SPREADING OF WAVE PACKETS IN THE ANDERSON MODEL ON THE BETHE LATTICE

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Abstract

The spreading of wave packets evolving under the Anderson Hamiltonian on the Bethe Lattice is studied for small disorder. The mean square distance travelled by a particle in a time t is shown to grow as t^2 for large t.

1 INTRODUCTION

The Anderson model [6] gives a description of the motion of a quantum-mechanical electron in a crystal with impurities. It is given by the random Schrödinger operator

$$H_\lambda \;=\; \tfrac{1}{2}\Delta + \lambda V \quad \text{on} \quad \ell^2(\mathbb{L})\,; \tag{1.1}$$

where $\mathbb{L}$ is either $\mathbb{Z}^d$ or the Bethe lattice $\mathbb{B}$ (same as Cayley tree - an infinite connected graph with no closed loops and a fixed number $K+1$ of nearest neighbors at each vertex ($K \geq 2$, so $\mathbb{B}$ is not the line $\mathbb{R}$); the distance between two sites x and y in $\mathbb{B}$ will be denoted by $d(x,y)$ and is equal to the length of the shortest path connecting x and y.) The (centered) Laplacian Δ is defined by

$$(\Delta u)(x) \;=\; \sum_{y} u(y)\;, \tag{1.2}$$

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where the sum runs over all nearest neighbors of x in $\mathbb{L}$, and V is a random potential, with $V(x)$, $x \in \mathbb{L}$, being independent, identically distributed random variables with common probability distribution μ. The real parameter λ is called the *disorder*.

It follows from ergodicity that the spectrum of the Hamiltonian H_λ is given by

$$\sigma(H_\lambda) = \sigma(\tfrac{1}{2}\Delta) + \lambda \operatorname{supp} \mu \tag{1.3}$$

with probability one [33, 9, 3], where $\sigma(\frac{1}{2}\Delta)$ equals $[-d, d]$ if $\mathbb{L} = \mathbb{Z}^d$ and $[-\sqrt{K}, \sqrt{K}]$ if $\mathbb{L} = \mathbb{B}$. The decomposition of $\sigma(H_\lambda)$ into pure point spectrum, absolutely continuous spectrum and singular continuous spectrum is also independent of the choice of V with probability one [26, 9, 3]. In the physics literature ([6, 32, 37, 1, 2, 29] and others) one finds the following picture: In one and two dimensions, as long as the potential is random (i.e., $\lambda \neq 0$), the model shows *exponential localization* (i.e., pure point spectrum with exponentially decaying eigenfunctions). In three and more dimensions both localized and *extended states* (i.e., absolutely continuous spectrum) are expected for small disorder, with the energies of extended and localized states being separated by the *mobility edge*; the model exhibits the *Anderson metal-insulator transition*. In the Bethe lattice the picture is similar to the multidimensional case ([1, 2, 27, 31, 30] and others), with perhaps a *mobility interval* instead of a *mobility edge* (see the discussion in [22]).

An alternative way to distinguish between localized and extended states is by looking at their time evolution. Let us consider a particle which is localized at the origin 0 of $\mathbb{L}$ (we fix an origin 0 for the Bethe lattice) at time $t = 0$, at time t it will be described by the wavefunction $\psi_t^{(\lambda)} = \mathrm{e}^{\mathrm{itH}_\lambda}\delta_0$, with δ_0 being the delta function at $x = 0$. A measure of the spread of the wave packet at time t is given by its mean square displacement

$$r_\lambda^2(t) \;=\; \sum_{x \in \mathbb{L}} |x|^2 |\psi_t^{(\lambda)}(x)|^2 \;=\; \sum_{x \in \mathbb{L}} |x|^2 \left|\mathrm{e}^{\mathrm{itH}_\lambda}(0, \mathrm{x})\right|^2 \;, \tag{1.4}$$

where $|x| = d(0, x)$ (it is not a norm in $\mathbb{B}$) and $\mathrm{e}^{\mathrm{itH}_\lambda}(0, \mathrm{x}) = \langle \delta_0, \mathrm{e}^{\mathrm{itH}_\lambda}\delta_\mathrm{x}\rangle$. We always have ([35], we give the argument in Lemma A.1)

$$r_\lambda^2(t) \;\le\; Ct^2 \;\text{ with }\; C \;=\; \|\tfrac{1}{2}\Delta\|^2 \;= \begin{cases} d^2 & \text{if } \mathbb{L} = \mathbb{Z}^d \,, \\ K & \text{if } \mathbb{L} = \mathbb{B} \,. \end{cases} \tag{1.5}$$

