$$\|F\|_\delta \equiv \sup_{\mathcal{T}_\delta} |F|$$

or, if F denotes several functions

$$\|F\|_\delta \equiv \max_i \|F_i\|_\delta$$

We first state a lemma that provides quantitative bounds for the solution to equations similar to (3.4).

Lemma 3.4. *Let L be a bounded analytic function on $\mathcal{T}_\delta$ and assume L has zero average over $\mathbf{T}^d$. For ω diophantine of type (K, τ) we can find a solution of the equation*

$$H(\phi) - H(\phi + \omega) = L(\phi)$$

unique, up to an additive constant, on $\mathcal{T}_\delta$. Moreover, the solution is bounded on any smaller domain $\mathcal{T}_{\delta-\eta}$ by

$$\|H\|_{\delta-\eta} \leq C_{K,\tau,d}\, \eta^{-\tau} \|L\|_\delta$$

for any $0 < \eta < \delta$.

A proof of Lemma 3.4 can be found in [Rüs76], [Ar88], [FB89].

In the process of constructing d approximate integrals in the neighborhood of the invariant surface we need to solve the equations

$$H_{\{i_m\}}(\phi) - H_{\{i_m\}}(\phi + \omega) = H_{\{i_{m-1}\}}(\phi + \omega) u(\phi, 0) + L_{\{i_m\}}(\phi)$$

where $L_{\{i_m\}}(\phi) = L^1_{\{i_m\}}(\phi) - L^2_{\{i_m\}}(\phi)$ with

$$L^1_{\{i_m\}}(\phi) = \sum_{j=1}^{m} \frac{1}{\{i_j\}!} \left(\frac{\partial}{\partial A}\right)^{\{i_j\}} H_{\{i_{m-j}\}}(\phi + \omega + Av(\phi, A))|_{A=0}$$

$$L^2_{\{i_m\}}(\phi) = \sum_{j=2}^{m} H_{\{i_{m-j}\}}(\phi) \frac{1}{\{i_j\}!} \left(\frac{\partial}{\partial A}\right)^{\{i_j\}} (A + A^{\{i_2\}} u(\phi, A))^{\{i_j\}}|_{A=0}$$

under the condition (3.5).

We will use induction to estimate bounds on the H's.

Theorem 3.5. *If the invariant surface is analytic in $\mathcal{T}_\delta$ and ω is diophantine of type (K,τ) then*

$$\|\tilde{H}_{\{i_m\}}\|_{\delta-m\eta} \leq ED^m$$
$$\max|\bar{H}_{\{i_m\}}| \leq ED^m$$

where $\bar{H} = \int_{\mathbf{T}^d} H d\phi$, $\tilde{H} = H - \bar{H}$, $\delta - k\eta > 0$ and $D = \tilde{K}\eta^{-1-\tau}$ for $\tilde{K}, E$ numbers that depend on the system, the invariant surface, the dimension and ω.

Proof. Using induction, the hypothesis holds for $m = 1$. Assuming that all $H_{\{i_m\}}$'s are determined completely up to order $m-2$ and up to an additive constant for $H_{\{i_{m-1}\}}$ and satisfy the bounds in the assumption we have

$$\sup_{\|A\|_d \leq \eta/2V} \|H_{\{i_{m-j}\}}(\phi+\omega+Av(\phi,A)\|_{\delta-(m-1/2)\eta} \leq \|H_{\{i_{m-j}\}}\|_{\delta-(m-1)\eta} \leq \|H_{\{i_{m-j}\}}\|_{\delta-j\eta}$$

where $V = \sup_{\mathcal{T}_\delta} |v(\phi,A)|$.

Using Cauchy estimates to bound derivatives with respect to A (see [PW94] for a justification of Cauchy estimates for the case of max norms in $\mathbb{C}^d$) we have

$$\sup_{\|A\|_d \leq \eta/2V} \|\frac{1}{\{i_j\}!}\left(\frac{\partial}{\partial A}\right)^{\{i_j\}} H_{\{i_{m-j}\}}(\phi+\omega+Av(\phi,A))|_{A=0}\|_{\delta-(m-1/2)\eta}$$
$$\leq \|H_{\{i_{m-j}\}}\|_{\delta-j\eta}\frac{(2V)^j}{\eta^j}$$

and

$$\sup_{\|A\|_d \leq \eta/2V} \|\frac{1}{\{i_j\}!}\left(\frac{\partial}{\partial A}\right)^{\{i_j\}} (A + A^{\{i_2\}}u(\phi,A))^{\{i_j\}}|_{A=0}\|_{\delta-(m-1/2)\eta} \leq \frac{1}{\eta^j}$$

