Numerical study of invariant sets of a volume-preserving map.

Stathis Tompaidis [1] [2]

IRMAR

Université de Rennes I

35042 Rennes Cedex

France

Abstract. We study the behavior of invariant sets of a volume-preserving map, that is a quasi-periodic perturbation of a symplectic map, using approximation by periodic orbits. We present numerical results for analyticity domains of invariant surfaces, behavior after breakdown and a critical function describing breakdown of invariant surfaces as a function of their rotation vectors. We discuss implications of our results to the existence of a renormalization group operator describing breakdown of invariant surfaces.

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[1] Partial support from NSERC grant OGP-0121848

[2] e-mail: `stathis@lie.univ-rennes1.fr`

1. Introduction

Existence and persistence of invariant sets of dynamical systems on which motion is, up to a smooth change of variables, quasi-periodic, is a problem that has attracted considerable attention for at least 100 years (see [Po93]). Such sets have many important, practical, applications as landmarks that organize the long-term behavior. For Hamiltonian systems that are close to integrable, the Kolmogorov-Arnol'd-Moser theorem guarantees that a majority of such invariant sets persists for small enough perturbation (see [Kol54, Ar63, Mo62] or [Ll93] for a self-contained introduction to KAM theory and a proof). Results similar to the KAM theorem can also be shown in the case of quasi-periodic perturbations of symplectic maps.

Unfortunately, the analytical estimates are very conservative compared to numerical indications, especially for high-dimensional systems. In this paper we investigate numerically the domains of existence of two-dimensional tori in a particular two-parameter family of volume-preserving maps and the behavior that occurs at breakdown. The maps we study are quasi-periodic perturbations of a family of symplectic maps. Our numerical algorithms are based on analytical results described in [T95, FL92-1]. The main idea is to use another landmark of long-term behavior, periodic orbits, to determine the existence and breakdown of tori.

For the case of two-dimensional systems, it was originally observed in [Gr79] that existence of invariant circles has a strong influence on periodic orbits close to the circle. In [T95] it was proven that in symplectic maps of any dimension, as well as in quasi-periodic perturbations of them, existence of an invariant torus implies that the behavior of the map in a neighborhood of the torus is close to that of an integrable map. We use this result as an indication of breakdown.

The system we will study is a three-dimensional model of a family of volume-preserving maps. Motion in one of the coordinates is rigid rotation with rotation number given by an appropriate diophantine number. The other two coordinates of the map are described by a perturbation of the standard map. Properties of the map and existence of tori have also been investigated in [ACS91]. Our rationale for studying this map (henceforth called the *rotating standard map*) is similar to that used in experimental physics, where one carefully prepares a sample in order to observe certain phenomena. In our case, the rotating standard map serves as a paradigm for phenomena that appear in higher-dimensional maps.

An important problem is to describe behavior at breakdown of invariant tori. In the two-dimensional case careful numerical experiments and analytical arguments suggest that the breakdown of invariant circles can be described by a fixed point, with a co-dimension one stable manifold, of a renormalization-group operator (see [McK82]). We investigate whether such an approach generalizes in the case of the rotating standard map. To consider such a generalization we use an algorithm (called the Jacobi-Perron algorithm) to approximate irrational points by rational ones. The algorithm reduces to the continued-fraction method for the case of invariant circles. Although it has not been widely used in the dynamical systems literature, it has important measure-theoretic and convergence properties that are useful in introducing a renormalization-group operator. In [Kos91] such an operator was constructed and it was shown that convergence, under repeated application of the operator, to a (trivial) fixed point implies existence of an invariant torus. We investigate whether breakdown can be understood in terms of a different fixed point.

The paper is organized as follows: in section 2 we present the rotating standard map. In section 3 we show existence of periodic orbits with any rotation vector and discuss efficient ways to compute them. In section 4 we present the Jacobi-Perron algorithm and describe its connection to an extension of modular transformations. In section 5 we compute domains of existence for a particular invariant torus and discuss whether breakdown can be understood in terms of a fixed point of a renormalization-group operator in certain regions of parameter space. Finally, in section 6 we generalize the notion of critical function.

2. Notation and preliminaries

We will study the two-parameter family of three-dimensional volume-preserving maps, the *rotating standard map*

$$F_{\epsilon,k} : \mathbb{R} \times \mathbf{T} \times \mathbf{T} \to \mathbb{R} \times \mathbf{T} \times \mathbf{T}$$

$$\begin{pmatrix} A_{n+1} \\ \theta_{n+1} \\ \phi_{n+1} \end{pmatrix} = F_{\epsilon,k} \begin{pmatrix} A_n \\ \theta_n \\ \phi_n \end{pmatrix} = \begin{pmatrix} A_n - \frac{1}{2\pi}(k + \frac{\epsilon}{2\pi}\cos(2\pi\phi_n))\sin(2\pi\theta_n) \\ \theta_n + A_{n+1} \mod 1 \\ \phi_n + \omega_2 \mod 1 \end{pmatrix} \tag{2.1}$$

The parameters ϵ, k will be considered complex. The value of the variable ω_2 determines if the standard map is perturbed periodically (for ω_2 rational) or quasi-periodically (for ω_2 irrational).

