LIE-POINT SYMMETRIES AND NONLINEAR DYNAMICAL SYSTEMS
(Symmetry and approximate symmetries of nonlinear equations: bifurcations, center manifolds, and normal form reduction)

Giampaolo Cicogna

Dipartimento di Fisica, Università di Pisa
Piazza Torricelli 2, 56126 Pisa, Italy

`cicogna@ipifidpt.difi.unipi.it`

Giuseppe Gaeta

Department of Mathematics, Loughborough University
Loughborough LE11 3TU, England

`g.gaeta@lut.ac.uk`

Abstract

Nonlinear symmetries of finite dimensional dynamical systems are related to nonlinear normal forms and center manifolds in the neighbourhood of a singular point. Certain abstract results can be used algorithmically to construct the normal forms and/or the center manifold up to a given order in the perturbation expansion. We also argue that for this task, approximate symmetries are as useful as exact ones.

1 Introduction

The purpose of this paper is to review some recent results [1–4] (see also [5] for a comprehensive discussion) concerning the symmetry analysis of *nonlinear dynamical systems*; more specifically, these are connected with perturbative analysis in the vicinity of a known solution. We will discuss how (nonlinear) symmetries are related to the Normal Forms [6–8] expansion in the neighbourhood of a singular point, and how these results are connected with the construction of Center Manifolds [8–10]; in the algorithmic procedures, one can also take advantages of the so called *approximate symmetries*, and actually we will argue that these are as useful as exact ones in a variety of problems.

It should be stressed that, contrary to other contributions in the present volume, we only deal with *finite dimensional* systems, i.e., with ODEs. We would like to be able to state that the results outlined here extend to evolution PDEs – and we believe this to be the case – but at the moment we are not able to give such extensions, neither to judge to what extent one could obtain generalizations of the present results. On the other side, the results presented in this paper are relevant to evolution PDEs at least in one respect, i.e., through *bifurcation theory* [10–12]; we will not discuss this here, but just refer to [13,14], where the basic results of equivariant bifurcation theory [15–19] are extended to the case of nonlinear (Lie-point) symmetries.

The paper is organized as follows: first of all we will fix notations and recall some basic definitions, notions and their use; in section 3 we will recall some general theorems obtained recently on these objects for systems with (nonlinear) Lie-point symmetries; section 4 is dedicated to recalling definitions and results on approximate symmetries. We will then shortly recall in section 5 the averaging theorems, which set a firm standpoint for the use of truncated expansions (and symmetries) in view of rigorous results. We will then be ready to discuss, in section 6, the algorithmic study and construction of symmetries (exact and approximate), of Normal Forms and of Center Manifolds. Finally, in section 7 we give some simple examples of application of the proposed procedure.

We will avoid to reproduce proofs available in the literature, and concentrate instead on one side on the meaning of the results, and on the other on their algorithmic implementation.

2 Notation and basic definitions

We deal with (smooth) dynamical systems, i.e., with systems of ODEs of the form

$$\dot{x} = f(x) \tag{1}$$

where $x \in M$, $f : M \to TM$, and M is a smooth manifold of dimension m that we think as immersed in R^n. Thus, we can also write (1) in components as

$$\dot{x}^i = f^i(x) \quad i = 1, \dots, n \qquad (2)\ ,$$

although we will use as much as possible the vector notation (1). Notice that in (1),(2) we have implicitly assumed that the system is time-autonomous; in the following we will only consider this case.

It is sometimes convenient to say that (1) defines a vector field X_f on M, whose components are the f^i's, i.e.,

$$X_f = f^i(x) \frac{\partial}{\partial x^i} \ . \tag{3}$$

The (Lie-point) symmetry algebra of (1) is defined as usual [20–22] as the algebra of vector fields on M whose first prolongation transforms solutions to (1) into – generally, different – solutions to (1). Here we are specially interested in the Lie-Point Time-Independent (LPTI) symmetry vector fields; their algebra will be denoted by $\mathcal{G}_f$. The LPTI vector fields are those of the form

$$X_\varphi = \varphi^i(x) \frac{\partial}{\partial x^i} \tag{4}$$

(see the above remark for their generality), and

$$\mathcal{G}_f = \{ X_\varphi \ : \ [X_\varphi, X_f] = 0 \ \} \tag{5}$$

where $[\cdot,\cdot]$ is the usual Lie bracket (commutator) of vector fields.

One can also introduce the Lie-Poisson bracket $\{\cdot,\cdot\}$ on functions, defined as

$$\{f,g\}^i = (f^j \cdot \nabla_j) g^i - (g^j \cdot \nabla_j) f^i \equiv X_f(g) - X_g(f) \tag{6}$$

which is obviously related to the commutator: indeed,

$$\{f,g\} = h \iff [X_f, X_g] = X_h \tag{7}$$

Thus, we also have

$$\mathcal{G}_f = \big\{ X_\varphi \ : \ \{\varphi, f\} = 0 \ \big\} \tag{8}$$

It should be stressed that in the case of first order ODEs as (1) one does often encounter *infinite dimensional* symmetry algebras. However, the set of symmetry vector fields has naturally, together with the algebra structure, the structure of a *module*. Indeed, if $\rho : M \to R$ is a constant of motion for (1), i.e., if

$$X_f(\rho) = 0 \tag{9}$$

and $X_\varphi \in \mathcal{G}_f$, it is immediate to check – from (5) or (8) – that also $X_{\rho\varphi} \in \mathcal{G}_f$. Thus, if we denote by $\mathcal{I}_f$ the algebra of constants of motion of (1), i.e., of real functions satisfying (9), we have that $\mathcal{G}_f$ is a module over the algebra $\mathcal{I}_f$ [22]. Notice that $\mathcal{G}_f$ can – and in general will - be infinite dimensional as an algebra, but finite dimensional as a module.

