REGULATED SMOOTHING FOR SCHRÖDINGER EVOLUTION

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In the case of the *free* Schrödinger evolution, the wave function $\psi(t,x) = (e^{it\Delta}\,\psi^0)(x)$ is infinitely differentiable with respect to $t>0$ and $x\in\mathbb{R}^n$, if ψ^0 is merely a distribution, but of compact support. The proof is evident, since

$$S_{-\Delta}(t,x,y) = e^{-in\frac{\pi}{4}}(4\pi t)^{-\frac{n}{2}}\, e^{i\,\frac{|x-y|^2}{4t}},$$

the kernel of $e^{it\Delta}$, is infinitely differentiable in $t>0$, x and y.

For a more general evolution operator $e^{-it\mathcal{A}}$, where $\mathcal{A} = -\Delta + V$ and $V = V(x)$ is a potential, the explicit form of the kernel $S_{\mathcal{A}}(t,x,y)$ is not usually available. Nevertheless, a few results on the regularity of $S_{\mathcal{A}}(t,x,y)$ are already known. In particular, if V is infinitely differentiable and bounded, together with all its derivatives, i.e.,

$$\sup_{x\in\mathbb{R}^n} |\partial_x^\alpha V(x)| < \infty, \qquad \forall \alpha\in\mathbb{Z}_+^n, \tag{1}$$

then $S_{\mathcal{A}}(t,x,y)$ is C^∞ in $\mathbb{R}_t^{>0}\times\mathbb{R}_x^n\times\mathbb{R}_y^n$. This was proved by S. Zelditch [Zel]; see also [OF], [Kit]. The same regularity of $S_{\mathcal{A}}(t,x,y)$ has been shown recently by W. Craig, T. Kappeler and W. Strauss [CKS], under the assumption that for all $x\in\mathbb{R}^n$,

$$V(x) \text{ is real, and } \quad |\partial_x^\alpha V(x)| \le c_\alpha \langle x\rangle^{\rho-|\alpha|}, \qquad \forall\alpha\in\mathbb{Z}_+^n, \tag{2}$$

with some ρ, $0\le\rho<1$ (as usual, $\langle x\rangle = \sqrt{1+|x|^2}$). In fact, they prove the smoothness of $S_{\mathcal{A}}(t,x,y)$ in the more general case of operators $\mathcal{A}$ with variable coefficients that

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stabilize at infinity and satisfy a non-trapping condition. The regularity of $S_{\mathcal{A}}(t,x,y)$ for some other types of Schrödinger equations and systems with variable coefficients and, possibly, boundary conditions, has been shown by L. Kapitanski and Yu. Safarov [KS], but their main concern is not in enlarging the class of admissible potentials.

Closely related to what we discuss are the constructions of the local (in time) parametrices for the Schrödinger equation suggested by D. Fujiwara [Fuj], H. Kitada [Kit], F. Treves [Tre]. The results of F. Treves, [Tre], imply that $S_{\mathcal{A}}(t,x,y)$ is smooth at least for small t, $0<t<t_1$, when $\mathcal{A}=-\Delta+Q(x)+V(x)$, where $Q(x)$ is a quadratic form (in x) and V satisfies (7) with $0\le\rho\le 1$ ($\rho=1$ is included). However, unlike the hyperbolic case, for Schrödinger equations, $S_{\mathcal{A}}(t,x,y)$ being smooth for $t\in[0,t_1]$ does not imply that it has no singularities for $t\in(t_1,2t_1]$. A counter-example is given by Mehler's formula ([MF], [Sim 1]) for the fundamental solution $S_{\mathcal{A}}(t,x,y)$ in the case $\mathcal{A}=-\Delta+|x|^2$ (the quantum harmonic oscillator):

$$S_{-\Delta+|x|^2}(t,x,y)=\frac{e^{-in\frac{\pi}{4}\mu(t)}}{|2\pi\sin 2t|^{\frac{n}{2}}}\,e^{i\{\cot 2t\,\frac{|x|^2+|y|^2}{2}\,-\,\csc 2t\cdot xy\}}\,,$$

where $\mu(t)=1+2k$ for $\frac{\pi}{2}k<t<\frac{\pi}{2}(k+1)$, integer k, and

$$S_{-\Delta+|x|^2}(t=\frac{\pi}{2}\,k,\ x,\ y)\ =e^{-ik\frac{\pi}{2}}\,\delta\,((-1)^k x-y).$$

Evidently, the singularities reappear. Moreover, their location and structure remain invariant under the perturbations of the Hamiltonian $-\Delta+|x|^2$ by the potentials $V(x)$ of class (2) with $\rho=0$, as was shown by S. Zelditch [Zel] and A. Weinstein [Wei].