We say that $\mathbf{x}$ is a periodic orbit of type (P/N), $P \in \mathbb{Z}^2$, $N \in \mathbb{N}^*(\equiv \mathbb{N} - \{0\})$, if $F^N_{\epsilon,k}(\mathbf{x}) = \mathbf{x}$ and $\tilde{F}^N_{\epsilon,k}(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}} + (P, 0)$, where $\tilde{F}_{\epsilon,k}, \tilde{\mathbf{x}}$ a (fixed) lift of $F_{\epsilon,k}, \mathbf{x}$ to the universal cover of $\mathbb{R} \times \mathbf{T} \times \mathbf{T}$. We will call N the period of the orbit. For N-vectors we will use the norm $\|\mathbf{v}\|_N = \sum_{i=1}^N |v_i|$.

We define the *rotation vector* of an orbit of $\tilde{F}_{\epsilon,k}$ as the 2-dimensional vector

$$\omega = \lim_{i \to \infty} \frac{\pi_1(\tilde{F}^i_{\epsilon,k}(A, \theta, \phi)) - (\theta, \phi)}{i}$$

if the limit exists, where π_1 the projection on the angle coordinates $\pi_1(A, \theta, \phi) = (\theta, \phi)$. For a periodic orbit of type (P/N) the rotation vector is $\omega = P/N$.

We are interested in the behavior of invariant sets with diophantine rotation vector. A diophantine rotation vector of type (K, τ) is a 2-vector ω, such that

$$|P \cdot \omega| \geq \frac{K}{\|P\|^\tau}, \quad P \in \mathbb{Z}^2, P \neq 0, K > 0$$

It is well known (see [Ar88]) that, for some K_0 and fixed $K > K_0, \tau > 1$ the set of vectors of type (K, τ) has positive Lebesgue measure in the unit square. In [T95] it was shown that existence of invariant sets on which motion is conjugate to rigid rotation with diophantine rotation vector has certain implications for the properties of periodic orbits in the neighborhood of the invariant set. As in [T95] we define the residue of a periodic orbit with period N

$$R(\mathbf{x}) = \frac{1}{6}\left[3 - \mathrm{Tr}(Df^N(\mathbf{x}))\right] \tag{2.2}$$

Considering the lift of map (2.1) to the universal cover of $\mathbb{R} \times \mathbf{T} \times \mathbf{T}$ we write (2.1) in Lagrangian formulation

$$\begin{aligned} \theta_{n+1} - 2\theta_n + \theta_{n-1} &= -\frac{1}{2\pi}\left(k + \frac{\epsilon}{2\pi}\cos(2\pi\phi_n)\right)\sin(2\pi\theta_n) \\ \phi_{n+1} - 2\phi_n + \phi_{n-1} &= 0 \end{aligned} \tag{2.3}$$

We will use equations (2.3) to numerically compute periodic orbits for $F_{\epsilon,k}$.

3. Existence and computation of periodic orbits

John Mather (see [Ma91]) extended several of the properties of twist maps of the annulus to finite compositions of twist maps. Consider a periodic perturbation of the standard map and the twist maps of the annulus

$$f_n \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p - \frac{1}{2\pi}\left(k + \frac{\epsilon}{2\pi}\cos(2\pi(\phi_0 + n\omega_2))\right)\sin(2\pi q) \\ q + p - \frac{1}{2\pi}\left(k + \frac{\epsilon}{2\pi}\cos(2\pi(\phi_0 + n\omega_2))\right)\sin(2\pi q) \end{pmatrix}$$

with generating functions

$$h_n(q, q') = \frac{1}{2}(q - q')^2 - V_n(q)$$

where

$$V_n(q) = \frac{1}{2\pi^2}\left(k + \frac{\epsilon}{2\pi}\cos(2\pi(\phi_0 + n\omega_2))\right)\cos(2\pi q).$$

The (finite) composition

$$f = f_1 \circ \cdots \circ f_N \tag{3.1}$$

is the rotating standard map with $B_n \equiv \omega_2 = M/N$ and with generating function h

$$h = h_1 * \cdots * h_N \tag{3.2}$$

where

$$h_1 * h_2(x_1, x_2) = \min_{\xi}\left(h_1(x, \xi) + h_2(x', \xi)\right)$$

The operation $*$ (named *conjunction* in [Ma91]) was used to show that many of the results for twist maps are preserved. In particular, for the rotating standard map as in (3.1) (i.e. a finite composition of twist maps), there exist configurations $\mathbf{x}$ for any $\omega \in \mathbb{R}$ such that $\mathbf{x}$ is an orbit of f with rotation number ω (see proposition 2.4 in [Ma91]). Moreover, if Γ is a curve invariant under f, on which motion is conjugate to rigid rotation with an irrational rotation number, then Γ consists of minimal configurations of h (for a discussion about minimal configurations see [Ma91]. For a proof see [Ma91] proposition 2.8). The above theorems guarantee the existence of minimal periodic orbits for our maps, with any period and any (rational) rotation vector.