Let us now assume that there is a point x_0 such that $f(x_0) = 0$, and that we are interested in the behaviour of the system around x_0. This means in particular that we can as well, for the sake of local analysis, assume to have a system defined in R^m. We can then expand f around x_0 (we take $x_0 = 0$ for ease of notation, and with no loss of generality), and we write

$$\dot{x} = \sum_{k=0}^{\infty} f_k(x) \tag{10}$$

where f_k is homogeneous of degree k in x, $f_0 \equiv 0$ (because of $f(0) = 0$), and we will find it useful in the following to denote $f_1(x)$ in a special way, i.e.,

$$f_1(x) = Ax \qquad f_1^i = A_{ij} x^j \tag{11}$$

We will assume that the matrix A is semisimple, i.e., that its eigenvalues have the same algebraic and geometric multiplicities. This also implies that A is normal, i.e., that $[A, A^+] = 0$. Let us consider the (linear) eigenspaces of A spanned by eigenvalues with real part, respectively, smaller than zero, greater than zero, and zero. These are called, respectively, the stable eigenspace E^s, the unstable eigenspace E^u, and the center eigenspace E^c. Under quite general hypotheses – which we do not want to discuss here, see e.g. [9,10] for a general discussion – there also exist a local stable manifold W^s, a local unstable manifold W^u, and a local center manifold W^c, with the property that W^i is tangent to E^i in the singular point x_0. The stable and unstable manifolds are unique, while the center manifold is not; however, if an analytic center manifold exists (which is not the case in general), this is unique.

While along the stable and unstable manifolds the local dynamics is essentially trivial, corresponding to exponential contraction and expansion, the "interesting" local dynamics takes place on the center manifold. Also, in bifurcation problems [9–12] one is reduced to study the dynamics on the center manifold. Thus, for these and other reasons [8–12] the determination of the center manifolds is an important task in the study of nonlinear dynamics. Unfortunately, this is not always an easy task; we will see in the next section that W^c is necessarily invariant under $\mathcal{G}_f$, which can help in its explicit construction.

More generally, we can aim at a classification of the possible (local) behaviour of the dynamical system (1) in the vicinity of the fixed point x_0, once the linear part A is fixed. That is, we ask what behaviour is possible if we do not know all of f, but only its linear (around x_0) part $f_1(x) = Ax$. This problem is closely related to that of simplifying the (local) expression for f. In both cases, we ask to transform f into a – simpler ! – form in order to have classes of equivalence of vector fields and behaviour of their flows.

It was shown by Poincaré that an appropriate way to proceed is to transform f into *Normal Form* by means of a series of (in general, only formal) near-identity changes of coordinates; we refer to [6–8] for a discussion of Normal Form theory and its applications.

More specifically, one considers changes of coordinates of the form

$$x = y + h_m(y) \tag{12}$$

with h_m homogeneous of degree m. Under such a change of coordinates, (10) is transformed into

$$\dot{y} = \sum_k \widetilde{f}_k(y) \tag{13}$$

with $\widetilde{f}_k = f_k$ for $k < m$, $\widetilde{f}_k$ being a complicated expression in terms of the f_j for $k > m$, and $\widetilde{f}_m$ being given by

$$\widetilde{f}_m(x) = f_m(x) + \{Ax, f_m(x)\} \tag{14}$$

where $\{\cdot,\cdot\}$ is the Lie-Poisson bracket defined above. It is customary to introduce the *homological operator* L_A associated to A, defined simply as

$$L_A(\cdot) = \{Ax, \cdot\} \tag{15}$$

so that (14) reads

$$\widetilde{f}_m = f_m + L_A(f_m) \tag{16}$$

and one can eliminate terms in the range of L_A by appropriately choosing the h_m (this is done by solving the homological equation). Thus, one considers transformations of the form (12) successively for $m = 2, 3, \ldots$, arriving at the *normal form* for (1),

$$\dot{x} = g(x) = Ax + \sum_{k=2}^{\infty} g_k(x) \tag{17}$$

Notice also that one can always add to the "appropriate" h_m a term δh_m belonging to $\mathrm{Ker}(L_A)$, leading to the same $\widetilde{f}_m$; this freedom will be useful in the following.

The Normal Form is therefore characterized by the fact that necessarily

$$g_k \in \mathrm{Ker}(L_{A^+}) = \mathrm{Ker}(L_A) \tag{18}$$

where in the last equality we have used the semisimplicity of A (the theory is more complicate for the Jordan decomposition of A having a nilpotent part).

Notice that (18) implies that X_g commutes with the vector field

$$X_A \equiv X_{Ax} = A_{ij} x^j \frac{\partial}{\partial x^i} \tag{19}$$

so that the determination of the Normal Form for (1) requires the determination of $\mathcal{G}_A$, where $\mathcal{G}_A$ stands for $\mathcal{G}_{Ax}$. Notice also that for X_g to commute with X_f, g must vanish on the points on which f does. As it will be remarked in the following, $\mathcal{G}_f \subseteq \mathcal{G}_A$.

