Random correlations for small perturbations of expanding maps

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October 1995

Dedicated to the memory of Wieslaw Szlenk

Abstract. We consider random compositions $F_{\sigma^n\omega} \circ \cdots \circ F_\omega$ of $\mathcal{C}^k$ expanding maps F_ω which are $\mathcal{C}^k$-close to a given $\mathcal{C}^k$ expanding map ($k > 1$) and not necessarily i.i.d. We study the random correlation functions $C_\omega(n)$ associated to the unique absolutely continuous stationary measures $F_\omega \nu_\omega = \nu_{\sigma\omega}$ and smooth test functions. We show $\mathcal{C}^{k-1}$ stability of the densities of the measures ν_ω, and good uniform bounds on the exponential rate of decay of random correlations as the smooth error level goes to zero. To do this, we let the associated random transfer operators $\mathcal{L}_{F_\omega}$ act on suitable cones of positive functions endowed with a Hilbert projective metric.

1. Introduction

When studying small random perturbations of a given expanding dynamical system $f : X \to X$, i.e., compositions $F_{\sigma^n\omega} \circ \cdots \circ F_\omega$ with each random variable F_ω "close" to f (see Section 2 for precise definitions), one approach is to consider the Markov chain with transition probability $P^\epsilon(x, E)$ given by the probability that $F_\omega(x) \in E$. (See [K1], [K2] for background on random dynamical systems.) The corresponding absolutely continuous invariant measure and its decay of correlations were investigated in [BY] where strong stability properties were obtained for independent identically distributed (i.i.d.) perturbations of expanding systems by studying an appropriate "integrated" transfer operator.

In a non i.i.d. setting, there is apparently no adequate integrated transfer operator to study. It is therefore not advantageous a priori to consider integrated correlation functions as in [BY] instead of the random correlations associated to the stationary measures $F_{\sigma^j\omega}\nu_{\sigma^j\omega} = \nu_{\sigma^{j+1}\omega}$ (see Section 2 for definitions, and also Remark 2.1). Such stationary measures were studied by Kifer ([K3, Theorem B], see also [Ko1, Ko2]), in particular for expanding F_ω. However, strong stability properties (i.e., convergence results for the densities of these measures) and rates of decay of correlations had not been considered yet. We obtain smooth strong stability of the densities of the stationary measures (Theorem A) uniform exponential decay of random correlations (Theorem B)

1991 *Mathematics Subject Classification.* 28D20 58F11; 58F30 60G10.

and finally (essentially) optimal upper bounds for these random rates of decay (Theorem C), for expanding systems.

Since we cannot use the integrated transfer operators but need to work with the family of random transfer operators used e.g. by Kifer in [K3], it has proved useful to take advantage of the flexibility of the Birkhoff cones techniques applied to dynamics by Ferrero and Schmitt [FS], and more recently extensively used by Liverani [Li]. This approach has been employed e.g. by Bogenschütz [Bo] in a random framework and should be suitable to study many perturbative situations. We expect the methods in this paper to be applicable with suitable modifications to other settings where enough (perhaps nonuniform or piecewise) hyperbolicity is present to guarantee exponential decay of correlations for the original dynamical system f.

Section 2 contains precise definitions and a statement of our results. After recalling some facts about Birkhoff cones and showing some preliminary bounds in Section 3, we prove Theorem A and Theorem B in Section 4 (using a "naive" family of cones). In Section 5, introducing "special" cones tailored for our dynamical system, we prove Theorem C. For the reader's convenience, the Appendix contains bounds from [BY].

Instead of rederiving the well-known (see e.g. [Ru]) quasicompactness properties of the transfer operator for f from Theorem B, we could have used them to construct immediately the "special" cones and apply them to the proof of Theorems A and B. Although this would have made the presentation slightly shorter, we have preferred to give a self contained exposition and use "naive" cones in Sections 3 and 4. (The only exceptions to this "self-contained" rule are the three results from Garrett Birkhoff stated in Section 3, and ergodicity properties only used in comments; in particular, we do not require the perturbative spectral results from [BY].) This exposition has the advantage of unifying the Hilbert metric and functional analysis approaches, in particular we recover upper bounds for the rate of decay of correlations due to Rychlik ([Ry], see also [Li]).

Work on this paper started during visits of the first two named authors to the Université de Bourgogne in Dijon, and we would like to thank the Laboratoire de topologie for its warm hospitality and support. The first named author is also very grateful to the SFB 170 at the University of Goettingen and to the IHES for their hospitality and financial support, and acknowledges useful discussions with Thomas Bogenschütz and inspiring questions from Claude-Alain Pillet. Finally, we are very much indebted to Pierre Collet who suggested to us the use of the special family of cones in Section 5.

2. Setting and statement of results

Random compositions and stationary measures.

Let X be a compact, connected, $\mathcal{C}^\infty$ Riemannian manifold, and let $f : X \to X$ be $\mathcal{C}^r$, with r of the form $r = (k, \gamma)$ (meaning that the kth derivative of f is γ-Hölder) for some $k \in \mathbb{N}$, $k \geq 1$ and $0 \leq \gamma \leq 1$, with $\gamma > 0$ if $k = 1$. (We shall say that $r' = (k', \gamma') \geq r$ if $k' \geq k$ and $\gamma' \geq \gamma$, and that $r' > 0$ if $k' + \gamma' > 0$.) We assume that f is locally λ_0-expanding for some $\lambda_0 > 1$ (i.e., for any $x \in X$ and $v \in T_x X$ we have

$\|Df(x)v\| \geq \lambda_0\|v\|$). We view the map f as fixed throughout; this is the dynamical system which we are randomly perturbing. (We refer e.g. to [KS] for properties of locally expanding maps, recalling only for now that such a map has a constant, finite, number of inverse branches.)

For small $\epsilon > 0$ we consider the ϵ-neighbourhood $B^r_\epsilon(f)$ of f in the space of all $\mathcal{C}^r$ transformations of X, endowed with the $\mathcal{C}^r$ metric (i.e., we consider the sum of the $\mathcal{C}^k$ distance and the C^γ distance between the k^{th} derivatives). We always assume that ϵ is small enough so that all maps in $B^r_\epsilon(f)$ are (locally) expanding. Let σ_ϵ be a bimeasurable automorphism of a Lebesgue space $(\Omega_\epsilon, \mathcal{D}_\epsilon)$ preserving a probability P_ϵ, and let $F_\epsilon : \Omega_\epsilon \to B_\epsilon(f)$ be a measurable map. When there is no ambiguity, we shall often write (Ω, σ) instead of $(\Omega_\epsilon, \sigma_\epsilon)$, and F_ω instead of $F_\epsilon(\omega)$. We shall consider the random orbits

$$F_{\sigma^n\omega} \circ F_{\sigma^{n-1}\omega} \circ \cdots \circ F_\omega(x) \tag{2.1}$$

for $\omega \in \Omega_\epsilon$ (which we view as chosen with P_ϵ) and $x \in X$ (which we view as chosen with Lebesgue measure dx on X), and let ϵ tend to zero. The simplest example is when $F_\epsilon(\Omega_\epsilon) = \{f\}$: We are then simply considering our original map. The simplest random example, corresponding to independent, identically distributed perturbations, is when Ω_ϵ is a two-sided shift space on the symbol set $B^r_\epsilon(f)$, σ_ϵ is the full shift, P_ϵ is a product measure, and $F_\epsilon(\omega) = \omega_0$.

Our main tool will be the (random) transfer operators

$$\mathcal{L}_\omega\varphi(x) = \mathcal{L}_{F_\omega}\varphi(x) = \sum_{y \in F_\omega^{-1}(x)} \frac{\varphi(y)}{|\det D_y F_\omega|} \tag{2.2}$$

(see e.g. [K3]) acting on the Banach space of $\mathcal{C}^{r-1}$ test functions $\varphi : X \to \mathbb{C}$ endowed with the $\mathcal{C}^{r-1}$ norm (where $r-1 = (k-1, \gamma)$) defined by

$$\|\cdot\|_{r-1} = \sup_{0 \leq j \leq k-1} \sup_X \|D^j\varphi(x)\| + |D^{k-1}\varphi|_\gamma \,,$$

where $|\cdot|_\gamma$ denotes the γ-Hölder semi-norm (we write $\|\psi\|_\gamma$ for the corresponding norm $\sup|\psi| + |\psi|_\gamma$) and we have chosen arbitrary norms on the successive tangent spaces. Observe that $\mathcal{L}^*_F(dx) = dx$ for each $F \in \Omega_\epsilon$ and small enough ϵ, where the $\mathcal{L}^*_F$ acts on functionals by duality: $(\mathcal{L}^*_F\mu)(\varphi) = \mu(\mathcal{L}_F\varphi)$. We simply write $\mathcal{L}$ for the transfer operator $\mathcal{L}_f$ associated to the map f itself. We may now state our first main result:

Theorem A (Strong stability of the random measures). *For small enough ϵ there is for each $\omega \in \Omega_\epsilon$ a probability measure on X equivalent with Lebesgue measure $\nu_\omega(dx) = h_\omega(x)dx$ such that $F_\omega\nu_\omega = \nu_{\sigma\omega}$. Each density h_ω is the unique $\mathcal{C}^{r-1}$ solution of $\mathcal{L}_\omega h_\omega = h_{\sigma\omega}$ with $\int h_\omega(y)\,dy = 1$ and we have*

$$\lim_{\epsilon \to 0} \sup_{\omega \in \Omega_\epsilon} \|h_\omega - h\|_{r-1} = 0\,, \tag{2.3}$$

where h is the unique $\mathcal{C}^{r-1}$ solution of $\mathcal{L}h = h$ with $\int h(y)\,dy = 1$.

Clearly, h is the density of an absolutely continuous invariant probability for f. By ergodicity, one may show ([KS]) that $h\,dx$ is the only such invariant measure for f.

What is new in Theorem A is the strong stability claim (2.3): the existence of the measures $h_\omega(x)\,dx$ was proved by Kifer in [K3] in a (similar) Hölder setting.