To compute such periodic orbits for the map (2.1) we use Newton's method in the space of finite length sequences. Given a periodic orbit with rotation vector $(P_1/N, P_2/N)$

we can eliminate the dependence on the ϕ variable in (2.3) and transform the equations to

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} = -\frac{1}{2\pi}\left(k + \frac{\epsilon}{2\pi}\cos(2\pi(\phi_0 + \frac{nP_2}{N}))\right)\sin(2\pi\theta_n) \tag{3.3}$$

Equations (3.3) can be interpreted as the Euler-Lagrange equations for a certain Lagrangian (see [Ma91], [KM89]). We introduce (formally) the action W for an orbit $\{\theta_i\}_{i=-\infty}^{\infty}$

$$W(\ldots, y_i, \ldots) = \sum_{n=-\infty}^{\infty} g(y_n, y_{n+1}, n)$$

where y is the lift of θ and

$$g(x, y, i) = \frac{1}{2}(x-y)^2 - \frac{1}{4\pi^2}\left(k + \frac{\epsilon}{2\pi}\cos(2\pi(\phi_0 + \frac{iP_2}{N}))\right)\cos(2\pi x).$$

Even though W is only formally defined, its gradient is well defined and a sequence of points is an orbit of (2.1) if and only if it is a critical point of W (see [KM89], [Gol94]).

$$\frac{\partial}{\partial y_l}W = 0 \iff \frac{\partial}{\partial y_l}[g(y_{l-1}, y_l, l-1) + g(y_l, y_{l+1}, l)] = 0, \quad \text{for all } l.$$

A minimal configuration for the generating function (3.2) corresponds to a critical point for W, but not necessarily vice versa, since minimax configurations may exist for certain values of the parameters.

For a periodic orbit of type $(P_1/N, P_2/N)$ we have $y_{N+1} = y_1 + P_1/N, y_0 = y_N - P_1/N$. We redefine the action W as

$$W_N(y_1, \ldots, y_N) = \sum_{n=1}^{N} g(y_n, y_{n+1}, n)$$

which leads to the system

$$\frac{\partial}{\partial y_l}[g(y_{l-1}, y_l, l-1) + g(y_l, y_{l+1}, l)] = 0, \quad l = 1 \ldots N \tag{3.4}$$

To solve (3.4), consider the operator

$$\mathcal{T} : \mathbb{R}^N \to \mathbb{R}^N, \quad \mathcal{T}[\mathbf{y}] = \frac{\partial}{\partial y_l}W_N, \quad l = 1, \ldots, N.$$

Given an initial guess for the coordinates of the periodic orbit $\mathbf{y} = (y_1, \ldots, y_N)$ we can improve it, setting $\mathbf{y}' = \mathbf{y} + \delta\mathbf{y}$. Ignoring terms of order $\delta\mathbf{y} \cdot \delta\mathbf{y}$ we have

$$D\mathcal{T}[\mathbf{y}]\delta\mathbf{y} = -\mathcal{T}[\mathbf{y}]$$

or,

$$A\delta\mathbf{y} = \begin{pmatrix} 2-a_1 & -1 & \ldots & 0 & -1 \\ -1 & 2-a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 2-a_{N-1} & -1 \\ -1 & 0 & \ldots & -1 & 2-a_N \end{pmatrix} \begin{pmatrix} \delta y_1 \\ \delta y_2 \\ \vdots \\ \delta y_{N-1} \\ \delta y_N \end{pmatrix} = -\mathcal{T}[\mathbf{y}] \tag{3.5}$$

where $a_i = (k + \frac{\epsilon}{2\pi}\cos(2\pi(\phi_0 + iP_2/N)))\cos(2\pi y_i)$.

This method of finding periodic orbits is very similar to the one used in [KM89] for the case of high-dimensional symplectic maps. The only difference is that the coefficients a_i in our case depend not only on the coordinates of the periodic orbit, but on the iteration number itself. Another method for numerically computing certain periodic orbits of maps with symmetries uses properties of periodic orbits with respect to symmetry lines of the map. We have not made use of symmetry lines in our numerical algorithm.

Our numerical implementation of the method for a particular $(P_1/N, P_2/N)$ periodic orbit used the following continuation algorithm:

- Choose a precision-cutoff value δ (in our computations we chose $\delta = 10^{-7}$).
- Choose a family of paths passing through $k = \epsilon = 0$.
- For a point along a path iterate Newton's algorithm, using as initial guess the periodic orbit computed at the previous point on the path.
- Proceed to the next point when the error between the computed periodic orbit and its first iterate is less than δ.

3.1. Iterative methods

The problem of solving the linear system (3.5) is greatly simplified by the fact that the matrix A is sparse, with only $3N$ nonzero entries. To solve systems of linear equations involving large, sparse matrices, iterative methods have been developed (similar problems appear frequently in finite element and finite difference discretizations of partial differential equations). Iterative methods have the advantage that, for sparse matrices, only order N computation steps and storage space are required.

Given a linear system

$$Au = b$$

an iterative method successively approximates the true solution $\bar{u}$ from an initial guess $u^{(0)}$. The iteration scheme is

$$u^{(n+1)} = Gu^{(n)} + k$$

where $G = I - Q^{-1}A, k = Q^{-1}b$, for a suitable matrix Q. The Q's are chosen to be easily invertible (for example diagonal, tridiagonal, or upper (lower) triangular matrices) so that their inversion can be done in order N computation steps.

An iterative method can be further speeded up by the use of an acceleration procedure. Such procedures are based on properties of the matrix A. For example, Chebyshev acceleration uses information about the – estimated – range of the eigenvalues of A, whereas the conjugate gradient method minimizes a certain function of A. For a more complete description, with many examples see [YG88, YH81]. Convergence of an iterative method is checked by monitoring the norm of the error.