In the case of free motion ($\lambda = 0$) one has $r_\lambda^2(t) \approx \mathit{const}\, t^2$. Anderson [6] argued that for large disorder, there are only localized states so the particle localizes near the origin, i.e. $\sup_t r_\lambda^2(t) < \infty$ with probability one (see the discussion in [14]). In three or more dimensions, the particle should diffuse with probability one for small disorder, and one should have [14, 28] $\mathbb{E}(r_\lambda^2(t)) \approx \mathit{const}\, t$, where $\mathbb{E}$ denotes expectation with respect to the probability distribution of the random potential. On the Bethe lattice, however, we should expect $\mathbb{E}(r_\lambda^2(t)) \approx \mathit{const}\, t^2$ at small disorder, since on the Bethe lattice both free motion and what should be diffusive motion (e.g., the simple random walk) have mean square displacements growing as t^2.

There are now many proofs of exponential localization for the Anderson Hamiltonian: for any disorder in one dimension (e.g., [15, 26, 8, 11] and others), and for large disorder or low energy in the multidimensional case (e.g., [14, 13, 10, 36, 28, 11, 5, 4, 19, 16, 12] and others) and on the Bethe lattice [5, 4, 16]. When the single site potential distribution μ has a bounded density *and compact support*, it is also proven that $\mathbb{E}(\sup_t r_\lambda^2(t)) < \infty$ for large disorder, both in the multidimensional case [28, 4] and on the Bethe lattice [4].

There are presently no rigorous results about extended states in $\mathbb{Z}^d$. In previous work [20, 21, 23] we proved the existence of extended states on the Bethe lattice for small disorder. In this article we show that

$$\limsup_{t\to\infty} \mathbb{E}\left(\frac{r_\lambda^2(t)}{t^2}\right) \;>\; 0 \tag{1.6}$$

on the Bethe lattice, for sufficiently small disorder. Combining this result with the uniform deterministic bound given by (1.5), we conclude (see Lemma D.1) that

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{r_\lambda^2(t)}{t^2} \;>\; 0\right) \;>\; 0 \tag{1.7}$$

for the Anderson model on the Bethe lattice with small disorder. It follows from ergodicity that

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{r_{x,\lambda}^2(t)}{t^2} \;>\; 0 \quad \text{for some } x \in \mathbb{B}\right) \;=\; 1\;, \tag{1.8}$$

where

$$r_{x,\lambda}^2(t) \;=\; \sum_{y\in\mathbb{B}} d(x,y)^2 |(\mathrm{e}^{\mathrm{itH}_\lambda}\delta_{\mathrm{x}})(\mathrm{y})|^2\;. \tag{1.9}$$

We will now describe our results in more detail. *Our assumption on μ is that its characteristic function $h(t) = \int \mathrm{e}^{-itv} d\mu(v)$ is differentiable on $(0,\infty)$, with $h'(t)$ absolutely continuous, $h'(t)$ and $h''(t)$ bounded on $(0,\infty)$.* These conditions are satisfied by any probability distribution μ with a finite second moment (e.g., uniform, Gaussian or Bernoulli distributions) and by the Cauchy distribution. Our main result is:

THEOREM 1.1 *Let $\mathbb{L} = \mathbb{B}$, then for sufficiently small λ we have*

$$\liminf_{t\to\infty} \frac{1}{t^3}\int_0^t \mathbb{E}(r_\lambda^2(s))\, ds \;>\; 0\;. \tag{1.10}$$

In particular, we have (1.6), (1.7) and (1.8).