Remark 2.1. Assume for a moment that Ω_ϵ is a two-sided shift space on the symbol set $B^r_\epsilon(f)$, σ_ϵ is the full shift and $F_\epsilon(\omega) = \omega_0$. If σ_ϵ is ergodic for P_ϵ, then the measure $\mu_\epsilon(dx, d\omega) = \nu_\omega(dx)P_\epsilon(d\omega)$ is an ergodic invariant probability measure for the skew-product $T_\epsilon(x,\omega) = (F_{\epsilon,\omega}(x), \sigma_\epsilon(\omega))$ on $X \times \Omega_\epsilon$ ([KK, Theorem 3.2]). By the Birkhoff ergodic theorem μ_ϵ is therefore the unique T_ϵ invariant probability $\nu'_\omega(dx)P_\epsilon(d\omega)$ with marginal P_ϵ on Ω_ϵ whose P_ϵ-disintegrations ν'_ω are (P_ϵ-almost all) equivalent with Lebesgue. Writing δ^U_u for the Dirac mass at a point $u \in U$, the Birkhoff theorem also implies that μ_ϵ satisfies $\frac{1}{n}\sum_{j=0}^{n-1} \delta^{X\times\Omega_\epsilon}_{T^j_\epsilon(x,\omega)} \to \mu_\epsilon(dx,d\omega)$ and

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta^{X}_{T^j_\epsilon(x,\omega)} \to \nu_\epsilon(dx) := \int_{\Omega_\epsilon} \nu_\omega(dx)\, P_\epsilon(d\omega)$$

for $dx \times P_\omega$ almost all (x,ω). Using [K3] it is not very difficult to check that *in the i.i.d. case* the invariant measure of the Markov chain mentioned in the introduction coincides with the measure $\nu_\epsilon(dx)$ (this is not true in general!).

Correlation functions and random correlation functions.

Writing $\nu(dx) = h(x)\,dx$ for the unique absolutely continuous f-invariant probability measure obtained from Theorem A, we introduce the correlation functions for (f,ν) associated to $\psi_1 \in L^1(dx)$, $\psi_2 \in L^\infty(X)$ as follows:

$$C_{\psi_1,\psi_2}(m) = \int_X \psi_1(f^m(x))\psi_2(x)\,d\nu(x) - \int_X \psi_1(x)\,d\nu(x)\int_X \psi_2(y)\,d\nu(y)\,, \quad m \in \mathbb{N}. \tag{2.4}$$

Note that by ergodicity, we have for Lebesgue almost all $x \in X$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\psi_1(f^{j+m}(x))\cdot\psi_2(f^j(x)) = \int_X \psi_1(f^m(x))\psi_2(x)\,d\nu(x)\,. \tag{2.5}$$

It is well known that under our assumptions the $C(n)$ decay exponentially fast (see e.g. [Ry, Ru]). We shall in fact recover this result as a consequence of Theorem B below, with the help of a useful and standard rewriting of (2.4):

$$C_{\psi_1,\psi_2}(m) = \int_X \psi_1(x)\Big[\mathcal{L}^m(\psi_2\,h)(x)\,dx - h(x)\cdot\Big(\int \psi_2(y)h(y)\,dy\Big)\Big]\,dx\,, \quad m \in \mathbb{N}. \tag{2.6}$$

For small enough ϵ and each $\omega \in \Omega_\epsilon$, define for $\psi_1 \in L^1(dx)$, $\psi_2 \in L^\infty(X)$ and all $m \ge 0$ the *random correlation*

$$\begin{aligned}\mathcal{C}^\epsilon_{\psi_1,\psi_2,\omega}(m) = &\int_X \psi_1(F_{\sigma^m\omega}\circ F_{\sigma^{m-1}\omega}\cdots\circ F_\omega(x))\,\psi_2(x)\,d\nu_\omega(x)\\ &- \int_X \psi_1(x)d\nu_{\sigma^{m+1}\omega}(x)\int_X \psi_2(y)d\nu_\omega(y)\end{aligned} \tag{2.7}$$

(where the measures ν_ω are given by Theorem A).

Remark 2.2. In the setting of Remark 2.1, we get for Lebesgue almost all $x \in X$ and P_ϵ almost all ω

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_1(T_\epsilon^{j+m}(x,\omega)) \cdot \psi_2(T_\epsilon^j(x,\omega)) = \int_{X\times\Omega_\epsilon} \psi_1(T_\epsilon^m(x,\omega))\psi_2(x)\,\nu_\omega(dx)\,P_\epsilon(d\omega)\,. \tag{2.8}$$

(Note in particular that $\psi_1(T_\epsilon^n(x,\omega)$ depends on ω.) The object of interest therefore appears to be

$$C^\epsilon_{\psi_1,\psi_2}(m) = \int_{\Omega_\epsilon} C^\epsilon_{\psi_1,\psi_2,\omega}(m) P_\epsilon(d\omega)\,. \tag{2.9}$$

However, except in the i.i.d. case where the integrated correlation $C^\epsilon_{\psi_1,\psi_2}(n)$ can be expressed in terms of the Markov chain and its invariant measure (see [BY]) and studied with the help of an integrated transfer operator, the function $C^\epsilon_{\psi_1,\psi_2}$ does not seem easier to handle than the random correlation (2.7).

The second main result follows:

Theorem B (Uniform bounds for the random rates of mixing). *Let $r' = (k',\gamma') > 0$ with $r' \le r-1$. For small enough ϵ and for each $\omega \in \Omega_\epsilon$ there is $\tau_{\epsilon,\omega,r'} < 1$ and a constant $K > 0$ so that for all $\psi_1 \in L^1(dx)$, $\psi_2 \in \mathcal{C}^{r'}(X)$, and all $n \ge 0$*

$$|\mathcal{C}^\epsilon_{\psi_1,\psi_2,\omega}(n)| \le K \cdot \tau^n_{\epsilon,\omega,r'} \Big(\int |\psi_1(x)|\,dx \Big) \|\psi_2\|_{r'}\,. \tag{2.10}$$

There is a constant $\tilde\tau_{r'} < 1$ such that

$$\limsup_{\epsilon\to 0} \sup_{\omega\in\Omega_\epsilon} \tau_{\epsilon,\omega,r'} \le \tilde\tau_{r'}\,. \tag{2.11}$$

Quasicompactness of the transfer operator.

We define for each $0 < r' \le r-1$, each sufficiently small ϵ and each $\omega \in \Omega_\epsilon$, the exponential *rate of decay $\bar\tau_{\epsilon,\omega,r'}$ of the random correlation C^ϵ_ω* in $\mathcal{C}^{r'}(X)$ to be the smallest number $\bar\tau_{\epsilon,\omega,r'}$ so that for any $\tau_{\epsilon,\omega,r'} > \bar\tau_{\epsilon,\omega,r'}$ (2.10) holds for some constant $K > 0$ (depending perhaps on $\bar\tau_{\epsilon,\omega,r'}/\tau_{\epsilon,\omega,r'}$) and all $\psi_1 \in L^1(dx)$, $\psi_2 \in \mathcal{C}^{r'}(X)$, $n \ge 0$.

Applying Theorem B to the case where $F_\epsilon(\Omega_\epsilon) = \{f\}$, and using (2.6) for suitable $\psi_1 > 0$ with $\int \psi_1(x)\,dx = 1$, we see that for all $0 < r' \le r-1$

$$\sup_{x\in X} \left| \mathcal{L}^n\varphi(x) - h(x) \cdot \int \varphi(y)\,dy \right| \le K \cdot \tilde\tau^n_{r'} \|\varphi\|_{r'}\,, \quad \forall \varphi \in \mathcal{C}^{r'}(X)\,, n \ge 0\,. \tag{2.12}$$

Since $\mathcal{L}^m h = h$, using $\tau < 1$ from the Yorke inequality in Lemma 4.2 below (and the fact that $\sup_X \mathcal{L}^n 1$ is uniformly bounded by (4.2), and therefore $\|\mathcal{L}^n\|_{r'}$ is uniformly bounded) we find a constant $\tilde K$ so that

$$\|\mathcal{L}^{2n}\varphi - h \cdot \int \varphi(y)\,dy\|_{r'} \le \tilde K \cdot \max(\tilde\tau^n_{r'}, \tau^n) \cdot \|\varphi\|_{r'}\,, \quad \forall \varphi \in \mathcal{C}^{r'}(X)\,, n \ge 0\,. \tag{2.13}$$

Therefore, for any $0 < r' \le r-1$ the spectral radius of $\mathcal{L}$ acting on $\mathcal{C}^{r'}(X)$ is equal to one, the only eigenvalue on the unit circle is the simple eigenvalue 1, and there are no other elements of the spectrum of modulus greater than $\max(\tilde\tau_{r'}, \tau)^{1/2}$ (in particular, the operator is quasicompact). Let us denote for $0 < r' \le r-1$

$$\tau_{r'} := \sup\{|z| \mid z \in \text{spectrum}\,(\mathcal{L} : \mathcal{C}^{r'}(X) \to \mathcal{C}^{r'}(X))\,, z \neq 1\} \tag{2.14}$$

then $\tau_{r'} \le \max(\tilde\tau_{r'}, \tau)^{1/2} < 1$ by (2.6), and for any $\bar\tau > \tau_{r'}$, there is a constant $K(\bar\tau) > 0$ so that for any $\psi_1, \psi_2 \in \mathcal{C}^{r'}(X)$

$$|C_{\psi_1,\psi_2}(n)| \le K \cdot \bar\tau^n \Big(\int |\psi_1|\, dx\Big)\, \|\psi_2\|_{r'} \quad \forall n \ge 0\,. \tag{2.15}$$

Note that, whenever $\tau_{r'}$ is the modulus of an eigenvalue of $\mathcal{L}$, it is the smallest number so that (2.15) holds for all $\bar\tau > \tau_{r'}$. (Use (2.6) for ψ_2 a corresponding eigenfunction and for suitable $\int \psi_1 = 1$.)

Finally, recalling that $\lim_{m\to\infty}(\mathcal{L}^m 1)^{1/m} \le 1$ by (4.2), we define

$$\theta_{r'} = \limsup_{m\to\infty} \Bigg(\sup_{x\in X} \sum_{y \in f^{-m}(x)} \frac{\|D(f_y^{-m})(x)\|^{(k'+\gamma')}}{|\det Df^m(y)|} \Bigg)^{1/m} \le \frac{\lim_m (\mathcal{L}^m 1)^{1/m}}{\lambda_0^{(k'+\gamma')}} < 1\,. \tag{2.16}$$

(for $y \in f^{-m}(x)$ we let f_y^{-m} be the corresponding local inverse branch in a neighbourhood of x).