Convergence of some iterative methods to the true solution has been rigorously demonstrated for the case that A is a symmetric, positive-definite matrix. Unfortunately we are not aware of any general convergence result for matrices that are not symmetric.

In our computations we have used the package `ITPACK 2C`, developed in the Center for Numerical Analysis in the University of Texas at Austin (for a description of `ITPACK 2C` see [KRYG82]). We used the Jacobi iterative method (where Q is the diagonal part of A) with either Chebyshev acceleration or conjugate gradient acceleration. We verified that the time to converge to a solution, within a specified precision, increased linearly with N for a type $(P_1/N, P_2/N)$ periodic orbit.

3.2. Direct methods

In [KM89] a direct algorithm was proposed to solve an equation similar to (3.5) in the case of high-dimensional symplectic maps. The algorithm had the advantages of an iterative method in that it required only order N steps and order N storage space. We implemented a similar method, taking advantage of the structure of the matrix A.

The method is based on the fact that A is very close to a tridiagonal matrix, namely it is *cyclic tridiagonal* (i.e. tridiagonal with two additional entries at the corners). We first solve the tridiagonal linear system in order N steps using the Thomas algorithm (see [YG88 vol.2 pg 587]). The algorithm takes order N computational steps to perform Gaussian elimination, due to the special form of a tridiagonal matrix. After finding the solution to the tridiagonal problem we can add corrections, due to the terms at the two corners, either using the Sherman-Morrison, or the Woodbury methods (see [FPTV92 pg.73-77] for a description of the methods and an implementation).

The algorithm can be easily extended to higher-dimensional cases. Our implementation is designed to be flexible, so that periodic orbits in different maps (either volume-preserving or symplectic) and even higher-dimensional systems can be computed by simply changing some map-dependent definitions in a file.

4. Jacobi-Perron approximation schemes

Approximation of irrational numbers by rationals has been important in the study of breakdown of invariant curves in twist maps of the annulus since the work of Greene (see [Gr79]). Greene conjectured that the behavior of periodic orbits with rotation numbers that are continued-fraction convergents of a diophantine, irrational number, determines the existence of an invariant curve with rotation number equal to the irrational. MacKay (see [McK82]) constructed a renormalization-group operator in spaces of analytic maps, that changes the rotation number of an invariant curve by eliminating the first continued fraction coefficient. For irrationals with periodic continued fraction expansions (and in particular the golden mean $\gamma = (\sqrt{5} - 1)/2$) he provided evidence, based on careful numerical work and analytical arguments, that the breakdown of invariant curves is described by a fixed point of the renormalization operator with a co-dimension one stable manifold.

4.1. Description of the algorithm

The Jacobi-Perron algorithm is one – of many – generalization of the continued fraction algorithm in higher dimensions. Detailed description of the algorithm and proofs of the results we present here can be found in [Ber71, Sch73, Kos91, Lag93].

Given a point $(\omega_1, \omega_2) \in (0,1) \times (0,1)$ the Jacobi-Perron convergents P_n/N_n,$P_n = (P_{1_n}, P_{2_n}) \in \mathbb{N}^2$, $N_n \in \mathbb{N}$ are recursively defined by

$$\begin{aligned} P_{n+1} &= k_{n+1} P_n + l_{n+1} P_{n-1} + P_{n-2} \\ N_{n+1} &= k_{n+1} N_n + l_{n+1} N_{n-1} + N_{n-2} \end{aligned} \tag{4.1}$$

where the integer coefficients k_{n+1}, l_{n+1} are determined by the Jacobi-Perron map

$$\begin{aligned} (\omega_1^{n+1}, \omega_2^{n+1}) &= (\{\frac{1}{\omega_2^n}\}, \{\frac{\omega_1^n}{\omega_2^n}\}) \\ (k_{n+1}, l_{n+1}) &= ([\frac{1}{\omega_2^n}], [\frac{\omega_1^n}{\omega_2^n}]) \end{aligned} \tag{4.2}$$

with initial values

$$\begin{aligned} &(\omega_1^0, \omega_2^0) = (\omega_1, \omega_2) \\ &P_0 = (0,0), \quad P_{-1} = (1,0), \quad P_{-2} = (0,1) \\ &N_0 = 1, \quad N_{-1} = N_{-2} = 0 \end{aligned}$$

The Jacobi-Perron algorithm is a linear simplex-splitting algorithm (see [Lag93]). For the case of points in the unit square a consequence is that if three successive Jacobi-Perron approximants define a triangle, all approximants of higher order (and the point being approximated) will lie inside that triangle. This property is shared with another commonly used algorithm, the Farey-tree approximation scheme (see [KO86]). Lagarias studied the rate of convergence of the Jacobi-Perron algorithm for a set of points of (Lebesgue) measure one in the unit square. Consider a point in the unit square w and a close-by point with rational coordinates $r = (p_1/q, p_2/q)$. The Roth exponent of r is defined as

$$\eta(r, w) = -\frac{\log \|w - r\|_2}{\log q}.$$

Let r_i be the i^{th} Jacobi-Perron approximant to a point w. The *best approximation exponent* for w using the Jacobi-Perron scheme is defined as

$$\eta_b \equiv \limsup_{i \to \infty} \{\eta(r_i(w))\}$$

and the *uniform approximation exponent* as

$$\eta_u \equiv \liminf_{i\to\infty}\{\min(\eta(r_i(w)), \eta(r_{i+1}(w)), \eta(r_{i+2}))\}.$$

The exponent η_b gives the rate of convergence for the best possible approximant towards a point, while the exponent η_u is an –asymptotic – estimate for the rate at which all the vertices of the triangle enclosing w approach w.