Notice that (1.6) follows immediately from (1.10). From (1.5) we know the uniform deterministic bound:

$$\frac{1}{t^3}\int_0^t r_\lambda^2(s)\, ds \;\le\; \frac{K}{3} \qquad \text{for any } t\;. \tag{1.11}$$

Thus we also get

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{1}{t^3}\int_0^t r_\lambda^2(s)\, ds \;>\; 0\right) \;>\; 0 \tag{1.12}$$

from (1.10) by Lemma D.1. Ergodicity again gives

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{1}{t^3}\int_0^t r_{x,\lambda}^2(s)\, ds \quad > \quad 0 \quad \text{for some } \ x \in \mathbb{B}\right) \quad = \quad 1\ . \tag{1.13}$$

This paper is organized as follows: In the next section we reformulate the problem in terms of Green's functions; Theorem 1.1 is shown to follow from Theorem 2.2. A review of the mathematical setting we use is given in section 3. Theorem 2.2 is proven in section 4. The appendices contain some results needed in our proofs.

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2 GREEN'S FUNCTIONS

We will now reformulate Theorem 1.1 in terms of the Green's function $G_\lambda(x, y; z) = \langle \delta_x, (H_\lambda - z)^{-1}\delta_y \rangle$. We start with the well known relation (see Lemma A.2)

$$\int_0^\infty \mathrm{e}^{-\eta t}\, \mathbb{E}(\mathrm{r}_\lambda^2(\mathrm{t}))\, \mathrm{dt} \quad = \quad \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \{\sum_{\mathrm{x}\in\mathbb{L}} |\mathrm{x}|^2\, \mathbb{E}(|\mathrm{G}_\lambda(0, \mathrm{x}; \mathrm{E} + \mathrm{i}\tfrac{\eta}{2})|^2)\}\, \mathrm{dE}\ , \tag{2.1}$$

valid for any $\eta > 0$. Similarly to (A.8), (A.9) and (A.10), we always have

$$\int_{-\infty}^{\infty} \mathbb{E}(|G_\lambda(0, x; E + i\tfrac{\eta}{2})|^2)\, dE \quad \le \quad \frac{2\pi}{\eta}\ , \tag{2.2}$$

$$\int_{-\infty}^{\infty} \{\sum_{x\in\mathbb{L}} |x|^2\, \mathbb{E}(|G_\lambda(0, x; E + i\tfrac{\eta}{2})|^2)\}\, dE \quad \le \quad \frac{4\pi}{\eta^3}\, \|\tfrac{1}{2}\Delta\|^2\ , \tag{2.3}$$

and

$$\sup_{E\in\mathbb{R}} \sum_{x\in\mathbb{L}} |x|^2\, \mathbb{E}(|G_\lambda(0, x; E + i\tfrac{\eta}{2})|^2) \quad \le \quad \frac{4}{\eta^4}\, \|\tfrac{1}{2}\Delta\|^2\ , \tag{2.4}$$

By a Tauberian Theorem (e.g., [34, Theorem 10.3]; notice the proof is also valid for lim inf), Theorem 1.1 follows from (2.1), (2.3) and

THEOREM 2.1 *Let $\mathbb{L} = \mathbb{B}$, then for sufficiently small λ we have*

$$\liminf_{\eta\downarrow 0}\, \eta^3 \int_{-\infty}^{\infty} \{\sum_{x\in\mathbb{B}} |x|^2\, \mathbb{E}(|G_\lambda(0, x; E + i\eta)|^2)\}\, dE \quad > \quad 0\ . \tag{2.5}$$

By Fatou's lemma, Theorem 2.1 follows immediately from the next theorem.

THEOREM 2.2 *Let $\mathbb{L} = \mathbb{B}$, then there exists $\lambda_0 > 0$, such that for any λ with $|\lambda| < \lambda_0$ we can find energies $E_\lambda^\pm \in (-\sqrt{K}, \sqrt{K})$, with $\lim_{\lambda\to 0} E_\lambda^\pm = \pm\sqrt{K}$, such that H_λ has purely absolutely continuous spectrum in the interval $I_\lambda = (E_\lambda^-, E_\lambda^+)$, and*

$$\liminf_{\eta\downarrow 0} \eta^3 \sum_{x\in\mathbb{B}} |x|^2 \, \mathbb{E}(|G_\lambda(0,x;E+i\eta)|^2) \quad > \quad 0 \quad \text{for all } E \in I_\lambda \,. \tag{2.6}$$

Purely absolutely continuous spectrum was proved in [20, 21], in the remainder of this article we will prove (2.6).