Thus, the potentials that grow quadratically at infinity may cause the fundamental solution $S_{\mathcal{A}}(t,x,y)$ to develop singularities in finite time. Not all quadratic potentials do that. For example, the kernel of $\exp\{-it(-\Delta-|x|^2)\}$ is

$$S_{-\Delta-|x|^2}(t,x,y)=\frac{e^{-in\frac{\pi}{4}}}{(2\pi\sinh 2t)^{\frac{n}{2}}}\,e^{i\{\coth 2t\,\frac{|x|^2+|y|^2}{2}\,-\,\frac{1}{\sinh 2t}\cdot xy\}}\,,$$

and has no singularities for $t>0$.

One of the facts that we will establish in the present paper is that if the potential grows slower than quadratically, then the corresponding fundamental solution is smooth for $t>0$. When the first draft of our paper was already written we learned that the smoothness of $S_{-\Delta+V}(t,x,y)$ for the potentials V satisfying (2) with $\rho<2$ had been proved independently by K. Yajima [Yaj 4]. He starts by constructing an approximate propagator - a parametrix - for the evolution equation, and then shows that both the approximate propagator and the complementary error-term are smooth. In the second part of his very interesting preprint [Yaj 4], Yajima shows that if the spatial dimension $n = 1$, then for certain potentials with *superquadratic* growth, the fundamental solution is nowhere smooth in $t > 0$, x and y. This is a remarkable result that still has to be better understood.

In the present paper we deal with the *subquadratic* potentials though of more general class than (2) (see Hypotheses (**V**) below). Our approach is entirely different from that of Yajima. We not only prove the C^∞ regularity of $S_{\mathcal{A}}(t,x,y)$ but also describe more precisely how the decay of the initial data converts into the improvement of the local regularity of the wave function. An important result in this spirit was proven by A. Jensen [Jen]: if the potential $V(x)$ is real-valued and bounded, together with its derivatives up to some order $k \geq 1$, then there exists a constant $C > 0$ such that the following estimate holds:

$$\|\langle x\rangle^{-k}\, e^{-it(-\Delta+V)}\, f\|_{H^k(\mathbb{R}^n)} \;\leq\; C\,(|t|^k + |t|^{-k})\, \|\langle x\rangle^k\, f\|_{L^2(\mathbb{R}^n)}, \tag{3}$$

for all $f \in \langle x\rangle^{-k}\, L^2(\mathbb{R}^n)$. This result was extended by T. Ozawa in [Oza 1] to include the potentials of the sort $V(x) = \sum_{j=1}^{m} c_j|x|^{-\mu_j}$, where each $\mu_j > 0$ and where these exponents obey certain restrictions involving the amount of smoothing, k, and the space dimension n. Note that because of the singularities of V, the value of k cannot be arbitrarily large. Some ideas behind the proofs in [Jen] and [Oza 1] go back to the works of C. Wilcox [Wil], W. Hunziker [Hun], C. Radin and B. Simon [RS], P. Perry [Per], and others, where the weighted Sobolev spaces *invariant* under the Schrödinger group $e^{-it(-\Delta+V)}$ were sought.

In what follows, we generalize Jensen's result in two directions: first, we allow the potentials to grow at infinity as $\langle x\rangle^\rho$ with $0 \leq \rho < 2$, and second, we can include a magnetic field with appropriate growth conditions also. The proof is quite simple, and to illustrate the basic idea, let us discuss at first the situation with no magnetic field involved.

We make the following assumption on the potential V:

Hypotheses (V):

The potential $V(\cdot)$ is infinitely differentiable, its imaginary and real parts satisfy

$$|\mathrm{Im}\, V(x)| \;\leq\; c_0, \quad |\mathrm{Re}\, V(x)| \;\leq\; c_0\, \langle x\rangle^\rho, \quad \forall x \in \mathbb{R}^n, \tag{4a}$$

while

$$|\partial_x^\alpha V(x)| \le c_{|\alpha|} \langle x\rangle^{\frac{\varrho}{2}\nu|\alpha|}, \quad \alpha \in \mathbb{Z}_+^n,\ |\alpha| > 0, \quad \forall x \in \mathbb{R}^n, \tag{4b}$$

with ρ and ν such that

$$0 \le \rho < 2, \qquad 0 \le \nu < 1. \tag{4c}$$

Consider now the initial-value problem for the Schrödinger equation in $\mathbb{R}^n$, $n \ge 1$,

$$\frac{1}{i}\frac{\partial}{\partial t}\psi(t,x) - \Delta\psi(t,x) + V(x)\psi(t,x) = 0, \qquad t > 0,\ x \in \mathbb{R}^n \tag{5a}$$

$$\psi(0,x) = \psi^0(x). \tag{5b}$$

To measure the regularity of solutions and handle the fact that the potential is growing, we need some special function spaces. For this purpose, define the scale $\{H^s_{(\rho)},\ s \in \mathbb{R}\}$ of Hilbert spaces generated from $H^0_{(\rho)} = L^2(\mathbb{R}^n)$ by the powers of the operator $\Lambda_\rho = \sqrt{-\Delta + \langle x\rangle^\rho}$ in the same way as the Sobolev spaces H^s are generated by the powers of $\Lambda_0 = \sqrt{-\Delta + 1}$.