Finally, we will often use for the symmetry vector fields X_s the representation

$$X_s = s^i(x) \frac{\partial}{\partial x^i} \quad ; \quad s(x) = Bx + \sum_{k=2}^{\infty} s_k(x) \tag{20}$$

3 Some abstract theorems

In this section, we recall some abstract theorems obtained in recent works. The first group of results concerns the Normal Form unfolding, i.e., the structure of the possible Normal Forms for a given linear part Ax (see previous section). As remarked above, this corresponds to the vector fields X_g in $\mathcal{G}_A$.

If the original dynamical system (1) admits some symmetry, this property is conserved upon the (formal) changes of variables needed to pass in normal forms. In particular, let us focus on a given symmetry $X_s \in \mathcal{G}_f$; the expansion (20) for $s(x)$ will also be modified due to the changes of coordinates generated by h_m, but in the transformation of $s(x)$ the relevant homological operator will be L_B rather than L_A. Thus, one can use the freedom to add a term in $\mathrm{Ker}(L_A)$ in order to simplify $\widetilde{s}$, at least for the part in $\mathrm{Ran}(L_B) \cap \mathrm{Ker}(L_A)$. One has the following theorem, proved in [1]:

Theorem 1. *Let X_f and X_s commute, with $f(x) = Ax + \sum f_k(x)$ and $s(x) = Bx + \sum_k s_k(x)$; let A and B be semisimple. Then, by means of a series of formal coordinates transformation of the form (12), f and s can be taken into a* joint normal form *(JNF) $\widetilde{f}(x) = Ax + \sum_k g_k(x)$, $\widetilde{s} = Bx + \sum_k r_k(x)$, such that both g_k and r_k (with $k \geq 2$) are in $\mathrm{Ker}(L_A) \cap \mathrm{Ker}(L_B)$.*

This can actually be extended in two directions. First of all, one can consider not a single symmetry vector field X_s, but a whole algebra, say with generators X_{s_j} and $s_j(x) = B_j x + \sum_k r_k^{(j)}$. On the other side, one could remove the condition that A and B are semisimple. Here we consider these two extension at the same time, quoting a result given in [2]; the notation A_s denotes the semisimple part of A, and $L_j \equiv L_{A_j}$.

Theorem 2. *Let us consider a $d-$dimensional algebra $\mathcal{G}$ of VFs spanned by $X_j = f_j^i(x) \frac{\partial}{\partial x^i}$ $(j = 1, \ldots, d)$, where $f_j(x) = A_{(j)} x + \sum_k f_k^{(j)}(x) = A_{(j)} x + F_{(j)}(x)$. Then:*

i) *If the algebra $\mathcal{G}$ is* solvable, *then all the nonlinear terms $F_{(j)}$ can be put in parallel NF, namely*

$$F_{(j)} \in \mathrm{Ker}\big(L^+_{(j)}\big) \qquad \text{for each } j = 1, \ldots, d$$

ii) *If the algebra $\mathcal{G}$ is* nilpotent *(in particular: abelian), then one can put all $F_{(j)}$ into a JNF, precisely (with obvious notations):*

$$F_{(j)} \in \Big(\bigcap_{b \neq a} \mathrm{Ker}\big(L_{(j),s}\big) \Big) \cap \mathrm{Ker}\big(L^+_{(j)}\big) \qquad \text{for each } j = 1, \ldots, d$$

iii) *In any solvable (resp.: nilpotent, or in particular abelian) subalgebra of a generic algebra $\mathcal{G}$, all nonlinear terms can be put in parallel NF as in* i *(resp.: in JNF as in* ii*).*

Notice that if (1) admits a LPTI symmetry X_g with $g(x) = Bx + G(x)$, and f, g are in JNF, then X_g is also a symmetry for the linear semisimple part of X_f, i.e., for $\dot{x} = A_s x$; the converse is not true, i.e., symmetries of this linear system are not necessarily symmetries for the full system. The linear semisimple part of g, B_s provides another symmetry (if $B_s \neq 0$) for the DS, i.e., we have $\{A_s x, g\} = 0$ and $\{B_s x, f\} = 0$. It is maybe worth remarking that the theorems quoted above also have the following consequence:

Theorem 3. *If the system (1) can be linearized, then it admits m independent* commuting *symmetries, which can be simultaneously taken into* linear *form by a coordinate transformation. If, in particular, the system has a diagonalizable A with real eigenvalues, then – once it is linearized and A is diagonal – the dilations along each direction x^i, are m linear commuting symmetries for the system. Conversely, if there is a coordinate system where the system admits m independent linear commuting symmetries X_{s_j} with $s_j = B_j x$ such that all B_j are semisimple, then the system can be linearized.*

We would like to mention that the relevance of the above results lies in that – for a system with symmetry – they greatly reduce the number of terms that can appear in the Normal Forms. These results extend and generalize the results known for the case of *linear* symmetries; it turns out that the presence of nonlinear symmetries is particularly effective in the reduction of terms (correspondingly, their presence is "more exceptional").

We do now pass to recall some results [3] concerning the relation between Center Manifolds for the system (1) and the symmetry algebra $\mathcal{G}_f$. With reference to the Corollary, we recall that the analytic Center Manifold, if it exists, is unique. We also recall that by globally invariant manifold it is meant a manifold which is transformed into itself, not necessarily with each point left fixed.

Theorem 4. *All the LPTI symmetries $X_s \in \mathcal{G}_f$ leave globally invariant both the stable and unstable local manifolds W^s, W^u; they also transform any Center Manifold into a Center Manifold (possibly the same).*

Corollary. The analytic Center Manifold, if it exists, is globally invariant under all the analytic LPTI symmetries in $\mathcal{G}_f$.