Another quantity that has been used to distinguish between localized and extended states is

$$D_\lambda(E) \quad = \quad \lim_{\eta\downarrow 0} \eta^2 \sum_{x\in\mathbb{L}} |x|^2 \, \mathbb{E}(|G_\lambda(0,x;E+i\eta)|^2) \quad . \tag{2.7}$$

$D_\lambda(E)$ is related to the D.C. conductivity by a Kubo-like formula [29, 14]. The belief is that $D_\lambda(E) = 0$ in the localization region and $D_\lambda(E) > 0$ in the region of extended states. The usual proofs of localization (e.g., [14, 13, 11, 5, 4]) all show $D_\lambda(E) \equiv 0$ in the energy intervals where they give localization. For the Bethe lattice (2.6) gives

COROLLARY 2.3 *Let $\mathbb{L} = \mathbb{B}$, and let λ_0, I_λ be as in Theorem 2.2. Then $D_\lambda(E) \equiv \infty$ in I_λ.*

3 THE MATHEMATICAL SETTING

We fix an arbitrary site in $\mathbb{B}$ which we will call the origin and denote by 0. Given two nearest neighbors sites $x, y \in \mathbb{B}$, we will denote by $\mathbb{B}^{(x|y)}$ the lattice obtained by removing from $\mathbb{B}$ the branch emanating from x that passes through y; if we do not specify which branch was removed we will simply write $\mathbb{B}^{(x)}$. Each vertex in $\mathbb{B}^{(x)}$ has degree $K+1$, with the single exception of x which has degree K.

Let $G_\lambda^{(0)}(x,y;z)$ denote the Green's function of the operator H_λ restricted to $\mathbb{B}^{(0)}$ with Dirichlet boundary conditions; we will write $G_\lambda(z)$ and $G_\lambda^{(0)}(z)$ for $G_\lambda(0,0;z)$ and $G_\lambda^{(0)}(0,0;z)$, respectively. For any $\lambda \in \mathbb{R}$, $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta > 0$, we define

$$\begin{aligned} \xi_{\lambda,z}(\varphi_+^2, \varphi_-^2) \quad &= \quad \mathbb{E}\left(\exp\left\{\frac{i}{4}\left(G_\lambda^{(0)}(z)\,\varphi_+^2 \;-\; \overline{G_\lambda^{(0)}(z)}\,\varphi_-^2\right)\right\}\right) &(3.1)\\ &= \quad \mathbb{E}\left(\exp\left\{\frac{1}{4}\left[i\mathcal{R}_\lambda^{(0)}(z)\,(\varphi_+^2 - \varphi_-^2) \;-\mathcal{I}_\lambda^{(0)}(z)\,(\varphi_+^2 + \varphi_-^2)\right]\right\}\right)\,, &(3.2) \end{aligned}$$

where $\mathcal{R}_\lambda^{(0)}(z) + i\mathcal{I}_\lambda^{(0)}(z)$ is the decomposition of $G_\lambda^{(0)}(z)$ into its real and imaginary parts; here $\varphi_\pm \in \mathbb{R}^2$ with $\varphi_\pm^2 = \varphi_\pm \cdot \varphi_\pm$. Notice $\mathcal{I}_\lambda^{(0)}(z) > 0$ since $\eta > 0$.

If $\lambda = 0$ we can calculate $G_0^{(0)}(z)$ obtaining [3]

$$\xi_{0,z}(\varphi_+^2, \varphi_-^2) \quad = \quad \mathrm{e}^{\frac{\mathrm{i}}{2\mathrm{K}}\{(-\mathrm{z}+\sqrt{\mathrm{z}^2-\mathrm{K}})\varphi_+^2 \,-\, \overline{(-\mathrm{z}+\sqrt{\mathrm{z}^2-\mathrm{K}})}\varphi_-^2\}}\,, \tag{3.3}$$

where we always make the choice Im $\sqrt{\ } > 0$. If $|E| < \sqrt{K}$, we have the pointwise limit

$$\xi_{0,E}(\varphi_+^2, \varphi_-^2) \;\equiv\; \lim_{\eta\downarrow 0} \xi_{0,z}(\varphi_+^2, \varphi_-^2) \;=\; \mathrm{e}^{\frac{1}{2\mathrm{K}}\{-\mathrm{iE}(\varphi_+^2-\varphi_-^2)-\sqrt{\mathrm{K}-\mathrm{E}^2}(\varphi_+^2+\varphi_-^2)\}} . \tag{3.4}$$

