Determining discrete symmetries of differential equations

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Summary We propose a method to determine a special class of discrete symmetries - which we call quantized - of differential equations, based on solution of a linear functional equation. By the same method, we can determine differential equations possessing a given quantized symmetry by solving a linear (standard, i.e. non functional) PDE. Geometrical motivation and interpretation are also given, together with simple but physically significant examples.

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1. Introduction

It is well known [1-6] that knowledge of continuous symmetries of differential equations can be of great use in finding particular or general solutions of the equations, or in simplifying them.

In the determination of *continuous* symmetries of DEs, we proceed by identifying the DE Δ with a manifold S_Δ in the appropriate *jet space* $\mathcal{J}$; we then look for a vector field η such that its prolongation η_* is tangent to S_Δ. In this way we are reduced to considerations on the *tangent space* to S_Δ, i.e. to a *linear* problem. Indeed, the condition $\eta_* : S_\Delta \to TS_\Delta$ gives a system of linear PDEs, the *determining equations* [1-6].

Knowledge of *discrete* symmetries would also be of great use in the study of DEs; unfortunately, in the determination of general discrete symmetries we cannot reduce to study the infinitesimal action of vector fields, and we end up in general with a nonlinear problem.

In the present note, we point out that for some class of discrete symmetries, which we will call *quantized* or *stroboscopic* for reasons to be clear in the following, we can still reduce to a linear problem, although considerably more difficult than the one to be solved for continuous symmetries. The method we propose has a clear geometric interpretation, and indeed we will present it geometrically, starting from the identification of Δ with the solution manifold S_Δ.

The class of discrete transformations we will study is that of transformations obtained by finite action of vector fields; i.e., if η is a vector field, we will consider transformations of the form $T_\lambda = e^{\lambda\eta}$, where $\lambda \in R$ is a *finite* parameter; we are interested in the case η is *not* a symmetry of Δ, but there exist values λ_0 such that $T_{\lambda_0} : S_\Delta \to S_\Delta$ (and $T_{\lambda_0} \neq I$). An example of this would be an equation which is invariant under a lattice of discrete translation, but not under continuous translations, see below.

It should be anticipated that the determining equation we obtain by the method proposed here is a *functional* equation, so that in general we cannot hope to find the most general solution, i.e. the complete set of discrete symmetries to a given DE. It should also be remarked that the problem is unavoidably underdetermined: indeed, two different vector fields η and η' such that $e^{\lambda_0\eta} = e^{\lambda_0\eta'}$ give raise to the same discrete transformation; we will further comment on this in the following.

It should also be mentioned that the problem of finding continuous symmetries for discrete equations, which is in some sense dual to the present one (see below), has been considered by several authors (see e.g. [7-12] and references therein); the method proposed here is however not related to methods used in such a problem.

In the following, we will proceed as follows. We will first consider the geometric problem of quantized transformations leaving invariant a manifold in R^{n+1}, and we will write explicitely the determining equations for these.

We will then pause to shortly discuss the geometrical interpretation of our method from a rather abstract point of view; this could be helpful in establishing relations with differential geometric properties of the manifold considered. Indeed, in this language we have to determine connections for a certain fiber bundle.

Coming back to our main idea, we will then specialize to the case of manifold in a jet space corresponding to a DE: in this case some extra structure is present, and this results in some simplification of the determining equation. Even with this, we are not able to give the complete solution to the determining equations, but we will show that by restricting the form of the vector field to be quantized we can give explicit solutions. We will then consider the inverse problem: that is, given a quantized symmetry, determine the most general DE (for assigned order and number of variables) invariant under that symmetry.

Finally, we will consider in detail some explicit examples, focusing in particular on simple discrete symmetries of wide use, such as translations, rotations and scale transformations.

We will suppose the reader is familiar with the problem and language of determining continuous symmetries of differential equations, and in general with the symmetry theory of differential equations [1-6].

In this note we will just present the main ideas underlying the approach we propose, and give some simple examples; a more thorough discussion will be presented elsewhere [13].