¿From the above estimates we deduce

$$\|L^1_{\{i_m\}}\|_{\delta-(m-1/2)\eta} \leq D^{m-1}E\frac{4V}{\eta}$$
$$\|L^2_{\{i_m\}}\|_{\delta-(m-1/2)\eta} \leq D^{m-1}E\frac{2}{\eta}$$

¿From the condition (3.5)

$$\|\bar{H}_{\{i_{m-1}\}}\| \le ED^{m-1}$$

for η fixed and E large enough.

Using Lemma 3.4 and fixing $\eta \le \delta/2k$ we have

$$\|\tilde{H}_{i_m}\|_{\delta-m\eta} \le ED^{m-1}\tilde{K}\eta^{-1-\tau} \le ED^m$$

which concludes the induction.

■

To conclude the proof of Theorem 2.2 we fix $\eta = \delta/2k$ and have $C_k \le \tilde{K}\left(\frac{k}{\delta}\right)^{k(1+\tau)}$ and, using a simple maximization argument over k,

$$\max_{k\in\mathbb{N}} \left(\frac{k}{\delta}\right)^{k(1+\tau)} B^k \le \exp[-(1+\tau)B^{-1/(1+\tau)}\delta e^{-1}]$$

Letting $B = \|N\omega - P\|_d^{1/2}$ concludes the proof of Theorem 2.2.

■

Remark. Theorem 2.2 is also valid for the case of complex maps with complex invariant surfaces, as long as the non-degeneracy condition (ii) is satisfied in a neighborhood of the invariant surface.

3.3. The volume-preserving case

The proof for the case of a quasi-periodic perturbation of a symplectic map is similar to the proofs of Theorem 2.1 and Theorem 2.2. We sketch the proof (referring to the proofs in sections 3.1 and 3.2) and emphasize the differences.

We study invariant sets of maps $f : \mathbf{T}^{d+e} \times \mathbb{R}^d \to \mathbf{T}^{d+e} \times \mathbb{R}^d$ on which motion is conjugate to rigid rotation with rotation vector $\omega = (\omega_1, \omega_2)$, $(\omega_1 \in \mathbf{T}^d$, $\omega_2 \in \mathbf{T}^e)$, with

$$f(\phi_1, \phi_2, A) = (f_1(\phi_1, \phi_2, A), \phi_2 + \omega_2)$$

where $f_1 : \mathbf{T}^{d+e} \times \mathbb{R}^d \to \mathbf{T}^d \times \mathbb{R}^d$ and $f_1(\cdot, \phi_2, \cdot)$ is symplectic.

The first part of the proof consists of constructing a normal form for f in a neighborhood of the invariant surface with rotation vector ω. As in Proposition 3.1 we can find a map h, defined in a neighborhood of the invariant surface, such that

$$h \circ f \circ h^{-1}(\phi_1, \phi_2, A) = (\phi_1 + \omega_1 + A_1 v(\phi_1, \phi_2, A_1), \phi_2 + \omega_2, A_1 + A_1^2 u(\phi_1, \phi_2))$$

with $v(\phi_1, \phi_2, 0) \neq 0$.

We can now construct d approximate integrals for f in a neighborhood of the invariant surface, by expanding and matching by orders as in Lemma 3.2. The difference at this point is that not only the properties of ω_1 (the rotation vector for the symplectic coordinates) but also the combined properties of ω_1 and ω_2 are important.

After constructing the approximate integrals, we perform a transformation (using a generating function in the "symplectic" coordinates, identity in the remaining coordinates) to substitute the approximate integrals for the original "actions".

The normal form for f in a neighborhood of the invariant surface is

$$f(\Phi_1, \phi_2, \tilde{A}_1) = (\Phi_1 + \omega_1 + \tilde{A}_1 \Delta(\tilde{A}_1), \phi_2 + \omega_2, \tilde{A}_1) + (E_1(\Phi_1, \phi_2, \tilde{A}_1), 0_e, E_2(\Phi_1, \phi_2, \tilde{A}_1))$$

where $\Delta(0, \omega_2) \neq 0$ and $\|E_{1,2}\| \leq C_k \|\tilde{A}_1\|_d^{k+1}$ in appropriate norms.