We now introduce the Banach spaces $\mathcal{K}_p$, $1 \le p \le \infty$, given by the completion of

$$\{g\colon\ [0,\infty)\times[0,\infty) \to \mathbb{C} \ \text{ of class } C^2 \ ;\ \ \|g\|_{\mathcal{K}} \equiv |\!|\!| g |\!|\!|_2 + |\!|\!| g |\!|\!|_p < \infty\} , \tag{3.5}$$

where

$$\begin{aligned} |\!|\!| g |\!|\!|_p^2 \;=\; & \|g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)} \;+\; \|2\partial_+ g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)} \\ & +\; \|2\partial_- g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)} \;+\; \|4\partial_+\partial_- g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)} , \end{aligned}$$

with $\partial_\pm g(\varphi_+^2,\varphi_-^2) \;=\; \frac{\partial}{\partial\varphi_\pm^2} g(\varphi_+^2,\varphi_-^2)$. We have [20, 21] $\xi_{\lambda,z} \in \mathcal{K}_\infty$ for all $\lambda \in \mathbb{R}$ and $z = E + i\eta$ with $\eta > 0$. The map $(\lambda, E, \eta) \to \xi_{\lambda, E+i\eta}$ is continuous from $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ to $\mathcal{K}_\infty$. If $|E| < \sqrt{K}$ we have $\xi_{0,E} \in \mathcal{K}_\infty$ and $\lim_{\eta\downarrow 0} \xi_{0,E+i\eta} \;=\; \xi_{0,E}$ in $\mathcal{K}_\infty$. (Notice that although $\xi_{0,E}$ can be defined for any $E \in \mathbb{R}$ by a pointwise limit, as in (3.4), $\xi_{0,E}$ is in the Banach space $\mathcal{K}_\infty$ if and only if $|E| < \sqrt{K}$.)

We will also use the Hilbert space $\mathcal{L}$, defined as the completion of

$$\{g\colon\ [0,\infty)\times[0,\infty) \to \mathbb{C} \ \text{ measurable};\ \ \|g\|_{\mathcal{L}} \equiv \|g(\varphi_+^2,\varphi_-^2)\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)} < \infty\} . \tag{3.6}$$

We can identify $\mathcal{L}$ with the subspace of $L^2(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)$ consisting of functions depending only on φ_+^2 and φ_-^2, and define $\mathcal{F}$ as the restriction to $\mathcal{L}$ of the Fourier transform on $L^2(\mathbb{R}^2\times\mathbb{R}^2, d^2\varphi_+ d^2\varphi_-)$, so $\mathcal{F}$ is the unitary operator on $\mathcal{L}$ given by

$$(\mathcal{F}g)(\varphi_+^2,\varphi_-^2) = \tfrac{1}{4\pi^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}(\varphi_+\cdot\varphi_+' - \mathrm{i}\varphi_-\cdot\varphi_-')}\, g({\varphi_+'}^2, {\varphi_-'}^2)\, d^2\varphi_+' d^2\varphi_-' . \tag{3.7}$$