Theorem C (Good bounds for the random rates of mixing). *For $0 < r' \le r-1$ let $\bar\tau_{\epsilon,\omega,r'}$ be the exponential rate of decay of the random correlation C^ϵ_ω in $\mathcal{C}^{r'}(X)$ and let $\tau_{r'}$, $\theta_{r'}$ be as defined in* (2.14), (2.16). *Then*

$$\limsup_{\epsilon\to 0} \sup_{\omega\in\Omega_\epsilon} \bar\tau_{\epsilon,\omega,r'} \le \max(\tau_{r'}, \theta_{r'})\,. \tag{2.17}$$

In dimension one, it is not difficult to adapt the methods in [CI, BJL] to show that when $0 < r' \le r-1$ the essential spectral radius $\rho^{r'}_{ess}(\mathcal{L})$ of $\mathcal{L}$ acting on $\mathcal{C}^{r'}(X)$ coincides with $\theta_{r'}$, so that in particular $\max(\theta_{r'}, \tau_{r'}) = \tau_{r'}$. It follows that Theorem C gives optimal upper bounds in dimension one (recall that [CI] also prove that any $0 < t < \theta_{r'}$ is an eigenvalue of $\mathcal{L}$).

We *conjecture* that the property $\theta_{r'} = \rho^{r'}_{ess}(\mathcal{L})$ (and therefore $\max(\theta_{r'}, \tau_{r'}) = \tau_{r'}$) holds in higher dimensions too for $r' \le r-1$.

We also *conjecture* that, if we assume further that $\tau_{r'} > \rho^{r'}_{ess}(\mathcal{L})$ for some $r' \le r-1$, then

$$\lim_{\epsilon\to 0}\big(\text{ess sup}_{\,\omega\in\Omega_\epsilon} \tau_{\epsilon,\omega,r'}\big) = \tau_{r'}\,. \tag{2.18}$$

Finally, we would like to point out that our results do not depend on the choice of the probability measure P_ϵ: what matters is that the random variables considered lie in the ball $B^r_\epsilon(f)$.

3. Transfer operators acting on function cones and Hilbert metrics

Transfer operators acting on cones.

First assume that $r = (1, \gamma)$ for some $0 < \gamma \le 1$. In this case, we work with the following family of convex cones $\Lambda_L = \Lambda_{L,\gamma}$, indexed by $L > 0$:

$$\Lambda_L = \{\varphi \in \mathcal{C}^0(X) \mid \varphi(x) > 0\,, \forall x \in X\,,\ \frac{\varphi(x)}{\varphi(y)} \le e^{Ld(x,y)^\gamma}\,,\ \forall x, y \in X\}\,. \tag{3.1}$$

As usual in applying projective methods, the first step consists in showing that our operators map suitable cones strictly inside themselves (recall that $\lambda_0 > 1$ is the expansion constant of f):

Lemma 3.1. *Assume that $r = (1, \gamma)$ with $0 < \gamma \le 1$. Fix some $1 > \xi > 1/\lambda_0^\gamma$. There are $\epsilon_0 > 0$ and $L_0 < \infty$, so that for any $\epsilon < \epsilon_0$, any $F \in B^{1+\gamma}_\epsilon(f)$ and all $L > L_0$*

$$\mathcal{L}_F \Lambda_L \subset \Lambda_{\xi \cdot L}\,. \tag{3.2}$$

(Bounds similar to Lemma 3.1 have been obtained in many settings, see e.g. [Li].)

Proof of Lemma 3.1. We start by observing that f satisfies the following stability property: let us fix some $\lambda_0 > \lambda > 1$ (assuming for further use that $\xi > 1/\lambda^\gamma$) then there is $\epsilon_0 > 0$ such that for any $F \in B^{1+\gamma}(\epsilon)$ with $\epsilon < \epsilon_0$ F is locally λ-expanding (in particular $\inf_X |\det DF| \ge \lambda^d$ where d is the dimension of X). It follows that for any $x, y \in X$ the sets $F^{-1}(x)$ and $F^{-1}(y)$ are in bijection in such a way that the distance $d(x', y')$ between two paired points is not greater than $d(x, y)/\lambda$. (First prove this for $d(x, y)$ sufficiently small and then use the fact that the metric on X comes from the Riemannian structure.) Moreover, by compactness, the γ-Hölder constants of the jacobians $|\det DF|$ for such F are uniformly bounded above by some $Q > 0$. It follows that for any F in $B^{1+\gamma}_\epsilon(f)$ and any $x', y' \in X$ we have (using $\sup \log'(u) \le 1/\inf u$)

$$\begin{aligned}\frac{|\det DF(x')|}{|\det DF(y')|} &= e^{\log|\det DF(x')| - \log|\det DF(y')|} \\ &\le e^{(|\det DF(x')| - |\det DF(y')|)/\lambda^d} \le e^{Qd(x',y')^\gamma/\lambda^d}\,.\end{aligned} \tag{3.3}$$

Fix $L > 0$ and φ in Λ_L. Then for any F in $B^{1+\gamma}_\epsilon(f)$ and any $x, y \in X$ we get by applying (3.3) and using the pairs $(x', y'(x')) \in F^{-1}(x) \times F^{-1}(y)$ described above:

$$\begin{aligned}\mathcal{L}_F\varphi(x) &\le \sum_{x' \in F^{-1}(x)} \frac{\varphi(y')e^{Ld(x',y')^\gamma}}{|\det DF(x')|} \\ &\le \sum_{x' \in F^{-1}(x)} \frac{\varphi(y')e^{Ld(x',y')^\gamma} e^{Qd(x',y')^\gamma/\lambda^d}}{|\det DF(y')|} \\ &\le \mathcal{L}_F\varphi(y) \cdot e^{(L+Q/\lambda^d)(d(x,y)^\gamma/\lambda^\gamma)}\,.\end{aligned} \tag{3.4}$$

Since $\xi > \lambda^\gamma$, it suffices to take $L_0 = (Q/\lambda^{d+\gamma})/(\xi - \frac{1}{\lambda^\gamma})$. $\square$

Consider now the case $r = (k, \gamma)$ with $k \geq 2$, assuming for simplicity that $\gamma = 0$ (the modifications needed for noninteger exponents above 2 are mostly left to the reader). We now define $\Lambda'_{L,\vec{M}} = \Lambda'_{L,\vec{M},k-1}$ for $L > 0$ and $\vec{M} = (M_1 > 0, \dots, M_{k-1} > 0)$ by

$$\begin{aligned}\Lambda'_{L,\vec{M}} = \{\varphi \in \mathcal{C}^{r-1}(X) \mid \varphi(x) > 0\,, \forall x \in M\,, \frac{\varphi(x)}{\varphi(y)} \leq e^{Ld(x,y)}\,, \forall x, y \in X \\ \|D^j\varphi(x)\| \leq M_j\varphi(x)\,, \forall x \in M\,, 1 \leq j \leq k-1\}\,.\end{aligned} \tag{3.5}$$

(When $\gamma > 0$, we add the condition that $D^{k-1}\varphi$ is γ-Hölder with Hölder constant bounded by some M_γ.)

Lemma 3.2. *Let $r = (k, 0)$ with $k \geq 2$ and fix some $1 > \xi > 1/\lambda_0$. There are $\epsilon_0 > 0$, $L'_0 > 0$, $\mathcal{M}_{0,1} > 0$ and functions $\mathcal{M}_{0,j} : \mathbb{R}_+^{j-1} \to \mathbb{R}_+$ for $2 \leq j \leq k-1$, so that for $\epsilon < \epsilon_0$, any $F \in B^r_\epsilon(f)$, and $L > L'_0$, $M_j > \mathcal{M}_{0,j}(M_1, \dots, M_{j-1})$ $(1 \leq j \leq k-1)$ we have*

$$\mathcal{L}_F(\Lambda'_{L,\vec{M}}) \subset \Lambda'_{\xi\cdot L,\xi\cdot\vec{M}}\,, \tag{3.6}$$

where we write $\xi\vec{M} = (\xi M_1, \dots, \xi M_{k-1})$.

Proof of Lemma 3.2. We may fix some $\lambda_0 > \lambda > 1$ (so that $\xi > 1/\lambda$) and assume that ϵ is small enough so that all maps F in $B^r_\epsilon(f)$ are λ-expanding, and that the jth derivatives, $1 \leq j \leq k-1$, of their jacobian $|\det DF|$ and also the norms of the jth derivatives D^jF for $1 \leq j \leq k$ are bounded by some uniform constant $Q > 0$. In particular, (3.3) and (3.4) hold for $\gamma = 1$, and since $\xi > 1/\lambda$, it suffices to take

$$L'_0 \geq \frac{Q/\lambda^{d+1}}{\xi - \frac{1}{\lambda}}\,, \tag{3.7}$$

to ensure that the second condition on the cones is satisfied.

Let us now write in detail the calculations for the first two derivatives. For any $F \in B^r_\epsilon(f)$, any $\varphi \in \Lambda'_{L,\vec{M}}$ and any $x \in X$ we have

$$\begin{aligned}\|D\mathcal{L}_F\varphi(x)\| &\leq \sum_{F(y)=x} \frac{\|D\varphi(y)\|}{\lambda|\det DF(y)|} + \sum_{F(y)=x} \frac{\varphi(y)\|D|\det DF(y)|\|}{\lambda|\det DF(y)|^2} \\ &\leq \sum_{F(y)=x} \frac{M_1\cdot\varphi(y)}{\lambda|\det DF(y)|} + \sum_{F(y)=x} \frac{\varphi(y)\|D|\det DF(y)|\|}{\lambda^{(1+d)}|\det DF(y)|} \\ &\leq \left(\frac{M_1}{\lambda} + \frac{Q}{\lambda^{(1+d)}}\right)\mathcal{L}_F\varphi(x)\,.\end{aligned} \tag{3.8}$$

We may thus take

$$\mathcal{M}_{0,1} = (Q/\lambda^{(1+d)})/(\xi - 1/\lambda)\,. \tag{3.9}$$

For $k \geq 3$ we have (using $\lambda > 1$ and $d \geq 1$)

$$\begin{aligned}
\|D^2 \mathcal{L}_F \varphi(x)\| &\leq \sum_{F(y)=x} \frac{\|D^2\varphi(y)\|}{\lambda^2 |\det DF(y)|} + \sum_{F(y)=x} \frac{3Q\,\|D\varphi(y)\|}{\lambda^2|\det DF(y)|} \\
&\qquad + \sum_{F(y)=x} \frac{(Q+3Q^2)\,\varphi(y)}{\lambda^3|\det DF(y)|} \qquad (3.10)\\
&\leq \left(\frac{M_2}{\lambda^2} + \frac{3QM_1}{\lambda^2} + \frac{Q+3Q^2}{\lambda^3}\right)\mathcal{L}_F\varphi(x)\,.
\end{aligned}$$

We may thus take

$$\mathcal{M}_{0,2}(M_1) \geq \frac{((Q+3Q^2)/\lambda + 3QM_1)/\lambda^2}{\xi - \frac{1}{\lambda^2}}\,.$$

For the general case $k \geq 4$, we use that we may bound $\|D^\ell \mathcal{L}_G \varphi(x)\|$ with $3 \leq \ell \leq k-1$ similarly as in (3.10), where each term on the right-hand side is either bounded by

$$\sum_{F(y)=x} \frac{p_j(Q)\cdot\|D^j\varphi(y)\|}{\lambda^\ell|\det DF(y)|} \leq \sum_{F(y)=x} \frac{p_j(Q)\cdot M_j\,\varphi(y)}{\lambda^\ell|\det DF(y)|}\,, \tag{3.11}$$

with $1 \leq j \leq \ell - 1$ and $p_j(\cdot)$ a polynomial (depending on λ), or by

$$\sum_{F(y)=x} \frac{p_0(Q)\cdot\varphi(y)}{\lambda^\ell|\det DF(y)|}\,, \tag{3.12}$$

where $p_0(\cdot)$ is a polynomial, except a single term of the form

$$\sum_{F(y)=x} \frac{\|D^\ell\varphi(y)\|}{\lambda^\ell|\det DF(y)|} \leq \sum_{F(y)=x} \frac{M_\ell\,\varphi(y)}{\lambda^\ell|\det DF(y)|}\,. \quad \square \tag{3.13}$$

Projective metrics in vector lattices.