2. Quantized symmetries for manifolds

Let us consider $X = R^n$, $Y = R^1$. Let us consider a manifold Γ in $M = X \times Y = R^{n+1}$ defined by the equation

$$y = f(x) \tag{1}$$

with $f : X \to Y$ a smooth function.

Let us further consider a vector field in M,

$$\eta = \varphi^i(x,y)\partial_i + \psi(x,y)\partial_y \tag{2}$$

Here and in the following

$$\partial_i \equiv \frac{\partial}{\partial x^i} \tag{3}$$

and summation over repeated indices is understood.

The infinitesimal action $e^{\varepsilon\eta}$ of η in M maps (x,y) to a new point (x',y'), where

$$x' = x + \varepsilon\varphi(x,y) \qquad ; \qquad y' = y + \varepsilon\psi(x,y) \tag{4}$$

so that the graph of f, $\Gamma = \{(x,f(x))\}$ is tranformed into a new curve $\Gamma_\varepsilon = \{(x,f_\varepsilon(x))\}$ which is the graph of the function

$$f_\varepsilon(x) = f(x) + \varepsilon[\psi(x,y) - (\varphi^i(x,y)\partial_i)f(x)] \tag{5}$$

Let us now introduce the function $F : R \times X \to R$ such that $F(\lambda,x) = f_\lambda(x)$ is the transform of $f(x)$ under $e^{\lambda\eta}$. Clearly, the infinitesimal transformation (5) yields that this F satisfies the *determining equation*

$$\frac{\partial F(\lambda,x)}{\partial\lambda} + \varphi^i(x,F(\lambda,x))\frac{\partial F(\lambda,x)}{\partial x^i} = \psi(x,F(\lambda,x)) \tag{6}$$

with initial condition

$$F(0,x) = f(x) \tag{7}$$

If η is a continuous symmetry of f, then Γ must be invariant under η, i.e. $f_\varepsilon(x) = f(x)$, or equivalently $(\partial F/\partial\lambda) = 0$. Indeed in this case (6) gives just the usual determining equations for symmetries of (1).

As mentioned in the introduction, we are interested in the case η is not a symmetry of (1), but there is a special value λ_0 such that $f_{\lambda_0} = f$, i.e. such that

$$F(\lambda_0,x) = F(0,x) \tag{8}$$

In other words, we are interested in determining η such that the solution to (6) with the initial condition (7) is *periodic* in λ (excluding the trivial case $\partial F/\partial\lambda = 0$). In this case, $\Lambda = e^{\lambda\eta} : M \to M$ maps Γ into itself and therefore qualifies as a discrete symmetry of (1), or equivalently of Γ.

It is now clear why we call such a symmetry, given by finite action of a vector field which is not a continuous symmetry of (1) itself, a quantized or stroboscopic symmetry, as anticipated in the Introduction. It should be stressed that we should also require that $\Lambda|_\Gamma$ is not the identity, or we would have a trivial discrete symmetry.

Notice that, multiplying η by a numerical constant, we can always set $\lambda_0 = 1$, or $\lambda_0 = 2\pi$. Thus, in the following we will in general consider λ_0 as fixed. The (2π) periodicity requirement in λ would suggest to expand $F(\lambda, x)$ as a Fourier series in λ. This is indeed possible, but is concretely useful only if (6) is linear, as it will be discussed in detail in [13].

We stress that in (6) we have to determine not only F, but φ and ψ as well. Thus, we are not dealing with a normal PDE, but with a functional PDE. The only data are the initial condition (7) and the (arbitrary) $\lambda_0 \neq 0$ appearing in (8). We could - and will - consider the inverse problem of determining the manifolds Γ which are invariant under the action of a given prescribed symmetry. In this case φ, ψ are given, and (6) is a regular PDE, not a functional one. It is not surprising that this inverse problem is much easier than the direct one, as we will see in the following.

Although we have considered quite special X, Y, M and Γ for ease of notation, it is simple to generalize the above discussion to the case X and Y are smooth submanifolds in real spaces; for our purposes we will not need to consider the case Γ is a generic smooth submanifold in $M = X \times Y$, and only need to consider Γ as the graph of a function f in an appropriate space.