We define linear operators

$$\mathcal{T} \;=\; 4\mathcal{F}\partial_+\partial_- \tag{3.8}$$

and

$$\mathcal{B}_{\lambda,z} = M\left(\mathrm{e}^{\mathrm{iE}(\varphi_+^2-\varphi_-^2)-\eta(\varphi_+^2+\varphi_-^2)}\mathrm{h}(\lambda(\varphi_+^2-\varphi_-^2))\right) , \tag{3.9}$$

where $M(g(\varphi_+^2,\varphi_-^2))$ denotes multiplication by the function $g(\varphi_+^2,\varphi_-^2)$. It turns out that $\mathcal{T}$ is unitary on $\mathcal{K}_2$ [7] and a bounded linear operator from $\mathcal{K}_1$ to $\mathcal{K}_\infty$, $\mathcal{B}_{\lambda,z}$ is a bounded linear operator on all $\mathcal{K}_p$, and $g \to g^n$ is a continuous map from $\mathcal{K}_\infty$ to $\mathcal{K}_1$ for any $n = 2, 3, \ldots$. Moreover, $\xi_{\lambda,z}$ satisfies the nonlinear equation in $\mathcal{K}_\infty$:

$$\xi_{\lambda,z} \;=\; \mathcal{T}\mathcal{B}_{\lambda,z}\xi_{\lambda,z}^K , \tag{3.10}$$

not only for all $\lambda \in \mathbb{R}$ and $z = E + i\eta$ with $\eta > 0$, but also for $\eta = 0$ if $\lambda = 0$ and $|E| < \sqrt{K}$. ((3.10) can be verified by writing explicitly the right hand side and performing all integrations.)

The following result is proven in [20, 21] and will play a major role in the proof of Theorem 2.2.

THEOREM 3.1 *For any E such that $|E| < \sqrt{K}$ there exist $\lambda_E > 0$ and $\varepsilon_E > 0$, such that the map $(\lambda, E', \eta) \to \xi_{\lambda,E'+i\eta} \in \mathcal{K}_\infty$, defined for*

$$(\lambda, E', \eta) \in \Big([(-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times (0, \infty)] \bigcup [\{0\} \times (E - \varepsilon_E, E + \varepsilon_E) \times \{0\}] \Big) \ ,$$

has a continuous extension to $(-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times [0, \infty)$ satisfying (3.10).

4 PROOF OF THEOREM 2.2

We start from the equation

$$\mathbb{E}(|G_\lambda\,(0, x; z)|^2) \ = \ \langle \xi_{\lambda,z} \,|\, \mathcal{V}_{\lambda,z}^{|x|}\, \xi_{\lambda,z} \rangle_{\lambda,z} \ \ , \tag{4.1}$$

where

$$\begin{aligned} \langle f \,|\, g \rangle_{\lambda,z} \ &= \ \tfrac{1}{\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(\varphi_+^2, \varphi_-^2) [\mathcal{B}_{\lambda,z} M(\xi_{\lambda,z}^{K-1}) g](\varphi_+^2, \varphi_-^2) d^2\varphi_+ d^2\varphi_- & (4.2) \\ &= \ \tfrac{1}{\pi^2} \langle \bar{f} \,, \mathcal{B}_{\lambda,z} M(\xi_{\lambda,z}^{K-1})\, g \rangle_{\mathcal{L}} & (4.3) \end{aligned}$$

is a bounded symmetric bilinear form on $\mathcal{L}$, and hence on $\mathcal{K}_\infty$, and $\mathcal{V}_{\lambda,z} = \mathcal{F}\mathcal{B}_{\lambda,z} M(\xi_{\lambda,z}^{K-1})$ is easily seen to be a contraction operator on $\mathcal{L}$. A derivation of (4.1) is given in appendix B.

Thus

$$\begin{aligned} J_\lambda\,(E + i\eta) \ &\equiv \ \sum_{x \in \mathbb{B}} |x|^2\, \mathbb{E}(|G_\lambda\,(0, x; E + i\eta)|^2) & (4.4) \\ &= \ \tfrac{K+1}{K} \sum_{r=1}^{\infty} r^2\, \langle \xi_{\lambda,z} \,|\, \mathcal{W}_{\lambda,z}^r\, \xi_{\lambda,z} \rangle \ \ , & (4.5) \end{aligned}$$

where $\mathcal{W}_{\lambda,z} = K\mathcal{V}_{\lambda,z}$.