Theorem 5. *Given any Center Manifold W_0^c of (1) there is, generically, some nontrivial LPTI symmetry $X_s \in \mathcal{G}_f$ leaving globally invariant this W_0^c; and conversely any LPTI symmetry of (1) leaves globally invariant some Center Manifold W_0^c.*

4 Approximate symmetries

We would now like to recall some definitions and results concerning approximate symmetries [4].

First of all, we recall that we say that X_g is an approximate symmetry of order n for the system (1) if

$$[X_f, X_g] = X_h \text{ with } h(x) = \sum_k h_k(x) \text{ where } h_k(x) \equiv 0 \text{ for } k \leq n$$

We also define in a similar way approximate constants of motion: we say that $\rho(x)$ is an approximate constant of motion of order k for the system (1) if

$$X_f(\rho) = \sum_k r_k(x) \text{ where } r_k(x) \equiv 0 \text{ for } k \leq n$$

For a given system (1), i.e., for given f, we will denote by $\mathcal{G}_f^{(n)}$ the set of approximate symmetries of order n, and similarly by $\mathcal{I}_f^{(n)}$ the set of approximate constants of motion of order n.

It is easy to check that the following lemmas hold [4]:

Lemma 1. The set $\mathcal{G}_f^{(n)}$ is a Lie algebra with the usual commutator of vector fields. The chain of Lie algebras $\mathcal{G}_f^{(k)}$ satisfies $\mathcal{G}_f^{(k+1)} \subseteq \mathcal{G}_f^{(k)}$.

Lemma 2. The set $\mathcal{I}_f^{(n)}$ is an (abelian) algebra. The chain of algebras $\mathcal{I}_f^{(k)}$ satisfies $\mathcal{I}_f^{(k+1)} \subseteq \mathcal{I}_f^{(k)}$.

Lemma 3. The set $\mathcal{G}_f^{(n)}$ has, beyond the structure of Lie algebra, the structure of a module over $I_f^{(n)}$.

The theorems given above can then be generalized to consider approximate symmetries; in particular we have [4] as generalization of the above theorems 1 and 3:

Theorem 6. *Let $X_s \in \mathcal{G}_f^{(n)}$, with $f(x) = Ax + \sum f_k(x)$ and $s(x) = Bx + \sum_k s_k(x)$; let A and B be semisimple. Then, by means of a series of formal coordinates transformation of the form (12), f and s can be taken into a* joint normal form *(JNF) up to order n; that is, they can be transformed into $\widetilde{f}(x) = Ax + \sum_k g_k(x)$, $\widetilde{s} = Bx + \sum_k r_k(x)$, such that both g_k and r_k are in* $\mathrm{Ker}(L_A) \cap \mathrm{Ker}(L_B)$ *for $k \leq n$.*

Theorem 7. *If the system (1) can be linearized up to order n, then it admits*

m independent commuting *approximate symmetries of order n, which can be simultaneously taken into* linear *form by a coordinate transformation. Conversely, if there is a coordinate system where the system admits m independent linear commuting approximate symmetries of order n, X_{s_j} with $s_j = B_j x$ such that all B_j are semisimple, then the system can be linearized up to order n.*

Notice that the fact that the system can be linearized up to order n means that in the Normal Form expansion the nonlinear terms are all of order higher than n.

We could also have generalizations of results for the non-semisimple case; these go along the same lines and are not reported here for the sake of simplicity. Also, in section 6 we will limit the algorithmic implementation of the results reported here to the semisimple case for the sake of simplicity, but it will be quite clear how to generalize these – at the price of having rather more complicate formulas and procedures to implement them, but no further conceptual difficulty – to the non-semisimple case.

5 Truncated equations and approximate solutions

The Normal Form method provides, in principle, a way to completely classify the possible local behaviour of a dynamical system around a fixed point with given linear behaviour; it also provides a constructive way to reduce the system to a simpler one, $\mathcal{C}^\infty$ equivalent to the original one. On the other side, it has two major drawbacks. First, it is only a formal method, in that all the series involved are in general only formal, and could fail to converge[1]; in general, we only have an *asymptotic*

series: that is, in practice, the normalising procedure should be carried over only up to some optimal order N and not to all orders. The second obvious drawback is that in any case, i.e., even if we are ensured of the convergence of all the relevant series, to obtain the complete Normal Form we should operate an *infinite* series of transformations.

Thus, for reasons of principle or just for practical ones, in actual computations we are obliged to perform the Normal Form expansion (or transformation) only up to some fixed order. In doing this, we will in general have an infinite series, and again in actual computations we will have to consider a truncation of such a series[2]. We have then to extract information on the solutions to the full system

[1] It should be remarked, however, that the symmetry properties of the system can be sufficient to ensure, in some cases, the convergence of the normalising transformation, see [23–25]

[2] Obviously, if we have some exact symmetry and/or conservation laws, we can take full

by the solutions to the truncated system. This is the fundamental problem of perturbation theory, and this is obviously not the place to discuss it (not to say that such a discussion would go beyond the expertise of the present authors); here we do only want to recall some results which are at the basis of such a theory, referring the reader e.g. to [6–8, 26,27] for a full discussion of averaging and related theories (such as averaging at all orders, or Nekhoroshev theory, and K.A.M. theory). The discussion and the theorems given below are quoted from [8]. The connection with the case at hand here is immediately obtained by the rescaling $x = \varepsilon\xi$.