It should be stressed that in introducing the function $F(\lambda, x)$ we are implicitely also passing to consider a vector field η' associated to η and acting in $M' = R \times M$, given explicitely by

$$\eta' = \eta + \partial_\lambda \equiv \varphi^i \partial_i + \psi \partial_y + \partial_\lambda \tag{9}$$

Then the graph $\Theta = (x, y, \lambda) : y = F(\lambda, x) \subset M'$ of $F(\lambda, x)$ is by construction an invariant submanifold of M' under η'. We are therefore reconducted to the problem of determining tangent vector field to a manifold, but now the manifold is not given a priori, but depends itself on the vector field.

3. A geometrical interpretation

We would like to stress that M' can be naturally seen as the total space of a fiber bundle B with base R (corresponding to the λ), fiber M and projection $\pi : (\lambda, x, y) \to \lambda$; η' is then a connection on B. When we are looking for solutions to (6), (7) which moreover satisfy the periodicity condition (8) - i.e. we are looking for quantized symmetries - we can consider the analogous bundle $\mathcal{B}$ with S^1 as base space, M as fiber and the same projection π.

Our problem can then be described in differential geometric language as the search for a connection ∇ on $\mathcal{B}$ such that there is a section Θ invariant under this connection and such that the restriction of Θ to $\pi^{-1}(0)$ is the prescribed Γ.

In the above language, the determination of manifolds invariant under a given quantized symmetry (the "inverse" problem, to be discussed below) amounts to the determination of sections of $\mathcal{B}$ invariant under a given connection ∇. Again, it is obvious that this is much easier than the "direct" problem, although in general it is not trivial at all.

It should be stressed that although Θ is an invariant manifold under the connection ∇, this does not imply that transporting a point $(x, f(x))$ around the base space S^1 by ∇ we get the same point. In general, we get a point $(x', f(x'))$ with $x' \neq x$, and the discrete transformation $\Lambda : \Gamma \to \Gamma$ is related to the holonomy of the connection ∇.

4. Quantized symmetries of differential equations

As already recalled, a DE Δ is naturally identified with a manifold S_Δ in an appropriate jet space $\mathcal{J}$ [1,5]. Therefore, if Γ is S_Δ and M is $\mathcal{J}$, the theory developed in the previous section can be applied to differential equations as well.

However, the fact that we are dealing with jet spaces makes that an extra structure (the contact structure) is now present. Due to this, we have some extra constraint on the functions appearing in (6): e.g., η should now be the prolongation to $\mathcal{J}$ of an underlying Lie-point vector field η_0 acting in the space M_0 of independent and dependent variables; due to this the φ^i and ψ are not arbitrary smooth functions, but must satisfy some relations, as we are now going to discuss.

Let us first consider the case of an autonomous ODE

$$u_t = f(u) \tag{10}$$

and time-independent Lie-point vector field

$$\eta_0 = \varphi(u)\partial_u \tag{11}$$

Now $M_0 = R^2 = \{(t,u)\}$, and (10) is identified with a manifold in $\mathcal{J} = M = R^3 = \{t, u, u_t\}$. The prolongation of $\eta_0 : M_0 \to TM_0$ to M is given by

$$\eta = \varphi(u)\partial_u + \Phi(u, u_t)\partial_{u_t} \tag{12'}$$

where $\Phi = \varphi_u u_t$ by the prolongation formula [1-6], so that on Γ we have

$$\Phi = \varphi_u f(u) \tag{12''}$$

Now u_t plays the role that was of y in section 2, and Φ corresponds to Ψ; we get (6) in the form

$$\frac{\partial F(\lambda,u)}{\partial\lambda} + \varphi(u)\frac{\partial F(\lambda,u)}{\partial u} = \Psi = \varphi_u F(\lambda,u) \tag{13}$$

(we have taken into account t-independence). Thus, Ψ is now determined by φ, and we have only one arbitrary function in η. Notice also that while the φ in the case of section 2 could depend on y, now the request that η_0 be a Lie-point vector field ensures that φ does not depend on u_t, and we end up with a determining equation linear in F.