We are in position now to state our result.

Theorem 1. *Assume that V satisfies the hypotheses* (**V**). *Let $\psi^0 \in H^s_{(\rho)}$, for some s, and, in addition, $\langle x\rangle^\ell \psi^0 \in H^{s-\nu\ell}_{(\rho)}$ for $\ell = 1, \ldots, m$, some integer $m \ge 1$, where ν is the same as in (4b). Then for any $t > 0$ the solution ψ of problem (5) has the following regularity properties:*

$$\langle x\rangle^{-\ell}\psi(t,\cdot) \in H^{s+\ell(1-\nu)}_{(\rho)}, \quad \ell = 0, 1, \ldots, m. \tag{6a}$$

Moreover, given $T > 0$, there exists a constant $c = c(T,m,V) > 0$ such that the following estimate holds:

$$\|\langle x\rangle^{-m} e^{-it(-\Delta+V)}\psi^0\|_{H^{s+(1-\nu)m}} \le c\sum_{\ell=0}^{m}(1+\frac{1}{t^\ell})\,\|\langle x\rangle^\ell\psi^0\|_{H^{s-\ell\nu}},\ \forall t,\ 0 < t \le T. \tag{6b}$$

Proof. Given $\psi^0 \in H^s_{(\rho)}$, there exists a unique solution $\psi \in C_{\mathrm{loc}}(\mathbb{R};\, H^s_{(\rho)})$ to (5). Moreover, on any finite time interval $[0,T]$,

$$\sup_{0\le\tau\le T}\|\psi(\tau)\|_{H^s_{(\rho)}} \le c\,\|\psi^0\|_{H^s_{(\rho)}}, \tag{7a}$$

with some constant $c > 0$ independent of ψ^0.

This is a particular case of the more general result that we need later anyway: for any real r, for any $u_0 \in H^r_{(\rho)}$ and for any $h(\cdot_t, \cdot_x) \in L^1_{\mathrm{loc}}(\mathbb{R} \to H^r_{(\rho)})$, there exists a unique solution $u(t,x)$ of the inhomogeneous problem,

$$\frac{1}{i}\frac{\partial}{\partial t}\, u(t,x) \;-\; \Delta u(t,x) \;+\; V(x)u(t,x) \;=\; h(t,x), \quad u(0,x) = u_0(x),$$

such that the corresponding mapping $u : \mathbb{R} \ni t \mapsto u(t,\cdot) \in H^r_{(\rho)}$ is (strongly) continuous. Moreover, for every $T > 0$, there exists a constant $\tilde{c}_T > 0$ such that the following energy estimate prevails:

$$\|u(t,\cdot)\|_{H^r_{(\rho)}} \;\le\; \tilde{c}_T \,\{\|u_0\|_{H^r_{(\rho)}} \;+\; \int_0^t \|h(\tau,\cdot)\|_{H^r_{(\rho)}}\, d\tau\}, \quad 0 \le t \le T. \tag{8}$$

For the proof see, e.g., [Kap].

In order to be able to apply Lemma 2.1 of [Kap] in our circumstances, one has to check that for every real k,

$$\Lambda_\rho^k \, V \, \Lambda_\rho^{-k} \;-\; \Lambda_\rho^{-k} \, V^* \, \Lambda_\rho^k \;:\; L^2 \to L^2 \tag{9a}$$

(i.e., the operator maps L^2 into itself and is bounded), where V is identified with the operator of multiplication by $V(x)$, and $V(x)^*$ is the complex conjugate of $V(x)$. Notice also that

$$\Lambda_\rho^k \, \langle x\rangle^\rho \, \Lambda_\rho^{-k} \;-\; \Lambda_\rho^{-k} \, \langle x\rangle^\rho \, \Lambda_\rho^k \;:\; L^2 \to L^2, \quad \forall k \in \mathbb{R}. \tag{9b}$$