To proceed, we need to recall basic definitions and results about projective metrics on positive cones in vector lattices (see [Bi2], and also [N] for a more recent exposition; [L] contains a lucid and short account). Let $\mathcal{E}$ be a topological vector space and $\preceq$ a continuous partial ordering (i.e., if $\psi, \varphi_n \in \mathcal{E}$, $\psi \preceq \varphi_n$ for all n, and $\lim_{n\to\infty}\varphi_n = \varphi$ then $\psi \preceq \varphi$). We call such a pair an *integrally closed vector lattice.* A subset $\Lambda \subset \mathcal{E}$ is called a *cone* if $\mu\varphi \in \Lambda$ for all $\varphi \in \Lambda$ and all $\mu \in \mathbb{R}^*_+$; the cone Λ is called *closed* if $\Lambda \cup \{0\}$ is closed. We define the Hilbert pseudo-metric Θ on the positive cone $\Lambda = \{\varphi \in \mathcal{E} \setminus \{0\} \mid 0 \preceq \varphi\}$ as follows: for $\varphi, \psi \in \Lambda$, set

$$\alpha(\varphi,\psi) = \sup\{\mu \in \mathbb{R}^+ \mid \mu\varphi \preceq \psi\}\,,\ \beta(\varphi,\psi) = \inf\{\mu \in \mathbb{R}^+ \mid \psi \preceq \mu\varphi\}\,, \tag{3.14}$$

(where we define $\alpha = 0$ and $\beta = \infty$ when the corresponding sets are empty), then let

$$\Theta(\varphi,\psi) = \log\frac{\beta(\varphi,\psi)}{\alpha(\varphi,\psi)}\,. \tag{3.15}$$

Our main tool will be:

Birkhoff's inequality ([Bi1], [Bi2]). *If $\mathcal{L}:\mathcal{E}\to\mathcal{E}$ is a linear map from the integrally closed vector lattice $\mathcal{E}$ into itself such that $\mathcal{L}(\Lambda)\subset\Lambda$ for the corresponding positive cone, then for any $\varphi,\psi\in\Lambda$ we have*

$$\Theta(\mathcal{L}\varphi,\mathcal{L}\psi)\le\tanh\left(\frac{\operatorname{diam}_\Theta(\mathcal{L}\Lambda)}{4}\right)\cdot\Theta(\varphi,\psi)\,. \tag{3.16}$$

(Where we use the notation $\operatorname{diam}_\Theta(A)=\sup_{\varphi,\psi\in A}\Theta(\varphi,\psi)$.*)*

We shall need two additional results from Birkhoff in the *special case* where ($\mathcal{E}=C^0(X),\preceq$) is an integrally closed vector lattice structure on the vector space $C^0(X)$ of complex valued continuous functions on a compact metric space X (endowed with the topology of uniform convergence), with the positive cone (Λ,Θ) having the property that $\varphi(x)>0$ for all $\varphi\in\Lambda$, and where we are given a Borel probability measure dm on X with full support:

Hilbert and uniform metrics ([Bi1]). *For $\varphi,\psi\in\Lambda$ with $\int_X\varphi\,dm=\int_X\psi\,dm=1$, we have*

$$\sup_X|\varphi(x)-\psi(x)|\le(e^{\Theta(\varphi,\psi)}-1)\sup_X\varphi(x)\,. \tag{3.17}$$

Completeness Lemma ([Bi1]). *For any $\psi_0\in\Lambda$ with $\int\psi_0\,dm=1$ the set*

$$\{\varphi\in\Lambda\mid\int_X\varphi\,dm=1\,,\ \Theta(\varphi,\psi_0)<\infty\} \tag{3.18}$$

is a complete metric space for the metric Θ.

The next step is to define the vector lattices which we shall use. We shall always take $\mathcal{E}$ to be $\mathcal{C}^0(X)$ with the uniform metric, and $dm=dx$ the normalised Lebesgue measure on X. Consider first the case $r=(1,\gamma)$ where $0<\gamma\le1$. Fixing $\xi>\lambda_0^\gamma$ and $L>L_0(\xi)$ as given by Lemma 3.1, we observe that Λ_L as in (3.1) is a closed convex cone in $C^0(X)$ and that $\Lambda_L\cap-\Lambda_L=\emptyset$. We define the order $\preceq_L$ by:

$$\varphi\preceq_L\psi\leftrightarrow\psi-\varphi\in\Lambda_L\cup\{0\}\,. \tag{3.19}$$

One proves that that the positive cone Λ associated with $\preceq_L$ coincides with Λ_L. A standard computation (see e.g. [FS; Ko1; Li, Lemma 2.2]) shows that, for any $\varphi,\psi\in\Lambda_L$:

$$\alpha_\gamma(\varphi,\psi)=\inf_{x\ne y\in X}\frac{e^{Ld(x,y)^\gamma}\psi(x)-\psi(y)}{e^{Ld(x,y)^\gamma}\varphi(x)-\varphi(y)}\le\inf_{x\in X}\frac{\psi(x)}{\varphi(x)}\,,$$

and

$$\beta_\gamma(\varphi,\psi)=\sup_{x\ne y\in X}\frac{e^{Ld(x,y)^\gamma}\psi(x)-\psi(y)}{e^{Ld(x,y)^\gamma}\varphi(x)-\varphi(y)}\ge\sup_{x\in X}\frac{\psi(x)}{\varphi(x)}\,,$$

so that

$$\Theta_{L,\gamma}(\varphi,\psi) = \log \sup_{\substack{x\neq y\in X\\ u\neq v\in X}} \frac{(e^{Ld(x,y)^\gamma}\psi(x)-\psi(y))(e^{Ld(u,v)^\gamma}\varphi(u)-\varphi(v))}{(e^{Ld(x,y)^\gamma}\varphi(x)-\varphi(y))(e^{Ld(u,v)^\gamma}\psi(u)-\psi(v))}\,. \tag{3.20}$$

Consider now integer $r \geq 2$. Similarly, we may define an order $\preceq'_{L,\vec{M}}$ using the cones $\Lambda'_{L,\vec{M}}$ in (3.5). We then get for $\varphi \in \Lambda'_{L,\vec{M}}$

$$\alpha'(\varphi,1) = \min\left(\alpha_1(\varphi,1),\ \inf_{\substack{x\in X\\ 1\leq j\leq k-1}} \frac{M_j}{M_j\varphi(x)+\|D^j\varphi(x)\|}\right),$$

and

$$\beta'(\varphi,1) = \max\left(\beta_1(\varphi,1),\ \sup_{\substack{x\in X\\ 1\leq j\leq k-1}} \frac{M_j}{M_j\varphi(x)-\|D^j\varphi(x)\|}\right),$$

so that

$$\begin{aligned}\Theta'_{L,\vec{M},k-1}(\varphi,1) = \max\Bigg(&\Theta_{L,1}(\varphi,1), \log \sup_{\substack{x,y\in X\\ 1\leq j,\ell\leq k-1}} \frac{M_\ell}{M_j}\cdot\frac{M_j\varphi(x)+\|D^j\varphi(x)\|}{M_\ell\varphi(y)-\|D^\ell\varphi(y)\|},\\ &\log \sup_{\substack{x\neq y\in X\\ u\in X\\ 1\leq j\leq k-1}} \frac{(e^{Ld(x,y)}-1)(M_j\varphi(u)+\|D^j\varphi(u)\|)}{(e^{Ld(x,y)}\varphi(x)-\varphi(y))M_j},\\ &\log \sup_{\substack{x\in X\\ u\neq v\in X\\ 1\leq \ell\leq k-1}} \frac{M_\ell(e^{Ld(u,v)}\varphi(u)-\varphi(v))}{(M_\ell\varphi(x)-\|D^\ell\varphi(x)\|)(e^{Ld(u,v)}-1)}\Bigg).\end{aligned} \tag{3.21}$$

Clearly, the next step is to get uniform bounds for the diameter of $\mathcal{L}_F\Lambda_L$ in Λ_L (see Lemma 2.3 in [Li] for a very similar bound), respectively $\mathcal{L}_F\Lambda'_{L,\vec{M}}$ in $\Lambda'_{L,\vec{M}}$:

Lemma 3.3.

(1) *For any $L>0$, $\xi<1$*

$$\operatorname{diam}_{\Theta_{L,\gamma}}\Lambda_{\xi L} \leq 2\log\frac{1+\xi}{1-\xi} + 2\xi L \operatorname{diam} X =: K(\xi,L)\,. \tag{3.22}$$

(2) *For any $r=(k,0)$ with $k\geq 2$, and any L, $\vec{M}$, $\xi<1$*

$$\operatorname{diam}_{\Theta'_{L,\vec{M},k-1}}\Lambda'_{\xi L,\xi\vec{M},k-1} \leq K(\xi,L)\,. \tag{3.23}$$

Proof of Lemma 3.3.