Thus, dealing with differential equations rather than algebraic manifolds gives indeed a somewhat simpler problem !

It is worth considering also the case of nonautonomous ODEs and time dependent vecor fields, i.e.

$$u_t = f(t,u) \tag{14}$$

$$\eta_0 = \tau(t,u)\partial_t + \varphi(t,u)\partial_u \tag{15}$$

In this case η is still given by (12'), but the prolongation formula yields $\Phi = \varphi_u u_t - \tau_t u_t - \tau_u (u_t)^2$, and on Γ we have

$$\Phi = \varphi_t + [\varphi_u - \tau_t] f(t,u) - \tau_u [f(t,u)]^2 \tag{16}$$

so that we end up with the determining equation

$$\frac{\partial F}{\partial\lambda} + \tau(t,u)\frac{\partial F}{\partial t} + \varphi(t,u)\frac{\partial F}{\partial u} = \Psi = \varphi_t + [\varphi_u - \tau_t]F - \tau_u F^2 \tag{17}$$

It should be stressed that for higher order ODEs the right hand side will not contain terms of order higher than one in F. This follows at once from the recursive structure of prolongation coefficients, i.e. by the general prolongation formula [1-6].

In the same way it is also easy to see, again by the prolongation formula [1-6], that for general DEs - ordinary or partial - of order n we will have on the left hand side a differential operator whose coefficients do not depend on F (while in the case of algebraic manifolds, see section 2, they could depend on F), applied to F, and on the right hand side an expression which contains terms of order not higher than one in F if $n \neq 1$, and not higher than two in F if $n = 1$.

We would like to point out that in the study of continuous Lie-point symmetries of first order ODEs such as (10), the determining equation is just $\varphi f_u - f \varphi_u = 0$. In the case of discrete equations (see 13), we get an equation of this form, but with $\partial F / \partial \lambda$ on the right hand side. Notice that indeed if $\partial F / \partial \lambda = 0$, then η_0 is a continuous symmetry of (10).

The fact that first order differential equations lead to more difficult determining equations than higher order ones, should not be a surprise. Indeed, the same happens also in the case of determining equations for continuous Lie-point symmetries [1,4,5].

The general procedure for writing the determining equations for discrete symmetries of higher order ODEs or evolution PDEs should by now be clear; it amounts to repeat the procedure illustrated in section 2 for manifolds and taking into account that $\eta : M \to TM$ is now the prolongation of an $\eta_0 : M_0 \to TM_0$. We stress that this only requires to apply the general prolongation formula [1-6].

5. Some special cases

We will now consider shortly, but explicitely, the case of second order ODEs and of evolution PDEs of order one or two in the spatial derivatives.

Autonomous second order ODEs

For an autonomous second order ODE

$$u_{tt} = f(u, u_t) \tag{18}$$

and time independent η_0 we have η_0 as in (11), but now η is the second prolongation of η_0, i.e.

$$\eta = \varphi \partial_u + \varphi_u u_t \partial_{u_t} + [\varphi_{uu} (u_t)^2 + \varphi_u u_{tt}] \partial_{u_{tt}} \tag{19}$$

and the determining equation is therefore

$$\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + \frac{\partial \varphi}{\partial u} u_t \frac{\partial F}{\partial u_t} = \frac{\partial^2 \varphi}{\partial u^2} (u_t)^2 + \frac{\partial \varphi}{\partial u} F \tag{20}$$

where $\varphi = \varphi(u)$ and $F = F(\lambda, u, u_t)$.