Our assumptions on the potential V guarantee (9a), but the rigorous justification would involve some basic calculus for certain classes of pseudodifferential operators in $\mathbb{R}^n$. Those pseudodifferential operators (PDOs) are defined by their global symbols from the corresponding classes of symbols $S^\mu = S^\mu(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, $\mu \in \mathbb{R}$, where $a(x,\xi)$ belongs to S^μ if $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, and for all α, $\beta \in \mathbb{Z}^n_+$,

$$|\partial_x^\alpha \partial_\xi^\beta \, a(x,\xi)| \;\le\; c_{\alpha,\beta} \begin{cases} \left(\lambda_\rho(x,\xi)\right)^{\mu-|\beta|}, & |\alpha| = 0, \\ \left(\lambda_\rho(x,\xi)\right)^{\mu-1-|\beta|} \, \langle x\rangle^{\rho-|\alpha|}, & |\alpha| \ge 1, \end{cases}$$

where $\lambda_\rho(x,\xi) = \sqrt{|\xi|^2 + \langle x\rangle^\rho}$. In fact, not only the justification of (9) but also several other steps in our proof require certain standard facts, such as the composition formula, parametrices, etc., for the operators of the classes $\operatorname{Op} S^\mu$. In addition, we have to know such facts as that $\operatorname{Op} a$ maps $H^r_{(\rho)}$ continuously into $H^{r-\mu}_{(\rho)}$, for all r, for $a \in S^\mu$. It is also useful to have in mind that an operator $\Lambda^\mu_\rho = (-\Delta + \langle x\rangle^\rho)^{\mu/2}$ belongs to $\operatorname{Op} S^\mu$, and its 'principal' symbol is $(|\xi|^2 + \langle x\rangle^\rho)^{\mu/2}$. Although we couldn't find the references where the appropriate calculus is developed exactly in the form we need it, we refer the reader to the papers [Bea], [Fei], [KT], [Par], where similar classes of PDOs are considered.

Returning to the proof of the theorem, denote, for brevity, $\mathcal{L} := \frac{1}{i}\frac{\partial}{\partial t} - \Delta + V$ and $\Upsilon_j = x_j + 2it\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$. Note, that

$$[\,\mathcal{L},\ \Upsilon_j\,] \;=\; -\,2\,i\,t\,\frac{\partial V(x)}{\partial x_j}\,. \tag{10}$$

Let us first prove the theorem in the case $m = 1$. Assume that $\psi^0 \in H^s_{(\rho)}$ and, in addition, $\langle x\rangle\psi^0 \in H^{s-\nu}_{(\rho)}$. Let ψ be the solution of (5), and denote $\psi_j := \Upsilon_j\psi$. Then ψ_j is the solution of the inhomogenious problem

$$\mathcal{L}\,\psi_j \;=\; -\,2\,i\,t\,\frac{\partial V(x)}{\partial x_j}\,\psi, \tag{11a}$$

$$\psi_j\big|_{t=0} \;=\; x_j\,\psi^0. \tag{11b}$$

When $t > 0$, the commutator (10), as an operator, has order at most ν in the Hilbert scale $\{H^r_{(\rho)}\}$. In other words, $[\mathcal{L}, \Upsilon_j] : H^r_{(\rho)} \to H^{r-\nu}_{(\rho)}$, or, equivalently, $\Lambda^{r-\nu}_\rho[\mathcal{L}, \Upsilon_j]\Lambda^{-r}_\rho : L^2 \to L^2$, for any r. This follows from the composition formula for PDOs, and some simple inequalities, the inspiring example of which is the following triviality (see (4b)):

$$|\nabla V| \;\le\; c_1\,\langle x\rangle^{\frac{\rho}{2}\,\nu} \;\le\; c_1\,\Big(\sqrt{|\xi|^2 + \langle x\rangle^\rho}\,\Big)^\nu.$$

Since $\psi \in C_{\mathrm{loc}}(\mathbb{R};\ H^s_{(\rho)})$, the right-hand side of (11a) is in $C_{\mathrm{loc}}(\mathbb{R};\ H^{s-\nu}_{(\rho)})$. Also, $\psi_j(0,\cdot) \in H^{s-\nu}_{(\rho)}$. Hence, $\psi_j \in C_{\mathrm{loc}}(\mathbb{R};\ H^{s-\nu}_{(\rho)})$, and (see (8))

$$\|\psi_j(t)\|_{H^{s-\nu}_{(\rho)}} \;\le\; \tilde{c}\,\{\|x_j\psi^0\|_{H^{s-\nu}_{(\rho)}} \;+\; \int_0^t 2\tau\|\frac{\partial V(x)}{\partial x_j}\,\psi(\tau)\|_{H^{s-\nu}_{(\rho)}}\,d\tau\}.$$

We also know that

$$\|\frac{\partial V(x)}{\partial x_j}\,\psi(\tau)\|_{H^{s-\nu}_{(\rho)}} \le c\,[\nabla V]_{s,\nu}\,\sup_{0\le\tau\le T}\|\psi(\tau)\|_{H^s_{(\rho)}} \;\le\; c\,[\nabla V]_{s,\nu}\,\|\psi^0\|_{H^s_{(\rho)}},$$

where $[\nabla V]_{s,\nu}$ is the maximum of the norms of the operators $\frac{\partial V(x)}{\partial x_j} : H^s_{(\rho)} \to H^{s-\nu}_{(\rho)}$. Thus,

$$\|\psi_j(t)\|_{H^{s-\nu}_{(\rho)}} \le \tilde{c}\,\{\|x_j\psi^0\|_{H^{s-\nu}_{(\rho)}} + t^2\,\|\psi^0\|_{H^s_{(\rho)}}\}. \tag{12}$$