(1) We use formula (3.20) for $\varphi,\psi\in\Lambda_{\xi L}$ and get

$$\begin{aligned}\Theta_{L,\gamma}(\varphi,\psi)&\le\log\sup_{\substack{x\neq y\in X\\ u\neq v\in X}}\frac{(e^{Ld(x,y)^\gamma}-e^{-\xi Ld(x,y)^\gamma})(e^{Ld(u,v)^\gamma}-e^{-\xi Ld(u,v)^\gamma})\psi(y)\varphi(v)}{(e^{Ld(x,y)^\gamma}-e^{\xi Ld(x,y)^\gamma})(e^{Ld(u,v)^\gamma}-e^{\xi Ld(u,v)^\gamma})\varphi(y)\psi(v)}\\&\le\log\left(\frac{(1+\xi)^2}{(1-\xi)^2}\,e^{2\xi L\operatorname{diam}X}\right).\end{aligned}\tag{3.24}$$

(2) We now use (3.21) which involves the maximum of four expressions. The first one was dealt with in (1) (noting that each 2 may be replaced by 1 in the right-hand-side of (3.24) when $\psi=1$). For the second, we clearly have for $\varphi\in\Lambda'_{\xi L,\xi\vec{M}}$:

$$\log\sup_{\substack{x,y\in X\\ 1\le\ell,j\le k-1}}\frac{M_\ell}{M_j}\cdot\frac{M_j(1+\xi)\varphi(x)}{M_\ell(1-\xi)\varphi(y)}\le\log\left(\frac{1+\xi}{1-\xi}\,e^{\xi L\cdot\operatorname{diam}X}\right).\tag{3.25}$$

The third and fourth expressions are quite similar. We only consider the third one, and see that for any $\varphi\in\Lambda'_{\xi L,\xi\vec{M}}$

$$\begin{aligned}&\log\sup_{\substack{x\neq y\in X\\ u\in X\\ 1\le j\le k-1}}\frac{(e^{Ld(x,y)}-1)(M_j\varphi(u)+\|D^j\varphi(u)\|)}{(e^{Ld(x,y)}\varphi(x)-\varphi(y))M_j}\\&\qquad\le\log\sup_{\substack{x\neq y\in X\in X\\ u\in X\\ 1\le j\le k-1}}\frac{(e^{Ld(x,y)}-1)(1+\xi)M_j\varphi(u)}{(e^{Ld(x,y)}-e^{\xi Ld(x,y)})M_j\varphi(x)}\\&\qquad\le\log\left(\frac{1+\xi}{1-\xi}e^{\xi L\cdot\operatorname{diam}X}\right).\quad\square\end{aligned}\tag{3.26}$$

4. Stability of random measures and bounds for the decay of random correlations

The following lemma is quite standard in our setting:

Lemma 4.1. *Consider the case $r=(1,\gamma)$ with $0<\gamma\le1$, and let L_0 and ξ be as in Lemma 3.1. For all sufficiently small ϵ and all $L>L_0$ there is a uniquely defined map $h:\Omega_\epsilon\to\Lambda_L$ (we write h_ω for $h(\omega)$ as usual) such that for each $\omega\in\Omega_\epsilon$*

$$\int_X h_\omega(x)\,dx=1\quad\textit{and}\quad\mathcal{L}_\omega h_\omega=h_{\sigma\omega}\,.\tag{4.1}$$

There is a constant $C > 0$ so that for each $\omega \in \Omega_\epsilon$, any $\varphi \in \Lambda_L$ with $\int \varphi(y)\, dy = 1$, and all n

$$\sup_X |h_\omega - (\mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega}\varphi)| \le C\Big(\tanh(\frac{K(\xi,L)}{4})\Big)^{n-1} \cdot K(\xi,L) \sup_X \varphi\,. \tag{4.2}$$

If $r = (k,0)$ with $k \ge 2$ then each h_ω is in fact an element of $\Lambda'_{L',\vec{M},k-1}$ for any $L' > L'_0$ and $M_j > \mathcal{M}_{0,j}$ as in Lemma 3.2, in particular each h_ω is $\mathcal{C}^{r-1}$.

(The new result is the fact that the h_ω are uniformly $\mathcal{C}^{r-1}$, see [K3] for the construction of the Hölder h_ω.)

Proof of Lemma 4.1. Fix $\epsilon < \epsilon_0$ and $L > L_0$. First observe that Lemmas 3.1 and 3.3(1) imply that for any $\xi > 1/\lambda_0^\gamma$ and $L > L_0$, any maps $\varphi, \psi : \Omega_\epsilon \to \Lambda_L$, and all $n \ge 1$, $m \ge 0$, $\omega \in \Omega_\epsilon$ we have

$$\begin{aligned}
&\Theta_{L,\gamma}(\mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega}(\varphi_{\sigma^{-n}\omega}), \mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n-m}\omega}(\psi_{\sigma^{-n-m}\omega}))\\
&\quad\le \Big(\tanh(\frac{K(\xi,L)}{4})\Big)^{n-1} \Theta_{L,\gamma}(\mathcal{L}_{\sigma^{-n}\omega}(\varphi_{\sigma^{-n}\omega}), \mathcal{L}_{\sigma^{-n}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n-m}\omega}(\psi_{\sigma^{-n-m}\omega}))\\
&\quad\le \Big(\tanh(\frac{K(\xi,L)}{4})\Big)^{n-1} K(\xi,L)\,.
\end{aligned} \tag{4.3}$$

Applying (4.3) to the constant maps $\varphi_\omega = \psi_\omega \equiv 1$, first for $n \to \infty$ and all fixed $m \ge 0$, and then for $n = 1$ and $m \to \infty$, we get that for each $\omega \in \Omega_\epsilon$ the sequence

$$\mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega} 1 \tag{4.4}$$

is Cauchy in the complete metric space $\{\varphi \in \Lambda_L \mid \int_X \varphi(x)\, dx = 1\,, \Theta(\varphi, \mathcal{L}_{\sigma^{-1}\omega}1) < \infty\}$ (recall the completeness lemma from Section 3). Setting h_ω to be the limit of this sequence, we have that $\int h_\omega\, dx = 1$ and $h_\omega \in \Lambda_L$ by definition, and we get $\mathcal{L}_\omega h_\omega = h_{\sigma\omega}$ by using the continuity of each operator $\mathcal{L}_\omega$. Claim (4.2) can be obtained by using the fact (to be proved below) that there is a constant $C > 0$ so that for any $\varphi \in \Lambda_L$ with $\int \varphi(y)\, dy = 1$ and all $m \ge 0$

$$\sup_X \mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-m}\omega}(\varphi) \le C \sup_X \varphi\,, \tag{4.5}$$

and then using again (4.3) (for $\varphi_\omega \equiv \varphi$, $\psi_\omega = h_\omega$, $m = 0$ and $n \to \infty$) and (3.17).

To show (4.5), we apply (4.3) for $n = 1$ and then use (3.17) to obtain

$$\sup_X |\mathcal{L}_{\sigma^{-1}\omega}(\varphi) - \mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-m-1}\omega}(\varphi)| \le (e^{K(\xi,L)} - 1) \sup_X \mathcal{L}_{\sigma^{-1}\omega}(\varphi)$$

finally, we observe that

$$\sup_X |\mathcal{L}_{\sigma^{-1}\omega}(\varphi)| \le \sup_X(\mathcal{L}_{\sigma^{-1}\omega}1) \cdot \sup_X \varphi\,.$$

To show uniqueness we note that if a family $\bar h_\omega \in \Lambda_L$ with $\int \bar h_\omega(y)\,dy = 1$ for all $\omega \in \Omega_\epsilon$ is a solution of $\mathcal{L}_\omega \bar h_\omega = \bar h_{\sigma\omega}$, then we have by construction

$$\bar h_\omega = \mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega}(\bar h_{\sigma^{-n}\omega}), \quad h_\omega = \mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega}(h_{\sigma^{-n}\omega}) \tag{4.6}$$

for each $n \geq 1$, so that, applying (4.3) to $m = 0$, $\varphi_\omega = \bar h_\omega$ and $\psi_\omega = h_\omega$ we get $\Theta_{L,\gamma}(h_\omega, \bar h_\omega) = 0$ for all ω. Since $\int h_\omega\,dx = \int \bar h_\omega\,dx = 1$ this implies $h_\omega = \bar h_\omega$ for all ω.

To prove the additional smoothness of h_ω when $k \geq 2$, simply apply Lemma 3.2 and Lemma 3.3(2) to show that the sequence (4.4) is also a Cauchy sequence in the complete metric space $\{\varphi \in \Lambda'_{L,\vec M} \mid \int_X \varphi(x)\,dx = 1\,, \Theta'(\varphi, \mathcal{L}_{\sigma^{-1}\omega}1) < \infty\}$ for the distance $\Theta'_{L,\vec M}$ (with suitable $L, \vec M$). Its limit, of course, is still h_ω. $\square$

We now prove the following well-known Yorke-type inequality:

Lemma 4.2 (Yorke inequality). *For $r = (k,\gamma)$ with $k \geq 1$, $0 \leq \gamma \leq 1$ and $k+\gamma > 1$, there are constants $K_0 > 0$, $K_1 > 0$, and $\tau < 1$ so that for all $\varphi \in \mathcal{C}^{r-1}(X)$ and all $n \geq 0$*

$$\sup_{1\leq j\leq k-1} \sup_X \|D^j \mathcal{L}^n \varphi\| \leq K_0 \tau^n \|\varphi\|_{k-1} + K_1 \cdot \sup_X |\varphi(x)|\,, \text{ if } \gamma = 0$$
$$\sup_{1\leq j\leq k-1} \sup_X \|D^j \mathcal{L}^n \varphi\| + |D^{k-1}\mathcal{L}^n\varphi|_\gamma \leq K_0 \tau^n \|\varphi\|_{r-1} + K_1 \cdot \sup_X |\varphi(x)|\,, \text{ if } \gamma > 0\,. \tag{4.7}$$

Proof of Lemma 4.2. We assume for simplicity that $\gamma = 0$, leaving the Hölder bounds for the reader. We first show that there are constant $t_1 < 1$, $K_{0,1}, K_{1,1} > 0$ so that for all $n \geq 0$ and all $\varphi \in \mathcal{C}^{r'}(X)$

$$\sup_X \|D^1 \mathcal{L}^n \varphi\| \leq K_{0,1} \cdot t_1^n \sup_X \|D^1\varphi\| + K_{1,1} \cdot \sup_X |\varphi(x)|\,. \tag{4.8}$$

To show (4.8), it suffices to find an integer $n_1 \geq 1$ and constants $t' < 1$, $K' = K'(n_1) > 0$ so that for all $\varphi \in \mathcal{C}^1(X)$

$$\sup_X \|D^1 \mathcal{L}^{n_1} \varphi\| \leq t' \sup_X \|D^1\varphi\| + K' \cdot \sup_X |\varphi(x)|\,. \tag{4.9}$$

(Then take $t_1 = (t')^{1/n_1} < 1$ and apply an inductive argument for arbitrary $n = qn_1 + r$ with $0 \leq r < n_1$, using a geometric series.) Finally, since $\sup_X \|D(f^{-n_1})(x)\| \leq 1/\lambda_0^{n_1}$ for each inverse branch of f^n, (4.9) follows from

$$\begin{aligned} D^1 \mathcal{L}^{n_1} \varphi(x) = &\sum_{f^{n_1}(y)=x} \frac{D\varphi(y) \circ D(f_y^{-n_1})(x)}{|\det D(f^{n_1}(y))|} \\ &+ \sum_{f^{n_1}(y)=x} \frac{\varphi(y)(D|\det D(f^{n_1})(y)|) \circ D(f_y^{-n_1})(x)}{|\det D(f^{n_1}(y))|^2}\,, \end{aligned} \tag{4.10}$$

together with the fact from (4.2) (or (4.5)) that $\mathcal{L}^m 1$ is uniformly bounded in m.