First order PDEs

For a first order PDE

$$u_t = f(t, x, u, u_x) \tag{21}$$

with $x \in R^q$, we write

$$\eta_0 = \tau(t, x, u) \partial_t + \xi^j (t, x, u) \partial_j + \varphi(t, x, u) \partial_u \tag{22}$$

where j runs from 1 to q, sum over repeated indices is understood, and $\partial_j = \partial/\partial x^j$. The first prolongation of this yields

$$\begin{aligned} \eta = \tau(t,x,u)\partial_t + \xi^j(t,x,u)\partial_j + \varphi(t,x,u)\partial_u \\ +(\varphi_t + \varphi_u u_t - \tau_t u_t - \tau_u (u_t)^2 - \xi^j_t u_j - \xi^j_u u_j u_t)\partial_{u_t} \\ +(\varphi_j + \varphi_u u_j - \tau_j u_t - \tau_u u_t u_j - \xi^i_j u_i - \xi^i_u u_i u_j)\partial_{u_j} \end{aligned} \tag{23}$$

and therefore the determining equation is

$$\begin{aligned} \frac{\partial F}{\partial \lambda} + \tau \frac{\partial F}{\partial t} + \xi^j \frac{\partial F}{\partial x^j} + \varphi \frac{\partial F}{\partial u} + (\varphi_j + \varphi_u u_j - (\tau_j + \tau_u u_j)F - \xi^i_j u_i - \xi^i_u u_i u_j)\frac{\partial F}{\partial u_j} \\ = \varphi_t - \xi^j_t u_j + (\varphi_u - \tau_t - \xi^j_u u_j)F - \tau_u F^2 \end{aligned} \tag{24}$$

Notice that now F appears also in the coefficients of F derivatives in the left hand side of the determining equation, contrary to what happens in the case of ODEs.

Autonomous first order PDEs

For autonomous first order PDEs

$$u_t = f(u, u_x) \tag{21'}$$

and η_0 independent of x and t,

$$\eta_0 = \varphi(u)\partial_u \tag{22'}$$

we get

$$\eta = \varphi(u)\partial_u + (\varphi_u u_x)\partial_{u_x} + (\varphi_u u_t)\partial_{u_t} \tag{23'}$$

and the determining equation is

$$\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + (\varphi_u u_x)\frac{\partial F}{\partial u_x} = \varphi_u F \tag{24'}$$

We stress that in this case the coefficient of F derivatives in the l.h.s. of the determining equation do not depend on F, and the determining equation is *linear*. By looking at the prolongation formula, we see at once that this is a general fact; i.e., for autonomous PDEs and η_0 of the form (22'), we always get linear determining equations.

Autonomous second order evolution PDEs

In the case of autonomous second order evolution PDEs,

$$u_t = f(u, u_x, u_{xx}) \tag{25}$$

and η_0 independent of x, t, i.e. as in (24'). In this case we get

$$\eta = \varphi\partial_u + \varphi_u u_x \partial_{u_x} + (\varphi_{uu}(u_x)^2 + \varphi_u u_{xx})\partial_{u_{xx}} + \varphi_u u_t \partial_{u_t} \tag{23''}$$

and the determining equation is therefore

$$\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + (\varphi_u u_x)\frac{\partial F}{\partial u_x} + (\varphi_{uu}(u_x)^2 + \varphi_u u_{xx})\frac{\partial F}{\partial u_{xx}} = \varphi_u F \tag{24''}$$

A general remark

We stress that if we have an autonomous evolution equation, the general form of η_0 transforming it into autonomous evolution equations is just $\eta_0 = \tau(t)\partial_t + \xi^j(x)\partial_j + \varphi(t,x,u)\partial_u$ [5,14]. Notice that for autonomous equations, the time and space translations correspond to continuous symmetries, so are not interesting in the present setting. Notice also that the above statement about linearity of the determining equations for autonomous PDEs does also apply to $\eta_0 = \varphi(t,x,u)\partial_u$.

6. Differential equations with prescribed discrete symmetries

As already stressed, our main equation (13) (or (6) in the case of general manifolds) yielding the condition to be satisfied to have a quantized symmetry, is a *functional* equation, as it requires to determine both the $F(\lambda,x)$ which is the "transported" of f under the vector field, *and* the φ describing the vector field (12) to be quantized to give a symmetry.

Our task is considerably simpler if we consider the "inverse problem", i.e. if we assign a priori a vector field, namely the φ, and we ask to determine the differential equations which admit this as a quantized symmetry.