Lagarias showed, using methods from ergodic theory (see [Lag93], also [Kos91]) that for the Jacobi-Perron algorithm η_b and η_u are constant in a set of measure one in the unit square. He conjectured that the constant values are in fact equal. In [Bald92] (see also [Kos91]) numerical methods were used to estimate $\eta_b = 1.374 \pm 0.002$, an estimate that, coupled with Lagarias' conjecture, suggests that the triangles formed from successive Jacobi-Perron approximants become, in the limit, needle-shaped.

Other (non measure-theoretic) properties of the Jacobi-Perron algorithm that are of interest are the following
• For all points in the unit square $k_n \geq 1, k_n \geq l_n \geq 0$.
• The triangle formed by three successive Jacobi-Perron approximants contains no rational point with denominator smaller than the largest denominator of the vertices.

In analogy with periodic continued-fractions with period 1, we introduce golden means of the Jacobi-Perron algorithm ω for which $k_n = k, l_n = l$. The polynomial $P_\omega(t) = t^3 - kt^2 - lt - 1$ is called the characteristic polynomial of ω. Golden means have the following properties:
∗ $k < \tau < k+1, 0 < |\tau_1|, |\tau_2| < 1$ where τ is the root of P_ω of maximal absolute value and τ_1, τ_2 the remaining roots.
∗ $(\omega_1, \omega_2) = (\tau - k, 1/\tau)$.
∗ $\|q_n\omega - (p_{1_n}, p_{2_n})\|_2 \leq C(\omega)\kappa^n$ where $\kappa = \max(|\tau_1|, |\tau_2|) < 1$.

In [Kos91], Kosygin constructed a renormalization-group operator in the space of symplectic maps of $\mathbb{R} \times \mathbb{R} \times \mathbf{T} \times \mathbf{T}$. The action of the operator on a map that has an invariant surface with rotation vector $(\omega_1, \omega_2) = ((k_1, l_1), (k_2, l_2), \ldots)$ produces a new map with an invariant surface with rotation vector $(\omega_1', \omega_2') = ((k_2, l_2), \ldots)$. For golden means of the Jacobi-Perron algorithm it was shown that, if the original map, under repeated action of the renormalization-group operation converges to a (trivial) map (which is a fixed point of the renormalization-group operator), then the original map admits an invariant surface, on which motion is conjugate to rigid rotation with rotation vector (ω_1, ω_2). Moreover, the trivial fixed point is attractive, i.e. maps in its neighborhood admit an invariant surface (this result can also be viewed as a renormalization-group proof of the KAM theorem for invariant surfaces with rotation vectors golden means of the Jacobi-Perron algorithm).

4.2. Connection between Jacobi-Perron algorithm and $\mathbf{PSL}_3(\mathbb{Z})$

The Jacobi-Perron algorithm has a natural connection with 3×3 matrix transformations. We introduce an extension of modular transformations of one complex variable to two complex variables

$$(z_1, z_2) \to \left(\frac{az_1 + bz_2 + c}{gz_1 + hz_2 + i}, \frac{dz_1 + ez_2 + f}{gz_1 + hz_2 + i}\right) \tag{4.3}$$

Successive transformations obey the rules of matrix multiplication of 3×3 matrices

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

where M and $-M$ are identified and (because of invariance of the transformation under scaling) $|\det M| = 1$. The group of transformations defined by (4.3) is isomorphic to the projective group $\mathbf{PSL}_3(\mathbb{Z})$. It is known that $\mathbf{PSL}_3(\mathbb{Z})$ is finitely generated by the elementary matrices $T_{ij} = I + e_{ij}, i \neq j, i, j = 1, 2, 3$ where e_{ij} the 3×3 matrix with the only nonzero entry the element i, j which is equal to 1. On the other hand the Jacobi-Perron operator can be viewed as a subgroup of $\mathbf{PSL}_3(\mathbb{Z})$ generated by the matrices

$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

T_1 represents translations in the first coordinate, T_2 translations in the second coordinate, and U generalized inversion.

Given a point (ω_1, ω_2) in the unit square, the Jacobi-Perron operator can be written as

$$(\omega_1^{(0)}, \omega_2^{(0)}) = (\omega_1, \omega_2)$$

$$(\omega_1^{(n+1)}, \omega_2^{(n+1)}) = T_1^{-k_n} T_2^{-l_n} U(\omega_1^{(n)}, \omega_2^{(n)}) \quad n \geq 1$$

where k_n, l_n the unique non-negative integers such that $(\omega_1^{(n+1)}, \omega_2^{(n+1)})$ is inside the unit square. The coefficients k_n, l_n form the Jacobi-Perron expansion of (ω_1, ω_2).

We can use the Jacobi-Perron operator to move between points with the same tail coefficients, since if $\omega^{(0)} = ((k_0, l_0), (k_1, l_1), \ldots)$, then $\omega^{(n)} = ((k_n, l_n), (k_{n+1}, l_{n+1}), \ldots)$. The difference with the lower dimensional case studied in [BPV90] is that the Jacobi-Perron transformation corresponds to a subgroup of $\mathbf{PSL}_3(\mathbb{Z})$, whereas the continued fraction algorithm corresponds to $\mathbf{PSL}_2(\mathbb{Z})$.