We consider an ODE

$$\dot{x} = \varepsilon f(x,t) + \varepsilon^2 g(x,t;\varepsilon) \tag{21}$$

with initial datum $x(0) = x_0$; we also assume that f is periodic in t with period T and denote the average (in t) by

$$f^0(y) = \frac{1}{T}\int_0^T f(y,t)dt \tag{22}$$

This defines the averaged equation

$$\dot{y} = \varepsilon f^0(y) \tag{23}$$

We denote by $\Phi(t;x_0)$ the flow of (21) with initial datum x_0, and by $\Phi^0(t;y_0)$ the flow of the averaged equation (23) with initial datum $y(0) = y_0$.

Theorem 8. *Let f, g and $\partial f/\partial x$ be defined, $\mathcal{C}^0$ and uniformly bounded on $(x,t) \in D \times [0,\infty)$ for D a domain in R^n; let g be Lipschitz continuous in x for $x \in D$. Let $x_0 \in D$, and let $y(t) = \Phi^0(t;x_0) \in D_0$ (D_0 an open neighbourhood contained in D) for $t \le (1/\varepsilon)$. Then, $|\Phi(t;x_0) - \Phi^0(t;x_0)| = O(\varepsilon)$ for $t \le O(1/\varepsilon)$.*

Actually, this theorem also holds in the case the average of f exists only in a generalized sense, i.e., as

$$f^0(y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(y,t)dt \tag{24}$$

and the averaged equation is again (23), with f^0 given by (24), and Φ^0 corresponds to its solution.

Theorem 9. *Let the same hypotheses as in theorem 8 be verified, with f^0 given by (24). Then, $|\Phi(t;x_0) - \Phi^0(t;x_0)| = O(\delta(\varepsilon))$ for $t \le O(1/\varepsilon)$, with $\delta(\varepsilon) = \sup_{x\in D}\sup_{0\le t\le C/\varepsilon} \varepsilon \cdot |\left[\int_0^t |f(x,s) - f^0(x)|ds\right]|$.*

---

advantage of them; e.g., we can make a preliminary change of variables so that some of the new ones – corresponding to "symmetry adapted" ones – are invariant under the time evolution.

We remark that the two above theorems deal with the case the "unperturbed" system – i.e., the one with $\varepsilon = 0$ is just the trivial one, $\dot{x} = 0$. This is good enough for the case we are considering, as via the above mentioned scaling the case of a stationary point is indeed mapped into this class of equations; however, it can be worth recalling what happens when we perturb a one-frequency periodic motion (the multi-frequency case is considerably more complicate, as resonances and small denominators must be taken into account [6,27]).

Thus, we consider – instead of (21) – the equation

$$\begin{aligned} \dot{x} &= \varepsilon f(x,\theta) + \varepsilon^2 g(x,\theta;\varepsilon) \\ \dot{\theta} &= \Omega(x) + \varepsilon \nu(x,\theta;\varepsilon) \end{aligned} \tag{25}$$

where $x \in D \subseteq R^n$ and $\theta \in S^1$. We consider now

$$f^0(x) = \int_0^{2\pi} f(x,\theta) d\theta \tag{26}$$

and Φ, Φ^0 represent the solutions, respectively, of (25) and of the averaged and truncated equation

$$\begin{aligned} \dot{x} &= \varepsilon f^0(x) \\ \dot{\theta} &= \Omega(x) \end{aligned} \tag{27}$$

Theorem 10. *Let f, g, ω, ν be $\mathcal{C}^1$ in $D \times S^1$; let $\Phi^0(t;x_0) \in D_0$ for $t \in [0, 1/\varepsilon)$, where D_0 is an open set contained in the interior of D. Then, $|\Phi(t;x_0) - \Phi^0(t;x_0)| = O(\varepsilon)$ for $t \le O(1/\varepsilon)$.*

The above theorems – and their generalizations [27] – give a rigorous foundation to the consideration of truncated Normal Form expansion, as they state that, if we are interested only in solutions over a long but finite time $t \le 1/\varepsilon^k$ and known with an error $O(\varepsilon)$, we can consider the normal form truncated at order k.

6 Algorithmic implementation

We want now to consider the algorithmic solution to three related problems, considered above: 1) transforming f into its Normal Form; 2) determining its symmetry algebra $\mathcal{G}_f$; 3) determining the invariant manifolds for the evolution under f.

We want to proceed in a perturbative way (but with analytical, not numerical, procedures), and we will work order by order in the expansions in powers of $(x - x_0)$.

If we use the expansions (10) for f and (17) for g, the commutation condition $[X_f, X_g] = 0$ reads

$$\sum_{j=0}^{k} \{f_j, g_{k-j}\} = 0 \quad \forall k \tag{28}$$

so that in the case of an approximate symmetry of order n the equation (28) does only hold for $k \leq n$. Let us look more closely to the terms in the sum of (28); as we know, $f_0 = g_0 = 0$, so that the terms with $j = 0$ and $j = k$ do not really matter. Isolating the terms with $f_1 = Ax$ and with $g_1 = Bx$, we get

$$[A, B] = 0 \tag{29}$$

in the case $k = 2$, and in general, for $k > 2$,

$$\{Ax, g_{k-1}\} - \{Bx, f_{k-1}\} = c_{k-2}(f, g) = -\sum_{j=2}^{k-2} \{f_j, g_{k-j}\} \tag{30}$$

Notice that if f_j and g_j are in $\mathrm{Ker}(L_A) \cap \mathrm{Ker}(L_B)$ for all $j \leq k-2$, then we are guaranteed that $c_{k-2}(f, g)$ is in $\mathrm{Ker}(L_A) \cap \mathrm{Ker}(L_B)$ as well.