Assuming that $t>0$ is fixed, we rearrange the equality $\langle x\rangle^{-1}\Upsilon_j\psi = \langle x\rangle^{-1}\psi_j$ as follows:

$$\frac{\partial}{\partial x_j}\,\Big(\frac{1}{\langle x\rangle}\psi\Big) = \frac{1}{2\,i\,t\,\langle x\rangle}\,\psi_j - \frac{x_j}{\langle x\rangle}\,\Big(\frac{1}{2\,i\,t} + \frac{1}{\langle x\rangle^2}\Big)\,\psi. \tag{13a}$$

Taking the $H^{s-\nu}_{(\rho)}$-norm of both sides, we obtain the estimate

$$\begin{aligned} \Big\|\frac{\partial}{\partial x_j}\,\Big(\frac{1}{\langle x\rangle}\psi(t)\Big)\Big\|_{H^{s-\nu}_{(\rho)}} &\le c\,\{\frac{1}{t}\|\psi_j(t)\|_{H^{s-\nu}_{(\rho)}} + (1+\frac{1}{t})\|\psi(t)\|_{H^{s-\nu}_{(\rho)}}\} \\ &\le c\,(1+\frac{1}{t})\,\{\|\psi^0\|_{H^s_{(\rho)}} + \|\langle x\rangle\,\psi^0\|_{H^{s-\nu}_{(\rho)}}\}, \end{aligned} \tag{13b}$$

where we have used (7), (12), and the fact that the operators of multiplication by $x_j\langle x\rangle^{-1}$ and $\langle x\rangle^{-k}$, $k>0$, are of order ≤ 0.

Now, for any $r\in\mathbb{R}$,

$$\big(\|\nabla_x f\|^2_{H^r_{(\rho)}} + \|\langle x\rangle^{\frac{\rho}{2}} f\|^2_{H^r_{(\rho)}}\big)^{\frac{1}{2}} \tag{14}$$

is an equivalent norm in $H^{r+1}_{(\rho)}$. The estimate (13b) then yields,

$$\Big\|\frac{1}{\langle x\rangle}\psi(t)\Big\|_{H^{s+1-\nu}_{(\rho)}} \le c\,(1+\frac{1}{t})\,\{\|\psi^0\|_{H^s_{(\rho)}} + \|\langle x\rangle\,\psi^0\|_{H^{s-\nu}_{(\rho)}}\}, \tag{15}$$

which proves the theorem in the case $m=1$.

Assume next that, in addition to $\psi^0\in H^s_{(\rho)}$ and $\langle x\rangle\psi^0\in H^{s-\nu}_{(\rho)}$, we also have $\langle x\rangle^2\psi^0\in H^{s-2\nu}_{(\rho)}$. Denote $\psi_{kj}=\Upsilon_k\psi_j=\Upsilon_k\,\Upsilon_j\,\psi$. The function $\psi_{kj}(t,x)$ solves the problem

$$\mathcal{L}\,\psi_{kj} = -\,2\,i\,t\,\frac{\partial V(x)}{\partial x_k}\,\psi_j - 2\,i\,t\,\frac{\partial V(x)}{\partial x_j}\,\psi_k + 4\,t^2\,\frac{\partial^2 V}{\partial x_k\partial x_j}\,\psi, \tag{16a}$$

$$\psi_{kj}\big|_{t=0} = x_k\,x_j\,\psi^0. \tag{16b}$$

Taking into account the information obtained in the first step, and the fact that multiplication by $\frac{\partial^2 V}{\partial x_k\,\partial x_j}$ is a bounded operator from $H^s_{(\rho)}$ into $H^{s-2\nu}_{(\rho)}$, we see that the right-hand side of (16a) is in $C_{\rm loc}(\mathbb{R};\,H^{s-2\nu}_{(\rho)})$, and its $H^{s-2\nu}_{(\rho)}$-norm is bounded by

$$\le \mathit{const}\,\{t\,\|\langle x\rangle\psi^0\|_{H^{s-\nu}_{(\rho)}} + t^2\,\|\psi^0\|_{H^s_{(\rho)}}\}.$$