5. Domains of existence of invariant surfaces

Existence of invariant surfaces on which motion is, up to a change of variables, rigid rotation, plays a significant role in determining long term dynamics in many physical applications. In the case of the rotating standard map the existence of two-dimensional invariant tori presents a complete barrier to phase-space diffusion. As was shown in [PW94] invariant tori in high-dimensional Hamiltonian systems also guarantee long-term stability for orbits in their neighborhood.

Several methods, both analytical and numerical, have been used in two-dimensional systems to determine the domain of existence of invariant curves (see [CC88, Ran87, LR91, BC90, FL92-2]). Unfortunately, estimates based on analytical methods are very conservative in the case of higher-dimensional systems. The, widely believed, most accurate among the numerical methods for two-dimensional systems is based on a conjecture of Greene concerning the behavior of periodic orbits approaching the invariant curve (see [Gr79] and also [FL92-1, McK92] for a rigorous, partial justification).

Results in [T95, FL92-1] provide justification for a similar criterion in higher-dimensional models. We reproduce the relevant results from [T95]

Theorem 5.1. *(Theorem 2.4 in* [T95]*)*
Let $f : \mathbf{T}^{d+e} \times \mathbb{R}^d \to \mathbf{T}^{d+e} \times \mathbb{R}^d \in C^r, r > 1$ (analytic) be a quasi-periodic skew-product of a symplectic map satisfying non-degeneracy conditions, over $\mathbf{T}^e$ such that $f|_{\mathbf{T}_e}$ is rigid rotation with a diophantine rotation vector. Assume that f admits a C^r (analytic) invariant surface Γ, homotopic to $\mathbf{T}^{d+e}$, on which the motion is C^r (analytically) conjugate to rigid rotation with rotation vector ω of type (K, τ). Moreover

assume that in a neighborhood of Γ, f *there are periodic orbits* $x_{(P/N)}$ *of type* (P/N) *for* $\|N\omega - P\|_{d+e}$ *small enough.*

Then, for $k \in \mathbb{N}, k < \frac{r-1}{\tau}$ *we can find* $D_k > 0$*, such that* $2d$ *of the eigenvalues* $\lambda_1, \ldots, \lambda_{2d}$ *of the derivative* $Df^N(x_{(P/N)})$ *satisfy*

$$|\lambda_i - 1| \leq D_k \|N\omega - P\|_d^{k/2} N, \ i = 1, \ldots, 2d$$

$$(|\lambda_i - 1| \leq \tilde{D}_1 N \exp(-\tilde{D}_2 \|N\omega - P\|_d^{\frac{-1}{2(1+\tau)}}), \ i = 1, \ldots, 2d)$$

The remaining e *eigenvalues are identically* 1.

Remark. For the rotating standard map the eigenvalues of the derivative are completely determined by the trace (since one eigenvalue is identically 1 and the map is volume-preserving). Therefore, instead of monitoring the eigenvalues we will study the behavior of the *residue*

$$R(\mathbf{x}) = \frac{1}{6}(3 - \mathrm{Tr}(Df^N(\mathbf{x})))$$

along a periodic orbit $\mathbf{x}$ with period N.

Remark. In the case of hyperbolic invariant sets semi-conjugate to rigid rotation, it was shown in [FL92-1] that the derivative along periodic orbits approaching the invariant set has exponentially increasing eigenvalues (increasing with the period of the orbit). Although the invariant sets of the rotating standard map are never hyperbolic (due to rigid rotation in the second angle coordinate) we have observed that in the absence of an invariant surface, the eigenvalues of periodic orbits approaching a limit set are exponentially increasing.

Theorem 5.1 (which also holds for complex maps and complex invariant sets) suggests the following algorithm:

- Fix a value $\alpha > 0$ (this value will be used as the cutoff criterion for determining breakdown).
- Choose a family of paths in the parameter space.
- Compute periodic orbits with rotation vectors close to the rotation vector of an invariant set of interest, along the paths, and determine the point along the path when the residue of the periodic orbit satisfies $|R(x)| > \alpha$.

Remark. The value of α plays only a minor role for periodic orbits with large period. The reason is that if an invariant curve exists, the eigenvalues of the derivative of the map approach 1 exponentially as the period increases, whereas if the invariant surface has "disintegrated" then at least one eigenvalue is expected to be exponentially large. The reasoning is valid as long as the breakdown of an invariant surface is well-defined, for example one value along the path in the parameter space.

Remark. The algorithm is based on continuation of periodic orbits using Newton's method in a space of finite sequences, as presented in section 3. Using numerically computed condition numbers we can determine the validity of the continuation scheme. As long as no eigenvalue of the derivative of the map along the periodic orbit is 1, the implicit function theorem guarantees the success of the continuation method for small enough steps along the path. Notice that one eigenvalue in the case of the rotating standard map is identically 1, due to rigid rotation in the second angle coordinate, but does not influence Newton's method, as described by (3.5).