It remains to show the bounds for $\sup_{1\le i\le j}\sup_X \|D^i\mathcal{L}^n\varphi\|$ with $2\le j\le k-1$. We do this by induction on j, showing first (just like (4.9)) that there are an integer n_j, and constants $t'<1$, $K'>0$ so that for all $\varphi\in\mathcal{C}^j(M)$:

$$\sup_X \|D^j\mathcal{L}^{n_j}\varphi\| \le t'\sup_X \|D^j\varphi\| + K'\cdot\big(\sup_{0\le i\le j-1}\sup_X \|\,D^i\varphi(x)\|\big)\,, \tag{4.11}$$

obtaining next constants $t_j<1$ and $K_{0,j}, K_{1,j}>0$ so that for all $n\ge 0$, $\varphi\in\mathcal{C}^j(M)$

$$\sup_X \|D^j\mathcal{L}^n\varphi\| \le K_{0,j}t_j^n\sup_X \|D^j\varphi\| + K_{1,j}\cdot\big(\sup_{0\le i\le j-1}\sup_X \|\,D^i\varphi(x)\|\big)\,, \tag{4.12}$$

and applying finally (4.12) to $\varphi=\mathcal{L}^n\psi$ in order to use the induction assumption (using a geometric series and replacing t_j by its square root in the process). $\square$

We may now prove Theorem A:

Proof of Theorem A. Let $\nu_\omega = h_\omega\,dx$ for each $\omega\in\Omega_\epsilon$, where $\epsilon<\epsilon_0$ and h_ω is given by Lemma 4.1. To check that $F_\omega\nu_\omega=\nu_{\sigma\omega}$ we use the fact that $\mathcal{L}^*_\omega dx = dx$ and get for any $\varphi\in L^1(dx)$

$$\int(\varphi\circ F_\omega)\,h_\omega\,dx = \int \mathcal{L}_\omega((\varphi\circ F_\omega)h_\omega)\,dx = \int \varphi\mathcal{L}_\omega h_\omega\,dx = \int \varphi\,h_{\sigma\omega}\,dx\,. \tag{4.13}$$

To show the strong stability result, we will first prove that

$$\lim_{\epsilon\to 0}\sup_{\omega\in\Omega_\epsilon}\sup_{x\in X}|h_\omega(x)-h(x)| = 0\,. \tag{4.14}$$

This will follow from (4.2) and the comparison bound (3.17) between the uniform and Hilbert metric, combined with the bounds (A.1) from [BY] in the Appendix. Indeed, for any large enough $n\ge 1$ and $\epsilon<\epsilon(n)$ we have

$$\begin{aligned}\sup_{x\in X}|h_\omega(x)-h(x)| &= \sup_X |h_\omega(x)-\mathcal{L}^n h(x)|\\ &\le \sup_X\big|h_\omega(x)-\mathcal{L}_{\sigma^{-1}\omega}\circ\cdots\circ\mathcal{L}_{\sigma^{-n}\omega}h(x)\big|\\ &\qquad + \sup_X\big|(\mathcal{L}_{\sigma^{-1}\omega}\circ\cdots\circ\mathcal{L}_{\sigma^{-n}\omega}-\mathcal{L}^n)h(x)\big|\\ &\le C(\tanh(K(\xi,L)/4))^{n-1}\cdot K(\xi,L)\sup_X h + c_{n,\epsilon}\|h\|_\gamma\,,\end{aligned} \tag{4.15}$$

where $c_{n,\epsilon}$ goes to zero with ϵ for each fixed n.

We will now bootstrap from (4.14) to get the smooth strong stability, treating first the case $r=(1,\gamma)$ with $\gamma<1$. We use the Yorke inequality in Lemma 4.2, apply the triangle inequality differently, and use this time the Hölder version of (A.2) in the

Appendix (from [BY]), obtaining for $\theta > \theta_\gamma$ (with θ_γ as defined in (2.16)), large enough n and $\epsilon < \epsilon(n)$:

$$\begin{aligned} |h_\omega(x) - h(x)|_\gamma &= |\mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega}(h_{\sigma^{-n}\omega}) - \mathcal{L}^n h|_\gamma \\ &\le \|\mathcal{L}_{\sigma^{-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma^{-n}\omega}(h_{\sigma^{-n}\omega}) - \mathcal{L}^n(h_{\sigma^{-n}\omega})\|_\gamma + |\mathcal{L}^n(h_{\sigma^{-n}\omega} - h)|_\gamma \\ &\le \theta^n \|h_{\sigma^{-n}\omega}\|_\gamma + K_0 \tau^n (\|h_{\sigma^{-n}\omega}\|_\gamma + \|h\|_\gamma) + K_1 \sup_X |h_{\sigma^{-n}\omega} - h| \,. \end{aligned} \tag{4.16}$$

To end the proof of Theorem A in the case $r = (1, \gamma)$, with the help of (4.16), just apply (4.14) and use the fact that the $\mathcal{C}^\gamma$ norms of the h_ω are uniformly bounded since they all lie in some Λ_L and their supremum norms are bounded (by (4.14), again).

The bootstrap for $r \ge 2$ may be treated similarly, using the $\mathcal{C}^{r-1}$ bound (A.2) from [BY] in the Appendix and fact that the h_ω all lie in some $\Lambda_{L,\vec{M}}$. $\square$

We may now prove Theorem B:

Proof of Theorem B. First consider the case where $\psi_1 \in L^1(dx)$ and $\psi_2 \in \Lambda_L$, respectively $\Lambda'_{L,\vec{M}}$. It follows that $\psi_2 h_\omega$ is in Λ_{L_1} (respectively $\Lambda_{L_1,\vec{M}_1}$) for some L_1 (and $\vec{M}_1$) and all ω. Writing

$$\begin{aligned} &\Big|\int_X \psi_1(F_{\sigma^n\omega} \circ \cdots \circ F_\omega(x)) \psi_2(x)\, h_\omega(x)\, dx - \int \psi_1(x) d\nu_{\sigma^{n+1}\omega}(x) \int \psi_2(y) d\nu_\omega(y)\Big| \\ &\quad = \Big|\int_X \psi_1(x) \mathcal{L}_{\sigma^n\omega} \circ \cdots \circ \mathcal{L}_\omega(\psi_2 h_\omega)(x)\, dx \\ &\qquad\qquad - \int \psi_1(x) h_{\sigma^{n+1}\omega}(x)\, dx \int \psi_2(y) h_\omega(y)\, dy\Big| \\ &\quad \le \left|\int_X \psi_1(x) \big|\mathcal{L}_{\sigma^n\omega} \circ \cdots \circ \mathcal{L}_\omega(\psi_2 h_\omega)(x) - h_{\sigma^{n+1}\omega}(x) \cdot \Big(\int \psi_2(y) h_\omega(y)\, dy\Big)\big|\, dx\right|, \end{aligned} \tag{4.17}$$

and using (4.2) from Lemma 4.1, we obtain the bound (2.10) with $\tau_{\epsilon,\omega,r'} = \tau_{\epsilon,r'} = \tanh(K(\xi, \max(L_1, L_0))/4)$ with ξ and L_0 given by Lemma 3.1, respectively Lemma 3.2 (note that only the supremum component of the norm of $\psi_2 h_\omega$ appears in the right-hand-side of (4.2)). Observe also that $\xi = \xi(\epsilon, r')$ and $L_i = L_i(\epsilon, r')$ converge to $\xi(0, r')$ and $L_i(0, r')$ ($i = 0, 1$) as $\epsilon \to 0$ by construction.

For the case of general $\psi_2 \in \mathcal{C}^{r'}(X)$ we write $\eta = \min(k' + \gamma', k + \gamma - 1, 1)$ and use standard arguments (see e.g. [Li]): First decompose each $\psi_2 h_\omega \in \mathcal{C}^{r'}$ into its real and imaginary parts, and then write each part as a difference of two nonnegative $\mathcal{C}^{r'}$ functions; it therefore suffices to consider nonnegative $\psi_2 h_\omega \in \mathcal{C}^{r'}$. For such a ψ_2 we set

$$\psi_{2,\omega} = (\psi_2 + \rho) \cdot h_\omega \,, \tag{4.18}$$

where

$$\rho = \frac{\sup_{\omega \in \Omega_\epsilon} |\psi_2 h_\omega|_\eta}{L_0 \inf_{x \in X, \omega \in \Omega_\epsilon} h_\omega(x)} > 0 \,. \tag{4.19}$$

Writing $\psi = \psi_2 h_\omega$ and $\phi = \psi_{2,\omega}$ for simplicity, we then have for any $x \neq y \in X$ (similarly as in (3.3)):

$$\begin{aligned}\frac{\phi(x)}{\phi(y)} &\leq \exp(\log(\psi(x) + e^{\xi L_0 d(x,y)^\eta} \rho h_\omega(y)) - \log(\psi(y) + \rho h_\omega(y))) \\ &\leq \exp\left(\frac{|\psi|_\eta d(x,y)^\eta + \rho h_\omega(y)(e^{\xi L_0 d(x,y)^\eta} - 1)}{\rho h_\omega(y)}\right) \\ &\leq \exp((L_0 + 2\xi L_0) \cdot d(x,y)^\eta)\,.\end{aligned} \tag{4.20}$$