Let us look at the solution to our second problem, i.e., the determination of $\mathcal{G}_f$ for given f. Clearly, all we have to do is to solve (28) order by order; this means, first determine the B satisfying (29), then the g_2 satisfying (30) for $g_1 = Bx$, and so on. Thus, we can solve our problem recursively, just by solving at each order the equation

$$L_A(g_{k-1}) = L_B(f_{k-1}) + c_{k-2}(f, g) \tag{31}$$

The solutions to this equation are obviously determined up to a term in $\mathrm{Ker}(L_A)$; on the other side the solution exists if and only if

$$L_B(f_{k-1}) + c_{k-2}(f, g) \in \mathrm{Ran}(L_A) \tag{32}$$

and we are by no means guaranteed that this condition is satisfied.

Thus, what we actually do is to determine $\mathcal{G}_f^{(1)}$; then, we determine $\mathcal{G}_f^{(2)}$ using the structure of $\mathcal{G}_f^{(1)}$, and so on. At any step, we find that in general not all the $X_g \in \mathcal{G}_f^{(k)}$ give raise to a $X_g \in \mathcal{G}_f^{(k+1)}$, the condition for this to happen being precisely (32). Some examples of explicit analytic computations (some of these requiring the use of algebraic manipulation programs) are given in [4].

We want to point out that it is actually convenient to solve problem 1) at the same time as we try to determine the symmetry algebra. We have seen above that a vector field is in Normal Form if and only if it commutes with its linear part (under the assumption this is semisimple); similarly, a set of vector fields (having semisimple linear parts) is in Joint Normal Form if and only if each one is commuting with the linear parts of all the other ones as well as its own. Similar statements hold for vector fields with non-semisimple linear part, but here we keep to semisimple linear part for ease of notation and discussion. We say that X_f with $f(x) = Ax + \sum_k f_k(x)$ is in Normal Form up to order n if and only if $\{Ax, f_k(x)\} = 0$ for all $k \le n$, and similarly for Joint Normal Forms.

Let us look again at our recursive procedure for determining (approximate) symmetries, taking into account the Normal Form construction as well. At the first step, we only have to solve (28) for B. With such a B, we have by definition that both $f_1 = Ax$ and $g_1 = Bx$ are in $\mathcal{K} \equiv \mathrm{Ker}(L_A) \cap \mathrm{Ker}(L_B)$.

We would like to recall that $[A, B] = 0$ implies (under our blanket hypothesis that A, B are semisimple) that $[L_A, L_B] = 0$; also, the semisimplicity of A ensures that the set P_k of homogeneous polynomials (in $x \in R^m$) of degree k has an invariant splitting as

$$P_k = [\mathrm{Ker}(L_A) \cap P_k] \oplus [\mathrm{Ran}(L_A) \cap P_k] \tag{33}$$

(see e.g. [1] for a more detailed discussion); i.e., $L_A : \mathrm{Ran}(L_A) \to \mathrm{Ran}(L_A)$. Obviously, the same considerations apply to L_B.

At the second step, we have to solve

$$L_A(g_2) = L_B(f_2) \tag{34}$$

as an equation for g_2 (at this stage $c_0(f, g) = 0$). However, we can first of all operate a change of variables of the form (12), so that f is brought to Normal Form at order 2, so that $f_2 \in \mathrm{Ker}(L_A)$. If a solution g_2 to (34) exists, the previous theorem 1 also ensures that we can, by properly choosing the h_2 which generates the transformation (12), actually also ensure $g_2 \in \mathrm{Ker}(L_B)$; now, applying L_A or L_B on (34) we see that necessarily also $f_2 \in \mathrm{Ker}(L_B)$, $G_2 \in \mathrm{Ker}(L_A)$.

Thus, if we build Joint Normal Forms up to order two, we have also built approximate symmetries of order two, and vice versa.

At higher orders, the $c_{k-2}(f, g)$ terms come into play. However, as remarked above, $\{\cdot, \cdot\} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$; proceeding as above, we have that if we can find a solution for g_k, it has to be in $\mathcal{K}$.

Thus, we can solve – algorithmically ! – the two problems at the same time (we suppose the fixed point x_0 has already been mapped into the origin of R^m):

1) Fix a X_f, i.e. $f(x) = \sum_{k=1}^{\infty} f_k(x)$; 2) Determine $A = (Df)(0)$; check that it is semisimple; 3) Determine the set $M(A)$ of matrices commuting with A, i.e., $[A, B] = 0$; 4) Choose a $B \in M(A)$, and correspondingly $g_1(x) = Bx$; 5) Solve the homological equation for h_k, with $k = 2$; 6) Implement the change of coordinates (12), using the h_k determined above; call again f_2 the

new expression for the second order term in the transformed variables; 7) Solve the equation (30) for g_k, with the g_ℓ chosen before for $\ell < k$; 8) Solve the homological equation – with L_B in the place of L_A and g in that of f, δh_k in that of h_k – requiring that $\delta h_k \in \mathrm{Ker}(L_A)$ (in this way, we get automatically that f_k and g_k are in $\mathcal{K}$); call $S_k(f,g)$ the set of such solutions; 9) Choose a $g_k \in G_k(f,g)$; 10) Repeat steps 5)–9) for $k = 3,4,\ldots,n$ up to the required order n.