Instead of studying the normal form for f itself we will study the extension $f^* : \mathbf{T}^{d+e} \times \mathbb{R}^{d+e} \to \mathbf{T}^{d+e} \times \mathbb{R}^{d+e}$ with

$$\begin{aligned} f^*(\Phi_1, \phi_2, \tilde{A}_1, A_2) =& (\Phi_1 + \omega_1 + \tilde{A}_1 \Delta(\tilde{A}_1), \phi_2 + A_2, \tilde{A}_1, A_2) \\ &+ (E_1(\Phi_1, \phi_2, \tilde{A}_1), 0_e, E_2(\Phi_1, \phi_2, \tilde{A}_1), 0_e) \end{aligned}$$

The map f^* is also area preserving and, for $A_2 \equiv \omega_2$, motion in the $\Phi_1, \phi_2, \tilde{A}_1$ coordinates under f^* is identical to motion in the $\Phi_1, \phi_2, \tilde{A}_1$ coordinates under f. The map f^* has the advantage that in a neighborhood of an invariant surface with rotation vector of type (K, τ) one can find periodic orbits (by simply changing A_2 to nearby rational numbers).

The bounds on the eigenvalues of the derivative follow from Lemma 3.3. The $2e$ eigenvalues corresponding to rotation in the ϕ_2, A_2 coordinates are identically 1.

Following arguments similar to section 3.2 we can also reproduce the proof for the analytic case. This concludes the proof of Theorem 2.4.

■

Remark. In the case of a general volume-preserving map $f : \mathbf{T}^d \times \mathbb{R} \to \mathbf{T}^d \times \mathbb{R}$ under conditions similar to the ones in Theorem 2.4 it is possible to construct one approximate integral in the neighborhood of the invariant surface. However no result similar to Theorem 2.4 is possible, since we have no control for the motion along the angle coordinates, similar to what we have for the symplectic case.

4. Conclusions

Our results in Theorem 2.1, Theorem 2.2 and Theorem 2.4 suggest that the eigenvalues of the derivative of a symplectic map along a periodic orbit, are in higher dimensions an analog of the residue (as used in Greene's criterion for two-dimensional twist maps). Based on this analogy, an efficient numerical algorithm can be implemented to indicate existence of a close-by invariant surface. Since convergence to the limit behavior (either 1 for the case of an invariant surface or ∞ for the case of a uniformly hyperbolic invariant set) is exponentially fast, relatively low period orbits can be used. In a separate paper we implement such an algorithm for the case of a quasi-periodic excitation of a two-dimensional symplectic map (see [T95]).

Periodic orbits can also be used to investigate behavior at breakdown. If transition can be described in terms of a fixed point of a renormalization group operator with a co-dimension one stable manifold, the eigenvalues of the periodic orbits scale with the period of the orbit and the distance from breakdown. Recently, Kosygin constructed a renormalization group operator and showed that if, under repeated action of the operator, the map converges to a – trivial – fixed point, then the original map admits an invariant surface (see [Kos91]). No such description is known for the behavior at breakdown. Numerical studies and analytical arguments suggest that if such a renormalization operator exists, there are regions in parameter space where behavior at breakdown is governed by dynamics more complex than a simple fixed point (see [MMS94, ACS91, T95]).

Another interesting problem is to determine the existence of lower-dimensional hyperbolic tori on which motion is conjugate to rigid rotation with a resonant rotation vector. One can separate phase space in the neighborhood of the low-dimensional torus to the center manifold of the torus and the hyperbolic directions. Arguments similar to the ones we used in this paper can be used to show that along the center manifold the map is close to an integrable normal form. Along the hyperbolic directions behavior can be described using arguments similar to [FL92]. The natural result appears to be that $2d^*$ eigenvalues (where d^* the dimension of the low-dimensional torus) of the derivative of the map along periodic orbits will approach 1, while the rest will approach $e^{\lambda_i T}$ where λ_i the non-zero Lyapunov exponents of the orbits on the low-dimensional torus and T the period. Unfortunately a numerical algorithm to estimate domains of existence of lower dimensional hyperbolic tori, would be difficult to implement, since we can not numerically isolate the eigenvalues that tend to 1, from eigenvalues that become exponentially large.

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