Hence, $\psi_{kj} \in C_{\rm loc}(\mathbb{R};\, H^{s-2\nu}_{(\rho)})$, and

$$\text{(17a)} \qquad \|\psi_{kj}(t)\|_{H^{s-2\nu}_{(\rho)}} \;\le\; \tilde{c}\,\{\|x_k\, x_j \psi^0\|_{H^{s-\nu}_{(\rho)}} \;+\; t^2\, \|\langle x\rangle \psi^0\|_{H^{s-\nu}_{(\rho)}} \;+\; t^3\, \|\psi^0\|_{H^{s}_{(\rho)}}\}.$$

Since $\langle x\rangle^{-1}\Upsilon_k \psi_j \;=\; \langle x\rangle^{-1}\psi_{kj}$, we obtain, in exactly the same fashion as we obtained (13b), an estimate for the $H^{s-2\nu}_{(\rho)}$-norm of $\nabla_x\,(\langle x\rangle^{-1}\psi_j(t))$ that, in turn, implies

$$\text{(17b)} \quad \|\frac{1}{\langle x\rangle}\psi_j(t)\|_{H^{s+1-2\nu}_{(\rho)}} \;\le\; c\,(1+\frac{1}{t})\,\{\|\psi^0\|_{H^{s-\nu}_{(\rho)}} + \|\langle x\rangle \psi^0\|_{H^{s-\nu}_{(\rho)}} + \|\langle x\rangle^2\, \psi^0\|_{H^{s-2\nu}_{(\rho)}}\}.$$

With this new information at hand, we write the equality $\langle x\rangle^{-2}\Upsilon_j \psi \;=\; \langle x\rangle^{-2}\psi_j$ in the form:

$$\text{(17c)} \qquad \frac{\partial}{\partial x_j}\,(\frac{1}{\langle x\rangle^2}\psi) \;=\; \frac{1}{2\,i\,t\,\langle x\rangle}\,\frac{1}{\langle x\rangle}\psi_j \;-\; \frac{x_j}{\langle x\rangle}\,(\frac{1}{2\,i\,t} \;+\; \frac{2}{\langle x\rangle^2})\,\frac{1}{\langle x\rangle}\psi,$$

and take the $H^{s+1-2\nu}_{(\rho)}$-norm of both sides. Using (15), (17) and (14), we obtain the estimate

$$\text{(17d)} \quad \|\frac{1}{\langle x\rangle^2}\psi(t)\|_{H^{s+2-2\nu}_{(\rho)}} \;\le\; c\,(1+\frac{1}{t^2})\,\{\|\psi^0\|_{H^{s}_{(\rho)}} + \|\langle x\rangle \psi^0\|_{H^{s-\nu}_{(\rho)}} + \|\langle x\rangle^2\, \psi^0\|_{H^{s-2\nu}_{(\rho)}}\}.$$

and, thus, prove the theorem in the case $m = 2$.

If $m > 2$, one should repeat the above argument several more times. □

An immediate corollary of Theorem 1 is the following result.

Corollary 2. *Assume that the potential V satisfies Hypotheses (**V**). Then the fundamental solution $S_{-\Delta+V}(t,x,y)$ is C^∞ in $t>0$ and $x,\, y \in \mathbb{R}^n$.*

If ψ^0 is a distribution of compact support, then the solution $\psi(t,x)$ of (5) is C^∞ in $t>0$ and $x \in \mathbb{R}^n$. □

As we already mentioned, the method we presented above works for the Schrödinger equation with magnetic field as well. Let us consider the following equation describing a nonrelativistic quantum spinless particle in time-independent electromagnetic field,

$$\text{(18a)} \qquad (\frac{1}{i}\frac{\partial}{\partial t} - A_0(x))\psi(t,x) + \sum_{k=1}^{n}(\frac{1}{i}\,\frac{\partial}{\partial x_k} - A_k(x))^2\,\psi(t,x) = 0,$$

$$\text{(18b)} \qquad \psi(0,x) = \psi^0(x).$$

Here $A_0, A_1, \dots A_n$ are some given real-valued functions of x, A_0 the electric and A_j $(j = 1, \dots, n)$ the magnetic potentials. To the best of the authors' knowledge, the regularity of the Schwartz kernel of the solution operator $S(t) : \psi^0(\cdot) \mapsto \psi(t, \cdot)$ for (18) has not been discussed in the literature. [We should mention, however, that the smoothing effect in the sense of space-time integrability properties of solutions (Strichartz-type estimates) for the Schrödinger equation with magnetic field was discussed in [Yaj 1,2].] Comparing (18) with (5), we may expect that smoothing takes place if the A_j's grow at infinity slower than $|x|$, and A_0 grows slower than $|x|^2$. Essentially, this turns out to be true. The linear growth of the A_j's is critical: if $n = 3$, $A_0 = 0$, and $\vec{A}(x) = \frac{1}{2}\vec{B}_0 \times \vec{x}$, so that the magnetic field $\vec{B}(x) = \mathit{curl}\,\vec{A}(x) = \vec{B}_0$ is constant, then the *magnetic* analogue of Mehler's formula (see [AHB, I], formula (3.5); and [FH]) shows that the fundamental solution $S(t, x, y)$ develops singularities in finite time. But, if the growth rate of $\vec{A}(x)$ is less than 1, then, *as we prove below*, there are no singularities for $t > 0$.