In figures 1a,1b,1c we follow periodic orbits along paths in the parameter plane. We start the paths at $\epsilon = k = 0$, where we know the coordinates of the periodic orbits. The initial value of ϕ_0 was taken to be 1/2. In figure 1a we investigated the domain of existence of the invariant curve with rotation vector ω with Jacobi-Perron expansion $(1,1)^\infty$ for real values of k, ϵ. We observe that for the boundaries for successive approximants crossed each other for many sets of parameter values (a similar phenomenon was observed in [FL92-2] for standard-like maps in the complex domain). The phenomenon is more pronounced for large values of k (see figure 1b). In [ACS91] a similar phenomenon was observed but for a different invariant surface (with rotation vector the spiral mean) using a different approximation scheme (an extension of the Farey-tree expansion). We observed however that for certain paths in the parameter plane the boundaries for successive approximants are well-ordered and their successive positions follow a power law. Such paths are the $\epsilon = 0$ path (corresponding to the well-understood case of the standard map) and the $k = 0$ path (see figure 2).

In figure 1c we present the domain of analyticity of the same invariant surface for a small, fixed value of k ($k = 0.2$) and for complex values of ϵ. The absence of crossings and the scaling observed in the figure suggests that behavior at breakdown could be understood in terms of a fixed point of a renormalization operator in a space of maps with complex variables.

For invariant surfaces that break down to invariant sets semi-conjugate to rotation, we can study the transition by monitoring the behavior of close-by periodic orbits. The natural question is whether an analog of Aubry-Mather theory applies to higher-dimensional systems and what are the invariant sets that an invariant surface breaks down to. The continuation algorithm allows to compute periodic orbits of high period close to breakdown. Based on the results in figure 1a we identified breakdown of the invariant surface with rotation vector $(1,1)^\infty$ at $\epsilon = 1.75, k = 0.2$ and $\epsilon = 3.55, k = 0$. We remark that although crossings between boundaries where different periodic orbits had residue 1 were common, the residue of each periodic orbit along a path seemed to behave regularly.

The transition occurring at breakdown was investigated in figures 3a, 3b, 3c, 3d, 3e. We observe that the invariant set develops gaps, analogous to the ones that appear as an invariant curve breaks down to an Aubry-Mather set in two-dimensional systems. Such gaps have been used to rigorously show (using a computer assisted proof) that breakdown has occurred (see [Mu89] for the case of a four-dimensional symplectic map). Notice that the largest gap occurs at $\theta = \phi = 0$ where the potential attains its maximum.

Remark. For the case of the rotating standard map we have shown in section 3 that periodic orbits for all rational rotation vectors exist. However we have not been able to rigorously determine whether they converge to an invariant set as their rotation vector tends to a diophantine rotation vector.

In figures 4a, 4b, 4c we investigated the behavior of the residue and the stability of periodic orbits along two paths in parameter space. The stability is quantified by the stability exponent of the periodic orbit defined, for an orbit x of period N as

$$\lambda = \frac{1}{N} \log |Df^N(x)|$$

where, by $|Df^N(x)|$, we mean the biggest eigenvalue of the matrix. The stability exponent tends, in the limit $N \to \infty$, to the Lyapounov exponent of the invariant set, as long as the orbit tends to the invariant set and motion on the invariant set is semi-conjugate to rigid rotation with diophantine rotation vector (in which case there is only one invariant measure and it makes sense to talk of Lyapounov exponents without specifying it). Notice that, due to nature of the map, one eigenvalue of the derivative is always one and the other two are reciprocal.

Remark. It is important to monitor the algorithm that computes the periodic orbits to guarantee that the type of periodic orbit we are following is the same. As was remarked in [OV92] periodic orbits of higher-dimensional symplectic maps may undergo bifurcations and follow a different (low-dimensional) torus from the one we are interested in. To avoid this problem we have monitored the size of the condition numbers for the solution of the linear system (3.5). When the condition numbers were large we decreased the stepsize of the algorithm. Moreover we have tried to choose periodic orbits with rotation vectors whose components are irreducible, to avoid bifurcations similar to the ones observed in [OV92]. As can be observed from the behavior of the stability exponent in figures 4b,4c there were indeed bifurcations for some of the periodic orbits with reducible components.

6. Critical function

In [Pe82] Percival introduced a critical function for two-dimensional twist maps. It is a function of the rotation number and its value corresponds to the perturbation strength at which the invariant curve breaks down (in the case of the standard map there are indications from renormalization-group theory that breakdown occurs at a specific value for each rotation number). Based on the renormalization-group description, Buric et al. [BPV90] observed that the critical function for the semi-standard map has transformation properties under the modular group that allow its rapid computation. These transformation properties have also been considered as additional evidence of an underlying renormalization-group transformation.

In analogy to [Pe82] we introduce a critical function $K(\omega)$ for the rotating standard map, from the space of rotation vectors ω to the parameter space (ϵ, k). Given a rotation vector ω, the values of the critical function are the values of the parameters at which breakdown occurs. Similar to the two-dimensional case, this function is discontinuous in a set of large measure (since it vanishes at all rationally dependent rotation vectors but is non-zero at a set of large measure).

We will use the value of the residue of a nearby periodic orbit as an indication of breakdown of an invariant surface.