Notice that step 7) could not be possible, i.e., at any level k the procedure could be forced to stop.

In this way, we build recursively the (tree of the) approximate symmetries of X_f up to order n.

A slightly different procedure is also possible:

1) Determine $A = (Df)(0)$ and correspondingly $f_1(x) = Ax$; check that A is semisimple; 2) Determine the set $M(A)$ of matrices commuting with A, i.e., $[A,B] = 0$; 3) Choose a semisimple $B \in M(A)$, and correspondingly $g_1(x) = Bx$; 4) Determine $\mathcal{K}_k \equiv \mathrm{Ker}(L_A) \cap \mathrm{Ker}(L_B) \cap P_k$ for all $k \le n$; 5) Choose f_k and g_k in $\mathcal{K}_k$ such that $f^{(k)} = \sum_{m=1}^k f_m$ and $g^{(k)} = \sum_{m=1}^k g_m$ satisfy eq.(30) up to order k, for $k = 2$; 6) Repeat step 5) up to $k = n$;

Notice that in step 4) we are actually building the Normal Form expansion up to order n, while steps 5)–6) give the construction of approximate symmetries. Once again, we stress that it could be impossible to construct approximate symmetries up to order n, i.e., – depending on the choices made for f_k and g_k – the procedure could stop at $k = \ell < n$.

Let us now focus on the problem of determining invariant manifolds; if we identify these manifolds with common (zero) level curves of some set of functions, say $w_1(x) = w_2(x) = \ldots = w_r(x) = 0$, we are led to look for a set $\{w_1,\ldots,w_r\}$ of functions $w_i : R^m \to R$ such that

$$X_f(w_i) = K_{i,j} w_j(x) \tag{35}$$

with some suitable matrix K.

Obviously, a special and simple case of this condition is the one where actually

$$X_f(w_i) = 0. \tag{36}$$

For the stable, unstable and center manifold, respectively, we require that they pass through x_0, and that in there they are tangent to the appropriate linear subspaces; thus we can choose the w_i to be equal, at first order, to an appropriate set of eigenvectors for A. We can then build the w_i order by order; if we write

$$w_i = C_{ij}x^j + \sum_{k=2}^{\infty} w_k^{(i)}(x) \tag{37}$$

we can then proceed to solve (35) – or (36) – order by order.

We do not want to discuss here the actual construction of invariant manifolds in full generality, and only consider the special case (36), i.e., nothing else than the perturbative construction of constants of the motion; in this case, we can actually consider a single function $\rho : R^m \to R$, and write

$$\rho(x) = \sum_{k=1}^{\infty} \rho_k(x) \tag{38}$$

By consider this series expansion and the one for f, the condition $X_f(\rho) = 0$ reads as in (36). We can then proceed order by order; again, the task is simplified if at each order k we first put f in Normal Form up to order k, and then solve

$$\sum_{j=1}^{k} (f_j \cdot \nabla)\rho_{k-j+1} = 0 \tag{39}$$

(where $\rho_1, \ldots, \rho_{k-1}$ have been previously determined) for ρ_k. Indeed, by having f in Normal Form up to order k we know automatically [5] that

$$I_f^{(k)} \subseteq I_A^{(k)} \tag{40}$$

so that we have to consider only a relatively small space of functions[3].

It should be mentioned that in concrete computations one would use an explicit basis for P_k, made of homogeneous monomials of order k; the use of such a basis transforms the PDEs to determine $P_k \cap \mathrm{Ker}(L_A)$, h_k, and so on, in *algebraic* equations. Clearly, the dimension of P_k increases rapidly with k, so that strong limitations exist to the viability of the procedure for higher order computations. Notice also that at several points we have some freedom left, e.g. in the determination of h_k (there could be free parameters), and any choice produces a different branch of the algorithm.

[3] In some cases, the presence of a symmetry can actually simplify this problem further, see [5].

7 An explicit example

In order to illustrate our discussion by following our procedure (up to order three) in an elementary example[4], consider the system

$$\begin{cases} \dot{x} = ax - y + 2xy + (x^2+y^2)y + x^3 \\ \dot{y} = x + ay + 2y^2 - (x^2+y^2)x + y^3 \end{cases} \tag{41}$$

for which obviously

$$A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \tag{42}$$

and $M(A)$ corresponds to matrices of the form (b, c real constants)

$$B = \begin{pmatrix} b & c \\ -c & b \end{pmatrix} . \tag{43}$$

It is easy to check explicitly that (for any value of a) $P_2 \cap \mathrm{Ker}(L_A) = \{0\}$, so that we can solve (uniquely !) the homological equation at order two,

$$L_A(h_2) = -f_2 = \begin{pmatrix} 2xy \\ 2y^2 \end{pmatrix} . \tag{44}$$

The explicit form of the solution is

$$\begin{cases} h_2^1 = \dfrac{2}{1+a^2}x^2 - \dfrac{2a}{1+a^2}xy \\ h_2^2 = \dfrac{2}{1+a^2}xy - \dfrac{2a}{1+a^2}y^2 \end{cases} \tag{45}$$

In the new coordinates, corresponding to the transformation generated by h_2, we get $f_2 = 0$, and the symmetry equation (30) at order two is simply

$$\{Ax, g_2\} + \{f_2, Bx\} \equiv \{Ax, g_2\} \equiv L_A(g_2) = 0 \tag{46}$$

which yields $g_2 = 0$.