Our assumptions on $A_0, A_1, \dots A_n$ are as follows.

Hypotheses (A)

The potentials $A_0, A_1, \dots A_n$ are infinitely smooth, $A_1, \dots A_n$ are real-valued, and

1) $A_0(x)$ satisfies Hypotheses ($\mathbf{V}$) with some ρ and ν obeying (4c);
2) Each A_j, $j = 1, \dots, n$, satisfies

$$(19a)\qquad |A_j(x)| \le \tilde{c}_0 \langle x \rangle^{\frac{\rho}{2}}, \qquad |\partial_x^\alpha A_j(x)| \le \tilde{c}_{|\alpha|} \langle x \rangle^{\frac{\rho}{2}(|\alpha|-1)}, \quad \forall \alpha \in \mathbb{Z}_+^n, \quad |\alpha| > 0,$$

for all $x \in \mathbb{R}^n$, with some constants $\tilde{c}$;

3) For $j, k = 1, \dots, n$, $j \ne k$, and for any $\alpha \in \mathbb{Z}_+^n$,

$$(19b)\qquad |\partial_x^\alpha F_{jk}(x)| \le c'_{|\alpha|} \langle x \rangle^{\frac{\rho}{2}\nu|\alpha|}.$$

Also, for any $k = 1, \dots, n$,

$$(19c)\qquad |\partial_x^\alpha \sum_{j=1}^n x_j F_{jk}(x)| \le c'_{|\alpha|} \langle x \rangle^{\frac{\rho}{2}\nu|\alpha|}.$$

[Here and further on we use the notation $F_{jk}(x) = \partial_{x_j} A_k(x) - \partial_{x_k} A_j(x)$.]

Theorem 1′. *Assume that the potentials $A_0, \dots, A_n$ satisfy the hypotheses ($\mathbf{A}$). If $\langle x \rangle^\ell \psi^0 \in H_{(\rho)}^{s-\nu\ell}$ for $\ell = 0, 1, \dots, m$, some integer $m \ge 1$, then the assertions of Theorem 1 hold for the solution $\psi(t, x)$ of (18).*

Proof. We use the same idea as in the proof of Theorem 1. As there, for the scale of Hilbert spaces we choose $\{H^r_{(\rho)}\}$, generated by the powers of the operator $\Lambda_{(\rho)} = \sqrt{-\Delta + \langle x \rangle^\rho}$. The hypotheses 1) and 2) in (**A**) are sufficient for showing that for every real k,

$$\Lambda^k_{(\rho)}\,(\frac{\partial}{\partial x_k}\,A_k \,+\, A_k\,\frac{\partial}{\partial x_k})\Lambda^{-k}_{(\rho)} \,-\, \Lambda^{-k}_{(\rho)}\,(\frac{\partial}{\partial x_k}\,A_k \,+\, A_k\,\frac{\partial}{\partial x_k})\Lambda^{k}_{(\rho)} \;:\; L^2 \to L^2,$$

and

$$\Lambda^k_{(\rho)}\,(A_0 - \sum_{1\le j\le n} |A_j|^2)\,\Lambda^{-k}_{(\rho)} \,-\, \Lambda^{-k}_{(\rho)}\,(A_0^* - \sum_{1\le j\le n} |A_j|^2)\,\Lambda^{k}_{(\rho)} \;:\; L^2 \to L^2,$$

as well. This and (9b), according to [Kap], Lemma 2.1, guarantee that the problem

$$(\frac{1}{i}\frac{\partial}{\partial t} - A_0(x))u(t,x) + \sum_{k=1}^{n}(\frac{1}{i}\,\frac{\partial}{\partial x_k} - A_k(x))^2\,u(t,x) = h(t,x), \quad u(0,x) = u_0(x),$$

has a (unique) solution in $C_{\mathrm{loc}}(\mathbb{R};\, H^r_{(\rho)})$ for every $u_0 \in H^r_{(\rho)}$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R};\, H^r_{(\rho)})$, with the energy estimate (8); r being arbitrary real.

Denote $\mathcal{L} := \frac{1}{i}\frac{\partial}{\partial t} + \sum_{k=1}^{n} D_k^2 - A_0$, where $D_k = \frac{1}{i}\,\frac{\partial}{\partial x_k} - A_k(x)$. The operators Υ_j used in the proof of Theorem 1 will be modified in a natural way to $\Upsilon_j = x_j + 2it(\frac{\partial}{\partial x_j} - i\,A_j(x)) = x_j - 2tD_j$.