LEMMA 4.1 *Let $\lambda \in \mathbb{R}$, $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta > 0$.*

(i) *For all $f, g \in \mathcal{L}$*

$$\langle f \,|\, \mathcal{W}_{\lambda,z}\, g \rangle_{\lambda,z} \ = \ \langle \mathcal{W}_{\lambda,z}\, f \,|\, g \rangle_{\lambda,z} \ . \tag{4.6}$$

(ii) *$\mathcal{W}_{\lambda,z}^2$ is a compact operator.*

(iii) *Let*

$$\theta_{\lambda,z} \ = \ -2(\partial_+ + \partial_-)\, \xi_{\lambda,z} \ \ . \tag{4.7}$$

Then $\theta_{\lambda,z} \in \mathcal{L}$ and

$$\mathcal{W}_{\lambda,z}\, \theta_{\lambda,z} \ = \ \theta_{\lambda,z} \ - \ \tfrac{4\eta}{K}\, \mathcal{W}_{\lambda,z}\, \xi_{\lambda,z} \ \ . \tag{4.8}$$

(iv) *For any $r = 0, 1, 2, \ldots$ we have*

$$\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; > \langle \xi_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; > \;\; 0 \tag{4.9}$$

and

$$\langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; > \langle \theta_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; > \;\; 0\,. \tag{4.10}$$

Notice that (4.8) is in [31].

Proof: (i) follows from $\mathcal{F}$ being unitary on $\mathcal{L}$ where we have $\overline{\mathcal{F}f} = \mathcal{F}\bar{f}$. (ii) is a consequence of $\mathcal{B}_{\lambda,z}\mathcal{F}\mathcal{B}_{\lambda,z}$ being a compact operator in $\mathcal{L}$ for $\eta > 0$, which can be shown as in [25, Proposition 9(i)].

To prove (iii), we use (3.10) and (3.8) to get

$$\theta_{\lambda,z} \;\; = \;\; -2(\partial_+ + \partial_-)\, \mathcal{T}\mathcal{B}_{\lambda,z}\xi^K_{\lambda,z} \;\; = \;\; -2(\partial_+ + \partial_-)\,[4\mathcal{F}\partial_+\partial_-\{\mathcal{B}_{\lambda,z}\xi^K_{\lambda,z}\}]\,. \tag{4.11}$$

Since [7, 25]

$$4\partial_\pm \mathcal{F} \partial_\pm \;\; = \;\; \mathcal{F}\,, \tag{4.12}$$

we have

$$\theta_{\lambda,z} \;\; = \;\; \mathcal{F}[-2(\partial_+ + \partial_-)\{\mathcal{B}_{\lambda,z}\xi^K_{\lambda,z}\}] \;\; = \;\; \mathcal{W}_{\lambda,z}\,\theta_{\lambda,z} + \tfrac{4\eta}{K}\,\mathcal{W}_{\lambda,z}\,\xi_{\lambda,z}\,, \tag{4.13}$$

as

$$-2(\partial_+ + \partial_-)\left(\mathrm{e}^{\mathrm{iE}(\varphi_+^2 - \varphi_-^2) - \eta(\varphi_+^2 + \varphi_-^2)}\mathrm{h}(\lambda(\varphi_+^2 - \varphi_-^2))\right) \;\; = \;\; 4\eta\left(\mathrm{e}^{\mathrm{iE}(\varphi_+^2 - \varphi_-^2) - \eta(\varphi_+^2 + \varphi_-^2)}\mathrm{h}(\lambda(\varphi_+^2 - \varphi_-^2))\right)\,. \tag{4.14}$$

We prove that

$$\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; > \;\; 0 \quad \text{and} \quad \langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; > \;\; 0 \tag{4.15}$$

in appendix B. Using (4.8), we have

$$\begin{aligned} \langle \xi_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; &= \;\; \langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \; - \; \tfrac{4\eta}{K}\langle \xi_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \xi_{\lambda,z} \rangle_{\lambda,z} && (4.16)\\ &< \;\; \langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z}\,, && (4.17) \end{aligned}$$

since $\langle \xi_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \xi_{\lambda,z} \rangle_{\lambda,z} > 0$ by (4.1). Similarly,

$$\begin{aligned} \langle \theta_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \;\; &= \;\; \langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z} \; - \; \tfrac{4\eta}{K}\langle \theta_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \xi_{\lambda,z} \rangle_{\lambda,z} && (4.18)\\ &< \;\; \langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\, \theta_{\lambda,z} \rangle_{\lambda,z}\,, && (4.19) \end{aligned}$$

since $\langle \theta_{\lambda,z} \,|\, \mathcal{W}^{r+1}_{\lambda,z}\, \xi_{\lambda,z} \rangle_{\lambda,z} > 0$ by (4.15) and (4.6). Thus (iv) is proven. ∎