It follows that each $\psi_{2,\omega}$ is in Λ_L for all $L > L_0(1+2\xi)$. Noting that $\rho \leq K \cdot |\psi_2|_\eta$ and applying (4.3) similarly as in the first case, we get the bound (2.10), with $\tau_{\epsilon,\omega,r'} = \tau_{\epsilon,\gamma} = \tanh(K(\xi, L)/4)$. $\square$

5. Good bounds for the decay rate of random correlations

To get better upper bounds, it will be useful to work with a more suitable family of cones, using the knowledge about the spectral properties of $\mathcal{L}$ acting on $\mathcal{C}^{r'}(X)$ for $0 < r' \leq r-1$ which we obtained from Theorem B in Section 2. More precisely, we will use the spectral decomposition following from (2.14):

$$\mathcal{L}\varphi(x) = h(x) \int \varphi(y)\,dy + \mathcal{P}\varphi(x)\,, \quad \forall \varphi \in \mathcal{C}^{r-1}(X)\,, \tag{5.1}$$

where h is the fixed function of $\mathcal{L}$, dx is the fixed functional of the dual of $\mathcal{L}$, and $\mathcal{P}$ is the bounded linear projector on $\mathcal{C}^{r'}(X)$ associated with the rest of the spectrum. In particular, for all $\tau > \tau_{r'}$ there is a constant $K > 0$ with $\|\mathcal{L}^n \mathcal{P}\|_{r'} = \|\mathcal{P}\mathcal{L}^n\|_{r'} \leq K \cdot \tau^n$. We define the following special family of cones for any $L > 0$:

$$\Lambda''_L = \Lambda''_{L,r'} = \{\varphi \in \mathcal{C}^{r'}(X) \mid \varphi(x) > 0\,, \forall x \in X\,, \|\mathcal{P}\varphi\|_{r'} \leq L \int_X \varphi(x)\,dx\}\,. \tag{5.2}$$

(The sets Λ''_L are closed convex cones in $C^0(X)$ endowed with the uniform metric.)

Lemma 5.1. *Let $0 < r' \leq r-1$ and let $\tau_{r'}$ be as in (2.14), and $\theta_{r'}$ as in (2.16). Fix some $\xi > \max(\tau_{r'}, \theta_{r'})$ and $L_0 > 0$. There is $N_0 \geq 1$ and for each $N \geq N_0$ an $\epsilon(N) > 0$ so that for all $L > L_0$, $\epsilon < \epsilon(N)$, and any $F_i \in B^r_\epsilon(f)$ $(1 \leq i \leq N)$,*

$$\mathcal{L}_{F_N} \circ \cdots \circ \mathcal{L}_{F_1}(\Lambda''_L) \subset \Lambda''_{\xi^N \cdot L}\,. \tag{5.3}$$

(Note that we are free to choose L_0 as small as we want!)

Proof of Lemma 5.1. The positivity condition is obviously satisfied. For the other one, just observe that we have for any $\xi > \tau > \tau_{r'}$, $\xi > \theta > \theta_{r'}$, and $N \geq N_0(\xi, \lambda)$ and

$\epsilon < \epsilon(N)$ (using (A.2) from the Appendix and the spectral decomposition (5.1) of $\mathcal{L}$):

$$
\begin{aligned}
&\|\mathcal{P}(\mathcal{L}_{F_N} \circ \cdots \circ \mathcal{L}_{F_1}\varphi)\|_{r'} \\
&\qquad \le \|\mathcal{P}\|_{r'} \cdot \|(\mathcal{L}_{F_N} \circ \cdots \circ \mathcal{L}_{F_1} - \mathcal{L}_F^N)\varphi\|_{r'} + \|\mathcal{P}\mathcal{L}_F^N \varphi\|_{r'} \\
&\qquad \le \theta^N \|\mathcal{P}\|_{r'} \|\varphi\|_{r'} + \|\mathcal{L}_F^N \mathcal{P}^2 \varphi\|_{r'} \\
&\qquad \le \theta^N \|\mathcal{P}\|_{r'} \big(\|\mathcal{P}\varphi\|_{r'} + \|h \int \varphi(x)\, dx\|_{r'}\big) + K\tau^N \|\mathcal{P}\varphi\|_{r'} \qquad (5.4) \\
&\qquad \le \theta^N \|\mathcal{P}\|_{r'} \cdot (L + \|h\|_{r'}) \cdot \int \varphi(x)\, dx + K\tau^N L \int \varphi(x)\, dx \\
&\qquad = \big(L(\theta^N \|\mathcal{P}\|_{r'} + K\tau^N) + \theta^N \|\mathcal{P}\|_{r'} \|h\|_{r'}\big) \int (\mathcal{L}_{F_N} \circ \cdots \circ \mathcal{L}_{F_1}\varphi)(x)\, dx\,.
\end{aligned}
$$

(We used that $\varphi = \mathcal{P}\varphi + h \int \varphi(y)\, dy$ and $\int (\mathcal{L}_{F_N} \circ \cdots \circ \mathcal{L}_{F_1}\varphi)(x)\, dx = \int \varphi(x)\, dx$.) $\square$

Lemma 5.2. *For any* $0 < L \le \inf_X h$, *and* $\xi < 1$

$$
\operatorname{diam}_{\Theta''_L} \Lambda''_{\xi L} \le 2 \log \frac{1+\xi}{1-\xi} =: K(\xi)\,. \qquad (5.5)
$$

Proof of Lemma 5.2. We start with the following useful observation: for any $\varphi \in \Lambda''_L$ with $0 < L \le \inf h$

$$
\sup_{x \in X} \frac{h(x)}{\varphi(x)} \le \sup_{x \in X} \frac{h(x)}{h(x)(\int \varphi(y)\, dy) - \sup_{u \in X} |\mathcal{P}\varphi(u)|} \le \frac{L}{L(\int \varphi(y)\, dy) - \|\mathcal{P}\varphi\|_{r'}}\,. \qquad (5.6)
$$

Under the same conditions, we have the analogous bound

$$
\inf_{x \in X} \frac{h(x)}{\varphi(x)} \ge \frac{L}{L(\int \varphi(y)\, dy) + \|\mathcal{P}\varphi\|_{r'}}\,. \qquad (5.7)
$$

Similarly as in Section 3, we easily check (using that $h \in \Lambda''_L$ and (5.6)–(5.7)) that for any $\varphi \in \Lambda''_L$

$$
\Theta''_L(\varphi, h) = \log \frac{L \int \varphi(x)\, dx + \|\mathcal{P}\varphi\|_{r'}}{L \int \varphi(x)\, dx - \|\mathcal{P}\varphi\|_{r'}}\,. \qquad (5.8)
$$

The proof is then analogous to the bound (3.25). $\square$

Proof of Theorem C. Similarly as in the proof of (4.2) we get by applying Lemma 5.1 and Lemma 5.2 that there is a constant $C > 0$ so that for sufficiently small ϵ, each $\xi > \max(\tau_{r'}, \theta_{r'})$, and all $0 < L \le \inf h$, there is an N_0 so that for every $\varphi \in \Lambda''_L$, each $\omega \in \Omega_\epsilon$ and all $n > N_0$

$$
\begin{aligned}
&\sup_X \big| \mathcal{L}_{\sigma^n \omega} \circ \cdots \circ \mathcal{L}_\omega \varphi - h_{\sigma^{n+1}\omega} \cdot \int_X \varphi(y)\, dy \big| \\
&\qquad\qquad \le C \Big(\tanh\big(\frac{K(\xi)}{4}\big) \Big)^n \cdot K(\xi) \sup_X \varphi\,,
\end{aligned} \qquad (5.9)
$$

(Note that we know that all the h_ω belong to some Λ''_L where L may be taken arbitrarily small for small enough ϵ.)

Observe now that for $\xi \to 0$, we have $K(\xi) = 2\log((1+\xi)/(1-\xi)) \sim 4\xi$, and thus

$$\tanh \frac{K(\xi)}{4} \sim \xi\,,\; \xi \to 0\,. \tag{5.10}$$

For any $\psi_2 \in \Lambda''_L$ so that $\psi_2 h_\omega$ all belong to some $\Lambda''_{\bar L}$ with $\bar L < \inf h$, we see from (5.9)–(5.10) that the upper bound for the exponential decay of the random correlations obtained from the cones Λ''_L can be made as close to $\max(\tau_{r'}, \theta_{r'})$ as we wish (just reread the first part of the proof of Theorem B having in mind the new family of cones).

To show the uniform bounds (2.17) for $\psi_2 \in \mathcal{C}^{r'}$, not necessarily in Λ''_L, we may suppose that ψ_2 is nonnegative and construct $\psi_{2,\omega} = (\psi_2 + \rho)h_\omega$ as in the second part of the proof of Theorem B, assuming that ϵ is small enough so that all the $h_\omega \in \Lambda''_{L_0}$ with $L_0 < \inf h/2$, and setting

$$\rho = \frac{\|\mathcal{P}\|_{r'}}{L_0} \cdot \|\psi_2\|_{r'}\,. \tag{5.11}$$

To finish, we check that $\psi_{2,\omega}$ is in Λ''_{2L_0}:

$$\begin{aligned}
\|\mathcal{P}((\psi_2+\rho)h_\omega)\|_{r'} &\le \|\mathcal{P}\psi_2\|_{r'} + \rho\|\mathcal{P}h_\omega\|_{r'} \\
&\le \|\mathcal{P}\|_{r'}\|\psi_2\|_{r'} + \rho L_0 \\
&= 2L_0\rho = 2L_0\rho \int h_\omega(y)\,dy \\
&\le 2L_0(\int (\psi_2(y) + \rho h_\omega(y))\,dy\,. \quad \square
\end{aligned} \tag{5.12}$$

Appendix: bounds for random perturbations of transfer operators

We include here for the reader's convenience some bounds adapted from [BY, Lemma 5]. The setting is the same as in Section 2, in particular $r' \le r-1$ and $\theta_{r'}$ is defined by (2.16), and we use $c_{n,\epsilon}$ to denote a constant depending only on n and ϵ, and tending to zero as $\epsilon \to 0$ for each fixed n.