Let ψ^0 satisfy the regularity assumption of the theorem and ψ be the corresponding solution of (18). Denote $\psi_j := \Upsilon_j\psi$. Then the ψ_j's satisfy

(20a)
$$\mathcal{L}\psi_j + \frac{2}{i}\sum_{k=1}^{n} F_{kj}(x)\psi_k = \frac{2}{i}\sum_{k=1}^{n}\big(x_k F_{kj}(x) - i\frac{\partial F_{kj}(x)}{\partial x_k}\big)\psi + 2it\frac{\partial A_0(x)}{\partial x_j}\psi\,,$$

(20b)
$$\psi_j\big|_{t=0} = x_j\,\psi^0.$$

We now view (20) as a *system* of equations for n functions $\psi_1,\ldots,\psi_n$. Taking into account the hypotheses 1) and 2) of (**A**), we see that the Cauchy problem

$$\mathcal{L}u_j + \frac{2}{i}\sum_{k=1}^{n} F_{kj}(x)\,u_k = h_j, \quad u_j\big|_{t=0} = u_j^0, \quad j = 1,\ldots,n,$$

is well posed in the scale of Hilbert spaces $H^r_{(\rho)} \times \cdots \times H^r_{(\rho)}$ (n factors), $r \in \mathbb{R}$. Because of the hypotheses 3) of (**A**), and since $\psi \in C_{\mathrm{loc}}(\mathbb{R};\, H^s_{(\rho)})$, the right-hand sides of (20a) lie in $H^{s-\nu}_{(\rho)}$. This implies that $\psi_j \in C_{\mathrm{loc}}(\mathbb{R};\, H^{s-\nu}_{(\rho)})$, and an estimate of the form (12) holds.

From the equality $\psi_j = (x_j + 2tA_j)\psi + 2it\frac{\partial}{\partial x_j}\psi$, we arrive at estimate (15), in complete analogy with the proof of Theorem 1. This proves Theorem 1′ in the case $m = 1$. For larger m, we consider consecutively $\psi_{kj} = \Upsilon_k \psi_j$, $\psi_{\ell kj} = \Upsilon_\ell \psi_{kj}, \ldots$, use the systems of equations they obey to show that $\psi_{kj} \in C_{\rm loc}(\mathbb{R};\, H^{s-2\nu}_{(\rho)})$, $\psi_{\ell kj} \in C_{\rm loc}(\mathbb{R};\, H^{s-3\nu}_{(\rho)}), \ldots$, and then return, step by step, to ψ, gaining regularity at the expense of weight. We skip the details. $\square$

The generalisation of Corollary 2 is straightforward and we do not discuss it.

An important property of problem (18) is its invariance with respect to the gauge transformations,

$$\text{(19a)} \qquad \psi \to \psi^g = e^{-ig}\psi,\ A_0 \to A_0^g = A_0 - \frac{\partial g}{\partial t},\ \ A_j \to A_j^g = A_j - \frac{\partial g}{\partial x_j},\ j = 1, \ldots, n,$$

where g is an arbitrary function of t and x. Obviously, the relation between the corresponding fundamental solutions is as follows:

$$\text{(19b)} \qquad S_{\mathcal{A}}(t,x,y) \;=\; e^{ig(t,x)}\, S_{\mathcal{A}^g}(t,x,y)\, e^{-ig(t,y)}.$$

This can be a source of information about smoothing in the situations where the magnetic potentials $\{A_j\}$ do not initially satisfy Hypotheses (**A**), but after the gauge transformation with an appropriate gauge $g(x)$, the potentials $\{A_j^g\}$ do satisfy those hypotheses. For example, we have the following result.

Corollary 3. *Assume that the potentials $A_0, \ldots, A_n$ are smooth, A_0 satisfies condition 1) of Hypotheses (**A**), and there exists a C^∞ gauge $g(x)$ such that the potentials A_j^g satisfy conditions 2) and 3) of Hypotheses (**A**). If the initial datum ψ^0 is such that*

$$\text{(20)} \qquad \langle x\rangle^\ell\, \psi^0 \in L^2, \quad \ell = 0,\, 1, \ldots, m,$$

then the solution ψ of (18) has the following regularity properties:

$$\text{(21)} \qquad \frac{1}{M_\ell(x)}\, \partial_x^\ell\, \psi(t,x) \in L^2, \quad \ell = 0,\, 1, \ldots, m,$$

uniformly in t on every finite interval $[a,b]$, $a > 0$. The weights $M_\ell(x)$ are defined as follows

$$M_\ell(x) \;:=\; \max_{\alpha,\beta\in\mathbb{Z}^n_+,\, |\alpha+\beta|\le\ell} \quad \langle x^\alpha\, \partial_x^\beta\, g(x)\rangle\,.$$

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