In this case we have to determine the f such that there exist a $F(\lambda,u)$ periodic (in λ) solution of (13) with initial datum $F(0,u) = f(u)$, namely the "initial data" in the space of differential equation which lead to periodic evolution under the vector field $\partial_\lambda + \varphi(u)\partial_u$. Although this is not a completely standard problem, it is nevertheless more tractable than the "direct problem", as it will be concretely shown also by the examples in the following section.

Indeed, if φ are given, then (13) - as an equation for F - is a quasilinear first order PDE, which can be solved in a standard way by the method of characteristics. Notice that for the "direct problem", eq.(13) can still be formally solved by characteristics, but now we should have also to determine the characteristic vector field (hence, the φ) at the same time as F.

7. Examples

We will present here some simple examples, dealing with symmetries of direct relevance in Physics; we will consider the case of (discrete) translations, (discrete) rotations, (discrete) scale transformations, and (discrete) conformal transformations. We will limit to consider first order autonomous ODEs, $u_t = f(u)$, for the sake of simplicity, and we look for a function $f(u)$ such that the equation admits a discrete symmetry, with associated vector field $\eta_0 = \varphi(u)\partial_u$. The case of PDEs would present the same kind of procedure; examples of this will be presented elsewhere [13].

A) Translations

As the first – and simplest – example, we just consider a translation, $\eta_0 = \partial_u$. The determining equation (13) is now

$$\frac{\partial F}{\partial \lambda} + \frac{\partial F}{\partial u} = 0 \tag{26}$$

The general solution of this equation is

$$F(\lambda,u) = h(\lambda - u) \tag{27}$$

with h a periodic function. The function $f(u)$ in our differential equation is equal to $F(\lambda,u)$ for $\lambda = 0$, i.e. we just obtain (with a sign change)

$$u_t = h(u) \tag{28}$$

with h periodic, e.g. $h(u) = \sin(u)$.

The result in this case is pretty obvious, but we see how the construction of this type of differential equations can be carried out in a straightforward way (at least in the simplest cases).

B) Rotations

An example of nonlinear determining equations is given by discrete rotations in the (u,t) plane, i.e; by $\eta_0 = t\partial_u - u\partial_t$. In this case, the determining equation is

$$\frac{\partial F}{\partial \lambda} - u\frac{\partial F}{\partial t} + t\frac{\partial F}{\partial u} \; = \; 1 + F^2 \tag{29}$$

The two invariants of this are

$$z = \lambda - \arcsin\left(u/\sqrt{t^2+u^2}\right) \quad ; \quad w = t^2 + u^2 \tag{30}$$

and the general solution is then

$$F(\lambda,t,u) = \frac{u + h(z,w)\ t}{t - h(z,w)\ u} \tag{31}$$

where again h should be periodic in λ, i.e. in z. By taking e.g. $h(z,w) = \sin(kz)$ (with $k \neq 0,1$) we have the differential equation

$$u_t = \frac{u + t\sin[k\ \arcsin(u/\sqrt{t^2+u^2})]}{t - u\sin[k\ \arcsin(u/\sqrt{t^2+u^2})]} \tag{32}$$

which is indeed invariant under the discrete rotation of angle $\alpha = (2\pi/k)$,

$$u \to u\cos(\alpha) - t\sin(\alpha) \quad ; \quad t \to u\sin(\alpha) + t\cos(\alpha) \tag{33}$$

C) Dilations

The next example we consider is that of a dilation, i.e. $\eta_0 = u\partial_u$; then the determining equation is

$$\frac{\partial F}{\partial \lambda} + u\frac{\partial F}{\partial u} = F(\lambda,u) \tag{34}$$

with general solution

$$F(\lambda,u) = h(ue^{-\lambda})u \tag{35}$$

in which again h should be periodic (but not constant) in λ, and $f(u)$ will be given by $f(u) = uh(u)$. We can take e.g.