In figure 5 we present the logarithm of a slice of the critical function for the rotating standard map, at a fixed value of $k = 0.2$. The figure was generated using periodic orbits and taking into account the symmetries of the critical function $\omega_1 \to 1 - \omega_1$, $\omega_2 \to 1 - \omega_2$. We chose the points at which to evaluate the critical function randomly, and then approximated the chosen points with a nearby (eventually) golden mean of the Jacobi-Perron algorithm (i.e. with Jacobi-Perron expansion $(\ldots,(1,1)^\infty)$). We truncated the Jacobi-Perron expansion to obtain rational points close to the (eventual) golden means. All the periods used were between 300 and 800. We used a continuation method to follow the periodic orbits up to the perturbation value of ϵ at which the residue becomes 1. We assigned that perturbation strength as the value of the critical function (the cases when breakdown has already occurred for $\epsilon = 0$ have been considered as if breakdown occurred at $\epsilon = 0$). Our procedure is similar to the one used in [BPV90] for the computation of the critical function for the semi-standard map.

Figure 5 took 30 hours of CPU time on an IBM RS/6000 370. It was plotted using `gnuplot`, by computing the values at the grid points using weighted interpolation (this process has the undesirable effect of "smoothing" the function in certain regions).

We have not been able to identify transformation properties of the critical function under the action of the Jacobi-Perron operator for the rotating standard map. We face two problems, the first being a possible problem for any multi-dimensional approximation algorithm, while the second pertaining to the Jacobi-Perron algorithm in particular:

(a) We do not know how many parameters are necessary to describe the critical fixed point of a renormalization-group algorithm. In hopes of simple behavior (similar to the one uncovered by MacKay in the case of the twist maps of the annulus) we have used one-dimensional parameter paths in the parameter space. Notice however that, as indicated in figure 1a, there are regions in parameter space where the renormalization group behavior is more complicated.

(b) The Jacobi-Perron algorithm is not symmetric with respect to the two coordinates. In the case of twist maps (when the rotation vector is a real number) it is possible to produce a linear combination of $\log K(\omega)$, $\log K(U\omega)$ to cancel the leading order logarithmic singularities of the critical function. It would be interesting to determine whether such cancelations are possible in the case of the rotating standard map or high-dimensional symplectic maps.

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9. Captions

Figure 1.
a) Domain of existence for the invariant surface with golden rotation vector $(1,1)^\infty$ for real values of the parameters. The periodic orbits chosen are Jacobi-Perron convergents to $(1,1)^\infty$. The different sets plotted correspond to the boundary where the residue of periodic orbits is 1. Set 1 has rotation vector $(44/81, 68/81)$, Set 2 $(81/149, 125/149)$, Set 3 $(149/274, 230/274)$, Set 4 $(274/504, 423/504)$, Set 5 $(504/927, 778/927)$, Set 6 $(927/1705, 1431/1705)$ and Set 7 $(1705/3136, 2632/3136)$.
b) Detail of figure 1a.
c) Domain of existence for the invariant surface with golden rotation vector $(1,1)^\infty$ for fixed $k = 0.2$, complex ϵ. Sets 1-7 same as in figure 1a.

Figure 2.
Scaling of the points where the residue becomes 1, for $k = 0$ for successive Jacobi-Perron approximants to $(1,1)^\infty$. The approximant order is denoted in the horizontal axis with the first approximant $(7/13, 11/13)$ and the last – eleventh – $(3136/5768, 4841/5768)$. The value of ϵ_{cr} is determined from the next approximant $(5768/10609, 8904/10609)$.

Figure 3.
a) A periodic orbit close to the invariant surface with rotation vector $(1,1)^\infty$. The invariant surface is close to breakdown ($k = 0.2, \epsilon = 1.7$). Periodic orbit rotation vector $(10609/19513, 16377/19513)$ and residue -0.831.
b) A periodic orbit close to an invariant set with rotation vector $(1,1)^\infty$ after breakdown ($k = 0.2, \epsilon = 1.8$). Periodic orbit rotation vector $(10609/19513, 16377/19513)$ and residue -7.01×10^{12}.
c) A little before breakdown. Rotation vector $(1,1)^\infty$, $k = 0, \epsilon = 3.5$. Periodic orbit rotation vector $(10609/19513, 16377/19513)$, residue -5.41×10^{-3}.
d) A little after breakdown. Rotation vector $(1,1)^\infty$, $k = 0, \epsilon = 3.6$. Periodic orbit rotation vector $(10609/19513, 16377/19513)$, residue -2.05×10^{3}.
e) Continuation of the periodic orbit in figure 3d. $k = 0, \epsilon = 4.0$. Residue -1×10^{78}.

Figure 4.
a) Residue of periodic orbits approximating an invariant set with rotation vector $(1,1)^\infty$ along the path $k = 0$. The different sets correspond to different periodic orbits. Set 1

$(149/274, 230/274)$, Set 2 $(274/504, 423/504)$, Set 3 $(504/927, 778/927)$, Set 4 $(927/1705, 1431/1705)$, Set 5 $(1705/3136, 2632/3136)$, Set 6 $(3136/5768, 4841/5768)$ and Set 7 $(5768/10609, 8904/10609)$.

b) Stability index λ of the periodic orbits in figure 4a.

c) Stability index λ of periodic orbits approximating an invariant set with rotation vector $(1,1)^\infty$ along the path $k = 0.2$. Sets 1-7 same as in figure 4a.

Figure 5.

Logarithm of the critical function at $k = 0.2$. The values of the function are computed using the residue of 6576 periodic orbits and interpolating on a 150×150 grid.