Lemma A.1. *Let $\theta > \theta_{r'}$ be given. Then there exists $N_0 \in \mathbb{N}$ such that for each $n \ge N_0$, there exists $\epsilon(n) > 0$ such that for each $\epsilon < \epsilon(n)$, and all $F_i \in B^r_\epsilon(f)$ $(i = 1, \ldots, n)$ one has for any $0 < r' \le r-1$ and each $\varphi \in \mathcal{C}^{r'}(X)$*

$$\sup_X |(\mathcal{L}_{F_n} \circ \cdots \circ \mathcal{L}_{F_1} - \mathcal{L}^n)\varphi| \le c_{n,\epsilon}\|\varphi\|_{r'}\,, \tag{A.1}$$

and

$$\|\mathcal{L}_{F_n} \circ \cdots \circ \mathcal{L}_{F_1} - \mathcal{L}^n\|_{r'} < \theta^n\,. \tag{A.2}$$

Proof of Lemma A.1. We will use the fact that all orbits of f are *strongly shadowable*: that is, if ϵ is small enough, then for a fixed x and a fixed n-tuple $\vec{F} = (F_1, \dots, F_n)$ with $F_i \in B^r_\epsilon(f)$, there is a natural bijection between the set $\{y \mid f^n(y) = x\}$ and the set $\{y_{\vec{F}} \mid F^n_{\vec{F}} y_{\vec{F}} := F_n \circ \cdots \circ F_1 y_{\vec{F}} = x\}$ so that paired points are not more than $O(\epsilon)$-distant. Moreover, if we write $g = 1/|\det Df|$, $\bar{g} = Df^{-1} \circ f$, and $G_i = 1/|\det DF_i|$, $\bar{G}_i = DF_i^{-1} \circ F_i$ (the choice of the inverse branch will always be clear from the context), then for each pair $(y, y_{\vec{F}})$ corresponding to a choice of an inverse branch of f^n at x we have (abusing notation slightly)

$$\begin{aligned} g(y) \cdot g(fy) \cdots g(f^{n-1}y) &= G_1(y_{\vec{F}}) \cdot G_2(F_1 y_{\vec{F}}) \cdots G_n(F^{n-1}_{\vec{F}} y_{\vec{F}}) \pm c_{n,\epsilon} \\ \bar{g}(y) \cdot \bar{g}(fy) \cdots \bar{g}(f^{n-1}y) &= \bar{G}_1(y_{\vec{F}}) \cdot \bar{G}_2(F_1 y_{\vec{F}}) \cdots \bar{G}_n(F^{n-1}_{\vec{F}} y_{\vec{F}}) \pm c_{n,\epsilon} \,. \end{aligned} \tag{A.3}$$

We first show (A.1), and (A.2) in the case $r = (2,0)$, $r' = (1,0)$. Let us compare $\mathcal{L}^N$ and $\mathcal{L}^N_{\vec{F}} = \mathcal{L}_{F_N} \circ \cdots \circ \mathcal{L}_{F_1}$ in the $\mathcal{C}^0$-norm, noting $|\varphi| = \sup_X |\varphi|$ and $|D\varphi| = \sup_X \|D\varphi\|$:

$$\begin{aligned} \varphi(y_{\vec{F}}) \cdot G_1(y_{\vec{F}}) \cdot G_2(F_1 y_{\vec{F}}) \cdots G_n(F^{n-1}_{\vec{F}} y_{\vec{F}}) &= (\varphi(y) \pm c_{n,\epsilon}|D\varphi|)(\prod_{j=0}^{n-1} g(f^j y) \pm c_{n,\epsilon}) \\ &= \varphi(y) \cdot g(y) \cdot g(fy) \cdots g(f^{n-1}y) \pm c_{n,\epsilon}(|\varphi| + |D\varphi|) \,. \end{aligned} \tag{A.4}$$

Hence, summing (A.4) over inverse branches we get

$$\sup_{x \in X} |(\mathcal{L}^n_{\vec{F}}\varphi)(x) - (\mathcal{L}^n\varphi)(x)| \le c_{n,\epsilon}\|\varphi\|_1 \,. \tag{A.5}$$

(Clearly, the above arguments may be adapted to the case $r = (1, \gamma)$ with $\gamma \le 1$.)

We now consider first derivatives (still for $r = (2,0)$), using the Leibnitz Theorem and decomposing each branch summand in $D(\mathcal{L}^n_{\vec{F}}\varphi)(x)$ into a first part A which is a sum of terms where some G factor is differentiated and a second part B where φ is differentiated. For the first part we have (abusing notation again)

$$\begin{aligned} A = \sum_{j=0}^{n-1} \varphi(y_{\vec{F}}) G_1(y_{\vec{F}}) \cdots [DG_{j+1}(F^j_{\vec{F}} y_{\vec{F}}) \bar{G}_j(F^j_{\vec{F}} y_{\vec{F}}) \cdots \\ \cdots \bar{G}_1(y_{\vec{F}})] G_{j+2}(F^{j+1}_{\vec{F}} y_{\vec{t}}) \cdots G_n(F^{n-1}_{\vec{F}} y_{\vec{F}}) \\ = \sum_j (\varphi(y) \pm c_{n,\epsilon}|D\varphi|)\big(g(y) \cdots [Dg(f^j(y)) \cdots] \cdots g(f^{n-1}y) \pm c_{n,\epsilon}\big) \\ = (\text{ the corresponding part for } D(\mathcal{L}^n\varphi)(x)\ ) \pm c_{n,\epsilon}(|\varphi| + |D\varphi|) \,. \end{aligned}$$

For the second part, we get

$$
\begin{aligned}
B &= D\varphi(y_{\vec F}) \cdot \prod_{j=0}^{n-1} \bar G_{j+1}(F^j_{\vec F} y_{\vec F}) \cdot \prod_{j=0}^{n-1} G_{j+1}(F^j_{\vec F} y_{\vec F}) \\
&= (D\varphi(y) \pm 2|D\varphi|) \cdot (\prod_{j=0}^{n-1} \bar g(f^j y) \pm c_{n,\epsilon}) \cdot (\prod_{j=0}^{n-1} g(f^j y) \pm c_{n,\epsilon}) \\
&= D\varphi(y) \prod_{j=0}^{n-1} (g(f^j y)\bar g(f^j y)) \pm c_{n,\epsilon}|D\varphi| \pm 2|D\varphi| \prod_{j=0}^{n-1} (g(f^j y)\|\bar g(f^j y)\|)\,.
\end{aligned}
$$

Recalling that $k-1+1=2$ and summing over inverse branches, we obtain if $n(\theta)$ is large enough

$$
\begin{aligned}
D(\mathcal{L}^n_{\vec F}\varphi) &= D(\mathcal{L}^n\varphi) \pm c_{n,\epsilon}\|\varphi\|_1 \pm 2\|\varphi\|_1 \sum_{y:f^n(y)=x} \prod_{j=1}^{n} (g(f^j(y))\|\bar g(f^j(y))\|) \\
&= D(\mathcal{L}^n\varphi) \pm c_{n,\epsilon}\|\varphi\|_1 \pm 2\|\varphi\|_1\theta^n\,.
\end{aligned}
\tag{A.6}
$$

For arbitrary differentiability $r=(k,\gamma)$, with $k+\gamma \ge 3$, we assume for simplicity that $\gamma = 0$. (The Hölder proof is very similar.) First note that for $j \le k-2$, the terms of the jth derivative $D^j(\mathcal{L}^n\varphi)$ involve only the ℓth derivative of φ for $\ell \le j$ so that

$$
\|D^j(\mathcal{L}^n_\epsilon\varphi) - D^j(\mathcal{L}^n\varphi)\| \le c_{n,\epsilon}\|\varphi\|_{j+1} \le c_{n,\epsilon}\|\varphi\|_{k-1}\,.
$$

The only potentially troublesome term is part B of $D^{(k-1)}(\mathcal{L}^n_\epsilon\varphi)(x)$, i.e.,

$$
\sum_{y_{\vec F}} D^{(k-1)}\varphi(y_{\vec F}) \prod_i G_i(F^i_{\vec F} y_{\vec F}) \big(\prod_i \bar G_i(F^i_{\vec F} y_{\vec F})\big)^{k-1}\,,
$$

but the same argument as above yields an additional error term of the type

$$
c_{n,\epsilon}\|\varphi\|_{k-1} + K\cdot\theta^n\|\varphi\|_{k-1}\,. \quad \text{(A.7)} \quad \square
$$

References

[BJL] V. Baladi, Y. Jiang, and O.E. Lanford III, *Transfer operators acting on Zygmund functions*, ETHZ preprint (1995), to appear Trans. Amer. Math. Soc.

[BY] V. Baladi and L.-S. Young, *On the spectra of randomly perturbed expanding maps*, (see also Erratum, Comm. Math. Phys. **166**, pp. 219–220 (1994)), Comm. Math. Phys. **156** (1993), 355–385.

[Bi1] G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc. **85** (1957), 219–227.

[Bi2] G. Birkhoff, *Lattice theory (3rd ed.)*, Amer. Math. Soc. (Amer. Math. Soc. Colloquium Publications), Providence, RI, 1967.

[Bo] T. Bogenschütz, *Equilibrium states for random dynamical systems*, Ph.D. Thesis, University of Bremen (1993).

[CI] P. Collet and S. Isola, *On the essential spectrum of the transfer operator for expanding Markov maps*, Comm. Math. Phys. **139** (1991), 551–557.

[FS] P. Ferrero and B. Schmitt, *Produits aléatoires d'opérateurs matrices de transfert*, Probability Theory and Related Fields **79** (1988), 227–248.

[KK] K. Khanin and Y. Kifer, *Thermodynamic formalism for random transformations and statistical mechanics*, Preprint (1994).

[K1] Y. Kifer, *Ergodic Theory of Random Perturbations*, Birkhäuser, Boston Basel, 1986.

[K2] Y. Kifer, *Random Perturbations of Dynamical Systems*, Birkhäuser, Boston Basel, 1988.

[K3] Y. Kifer, *Equilibrium states for random expanding transformations*, Random Comput. Dynamics **1** (1992), 1–31.

[Ko1] A. Kondah, *Perturbations aléatoires de systèmes dynamiques unidimensionnels*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), 697-699.

[Ko2] A. Kondah, *Les endomorphismes dilatants de l'intervalle et leurs perturbations aléatoires (Thèse de doctorat, Université de Bourgogne)* (1991).

[KS] K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Math. **23** (1969), 83–92.

[L] C. Liverani, *Decay of correlations*, Ann. of Math. (2) **142** (1995), 239–301.

[N] R.D. Nussbaum, *Hilbert projective metric and iterated nonlinear maps, I*, Amer. Math. Soc. **391** (Memoirs Amer. Math. Soc.), 1988.

[Ru] D. Ruelle, *The thermodynamic formalism for expanding maps*, Comm. Math. Phys. **125** (1989), 239–262.

[Ry] M. Rychlik, *Regularity of the metric entropy for expanding maps*, Trans. Amer. Math. Soc. **315** (1989), 833–847.

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