$$h(z) = \sin(2\pi - \log z) \tag{36}$$

which yields

$$f(u) = -u\sin(\log u) \tag{37}$$

D) Conformal Transformations

In the last example we consider a conformal transformation, i.e. we use a nonlinear vector field, $\eta_0 = u^2\partial_u$. The deermining equation is in this case

$$\frac{\partial F}{\partial \lambda} + u^2\frac{\partial F}{\partial u} = 2uF(\lambda,u) \tag{38}$$

with solution

$$F(\lambda,u) = h\left(\lambda + \frac{1}{u}\right)u^2 \tag{39}$$

where again h should be periodic (but not constant) in λ. Choosing, e.g.,

$$h(z) = \sin(z) \tag{40}$$

we have for $f(u)$, i.e. for the differential equation,

$$f(u) = u^2 \sin\left(\frac{1}{u}\right) \tag{41}$$

8. Some final remarks

We would like at this point to present some short remarks about the applications and possible extensions of the method proposed here. Some of these will be developed in a future work [13].

While continuous symmetries allow for a reduction in the order of an ODE, this is not the case for a discrete symmetry (just think of a system invariant under left-right inversion), so that in this sense the information about discrete symmetries is less valuable than the one on continuous symmetries. On the other side, some kind of symmetry reduction can still be applied even for discrete symmetries.

First of all, in the case of ODEs a discrete symmetry implies in general the invariance (under the flow of the ODE) of certain subspaces, pointlike invariant under the symmetry [15]; one can then consider the restriction of the ODE to such subspaces, and on this apply the usual symmetry reduction.

Moreover, in the case of PDEs a continuous symmetry does not allow to reduce the equation, but only to consider specific classes of solutions, invariant under the given symmetry. From this point of view, discrete symmetries are equivalent to continuous ones, in the sense we can still look for solutions which are invariant under the given (discrete) symmetry of the equation.

As it is well known (see e.g. [6]), it is possible that a given solution (or a given set of solutions) is also invariant under a transformation which is not a symmetry of the equation; one says then that this transformation is a *conditional symmetry* [6,7] of the equation. Here again, discrete symmetries would play the same role as continuous ones, in that one can consider discrete conditional symmetries and determine solutions which are invariant under them.

It is clear that in the present work we have considered only discrete symmetries corresponding to diffeomorphisms (in the base space M) which are homotopic to the identity, i.e. which can be obtained as a continuous deformation of the identity, so that we cannot deal with general discrete tranformations. On the other side, if we consider the different homotopy classes of diffeomorphisms of M and fix a representative g_0 – simple enough, e.g. an inversion – for each of these, we could then apply the ideas of the present work to discrete transformations which are obtained as continuous deformations of this g_0.

Finally, we would like to mention that in recent years some attention has been devoted to a problem which is in some sense dual to the one considered here, i.e. to the study of continuous symmetries of discrete equations [8-12]; the problem of determining the differential equations which admit a given discrete symmetry can simply be mapped into this one by reversing the role of the discrete map (to be considered as the time evolution rather than the symmetry) and of the continuous equation (to be considered as the symmetry rather than the time evolution law). Notice, however, that not all the differential equations can be considered as (Lie-point) vector fields, and not all the discrete maps describing time evolution are of the class considered here.

9. Conclusions

We have discussed "quantized" symmetries of manifolds in R^n, and found the equations these have to satisfy. On the basis of this discussion we have considered the case of the manifolds identifying a differential equation in the appropriate jet space, and found the corresponding equations to be satisfied; these are the determining equations for discrete (quantized) symmetries of a differential equation.

Such equations are functional ones; they can also be used to determine the differential equations possessing a given discrete (quantized) symmetry, in which case they are standard, i.e. non functional, linear PDEs.

We have considered some simple cases, discussing quantized translations, dilations, and conformal transformations; these examples show that our method is concretely viable.

The computation of discrete symmetries of differential equations is a very interesting problem from the point of view of explicit solutions and properties of these equations. The method we present here allows a systematic approach to this problem using well known techniques from the computation of continuous symmetries. Although the determining equations are very undetermined and not very easy to solve, they can be used to find explicit discrete symmetries or particular equations having these symmetries, and provide insights into the structure of differential equations with these properties.

In a future work [13] we will present a more complete analysis together with some further results and applications to integrable models.

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