In these coordinates, we expect that the term f_3 assumes a more complicate expression. However, notice that $P_3 \cap \mathrm{Ker}(L_A)$ is empty for $a \neq 0$ (so that we get $g_3 = 0$ as well), while for the more interesting case $a = 0$ it is spanned by

$$\alpha = \begin{pmatrix} (x^2+y^2)x \\ (x^2+y^2)y \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} -(x^2+y^2)y \\ (x^2+y^2)x \end{pmatrix} , \tag{47}$$

i.e., by terms of the form $(x^2+y^2)Bx$ (this corresponds to a general result, see [5]).

[4] Other explicit examples are given in [1–5].

In the present case, when we consider the new coordinates, solve the homological equation at order three, and pass again to new coordinates corresponding to the h_3, we get

$$f_3 = (x^2 + y^2) \begin{pmatrix} y \\ -x \end{pmatrix} \tag{48}$$

This can also be seen without explicit computations: both f_2 and h_2 are in $\mathrm{Ran}(L_A)$, and so is therefore $\{h_2, f_2\}$, which is the term added to f_3 as a result of the change of coordinates corresponding to h_2.

The symmetry equation at order three is therefore

$$\{Ax, g_3\} + \{f_2, g_2\} + \{f_3, Bx\} = L_A(g_3) - L_B(f_3) = 0 \tag{49}$$

If in B, see (43), we choose $b \neq 0$, this has no solutions (i.e., the corresponding approximate symmetry is of order two only); for $b = 0$ we get $L_B(f_3) = 0$, and g_3 can be chosen as any linear combination of the α, β in (47). Notice that if $g_3 = -b\beta$, we get $X_\varphi = -bX_f$ (up to order three).

Pursuing the computations to higher orders, we would be going through the same situation for even and odd orders, respectively.

References

[1] G. Cicogna and G. Gaeta: "Poincaré normal forms and Lie point symmetries", *J. Phys. A (Math. Gen.)* **27** (1994), 461.

[2] G. Cicogna and G. Gaeta, "Normal forms and nonlinear symmetries", *J. Phys. A (Math. Gen.)* **27** (1994), 7115.

[3] G. Cicogna and G. Gaeta, "Symmetry invariance and center manifolds for dynamical systems", *Nuovo Cimento B* **109** (1994), 59.

[4] G. Cicogna and G. Gaeta, "Approximate symmetries in dynamical systems", *Nuovo Cimento B* **109** (1994), 989.

[5] G. Cicogna and G. Gaeta, "On symmetry and normal form theory", Preprint 1995.

[6] V.I. Arnold, *Geometrical methods in the theory of differential equations*; Springer, Berlin, 1988.

[7] V.I. Arnold and Yu.S. Il'yashenko, "Ordinary differential equations"; in *Dynamical Systems - I*, D.V. Anosov and V.I. Arnold eds., E.M.S., Springer, Berlin, 1988.

[8] F. Verhulst, *Nonlinear differential equations and dynamical systems*, Springer, Berlin 1990.

[9] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer, New York, 1983.

[10] D. Ruelle, *Elements of differentiable dynamics and bifurcation theory*, Academic Press, London.

[11] S.N. Chow and J. Hale, *Methods of bifurcation theory*, Springer, New York, 1982.

[12] J.D. Crawford, "Introduction to bifurcation theory", *Rev. Mod. Phys.* **63** (1991), 991.

[13] G. Cicogna and G. Gaeta, "Lie-point symmetries in bifurcation problems", *Ann. Inst. H. Poincaré* **56** (1992), 375.

[14] G. Cicogna and G. Gaeta, "Nonlinear symmetries in bifurcation theory", *Phys. Lett. A* **172** (1993), 361.

[15] D.H. Sattinger, *Group Theoretic Methods in Bifurcation Theory*, LNM 762, Springer, Berlin, 1979; *Branching in the Presence of Symmetry*, S.I.A.M., Philadelphia, 1984.

[16] L. Michel, "Symmetry defects and broken symmetry. Configurations. Hidden symmetry", *Rev. Mod. Phys.* **52** (1980), 617.

[17] M. Golubitsky, D. Schaeffer and I. Stewart, *Singularities and groups in bifurcation theory - vol. II*, Springer, Berlin, 1988.

[18] G. Gaeta, "Bifurcation and symmetry breaking", *Phys. Rep.* **189** (1990), 1.

[19] J.D. Crawford and E. Knobloch, "Symmetry and symmetry-breaking bifurcations in fluid mechanics", *Ann. Rev. Fluid Mech.* **23** (1991), 341.

[20] P.J. Olver, *Applications of Lie groups to differential equations*, Springer, Berlin, 1986.

[21] G.W. Bluman, S. Kumei, *Symmetries and differential equations*, Springer, Berlin, 1989.

[22] G. Gaeta, *Nonlinear symmetries and nonlinear equations*, Kluwer, Dordrecht, 1994.

[23] A.D. Bruno, *Local methods in nonlinear differential equations*, Springer, Berlin, 1989.

[24] A.D. Bruno and S. Walcher, *J. Math. Anal. Appl.* **183** (1994), 571.

[25] G. Cicogna, "Symmetries in dynamical systems and convergent normal forms", *J. Phys. A* **28** (1995), L179.

[26] J. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Springer, Berlin, 1985.

[27] V.I. Arnold, ed., *Dynamical Systems - III: Classical Mechanics*, E.M.S., Springer, Berlin, 1993.