LEMMA 4.2 *Let $\lambda \in \mathbb{R}$, $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta > 0$. Then*

$$J_\lambda\,(E + i\eta) = \tfrac{(K+1)}{4\eta}\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} + \tfrac{K(K+1)}{16\eta^2}\left(\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} + 2\sum_{r=0}^{\infty}\langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z}\right). \quad (4.20)$$

Proof: Using (4.8), we have

$$\sum_{r=1}^{R} r^2\,\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\xi_{\lambda,z}\rangle_{\lambda,z} \;=\; \tfrac{K}{4\eta}\sum_{r=1}^{R} r^2\,\langle \xi_{\lambda,z} \,|\, \mathcal{W}^{r-1}_{\lambda,z}\,(I - \mathcal{W}_{\lambda,z})\,\theta_{\lambda,z}\rangle_{\lambda,z} \quad (4.21)$$

$$=\; \tfrac{K}{4\eta}\left\{\sum_{r=0}^{R-1}(2r+1)\,\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \;-\; R^2\langle \xi_{\lambda,z} \,|\, \mathcal{W}^R_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z}\right\}. \quad (4.22)$$

Since $\mathcal{W}^2_{\lambda,z}$ is a compact operator and $\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z}$ is not constant and has a limit as $r \to \infty$ by (4.9), we can apply Theorem C.1 to conclude that $\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \to 0$ exponentially fast, so we have

$$J_\lambda\,(E + i\eta) \;=\; \tfrac{K+1}{4\eta}\sum_{r=0}^{\infty}(2r+1)\,\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z}\,. \quad (4.23)$$

Repeating the procedure, we get

$$\sum_{r=1}^{R}(2r+1)\,\langle \xi_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \;=\; \tfrac{K}{4\eta}\sum_{r=1}^{R}(2r+1)\,\langle (I - \mathcal{W}_{\lambda,z})\,\theta_{\lambda,z} \,|\, \mathcal{W}^{r-1}_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \quad (4.24)$$

$$=\; \tfrac{K}{4\eta}\left\{\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} \;+\; 2\sum_{r=0}^{R-1}\langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \;-\; (2R+1)\langle \theta_{\lambda,z} \,|\, \mathcal{W}^R_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z}\right\} \quad (4.25)$$

As before, we use (4.10), the compactness of $\mathcal{W}^2_{\lambda,z}$ and Theorem C.1 to let $R \to \infty$ in (4.25). Combining the resulting expression with (4.25) we get (4.20). ∎

We now need the following extensions of (4.8).

LEMMA 4.3 *Let $\lambda \in \mathbb{R}$, $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta > 0$. Then for any $r = 1, 2, \ldots$ we have*

$$\mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z} \;=\; \theta_{\lambda,z} \;-\; \tfrac{4\eta}{K}\sum_{s=1}^{r}\mathcal{W}^s_{\lambda,z}\,\xi_{\lambda,z}\,. \quad (4.26)$$

and

$$\langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \;=\; \langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} \;-\; \tfrac{4\eta}{K}\,r\,\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} \;+\; \tfrac{16\eta^2}{K^2}\sum_{s=1}^{r}(r - s + 1)\,\langle \xi_{\lambda,z} \,|\, \mathcal{W}^s_{\lambda,z}\,\xi_{\lambda,z}\rangle_{\lambda,z}\,. \quad (4.27)$$

Proof: (4.26) follows immediately from (4.8) by induction. Using (4.26) twice, we get

$$\begin{aligned}\langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} &= \langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} - \tfrac{4\eta}{K}\langle \xi_{\lambda,z} \,|\, \sum_{u=1}^{r} \mathcal{W}^u_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \\ &= \langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} - \tfrac{4\eta}{K}\, r\, \langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} + \tfrac{16\eta^2}{K^2} \sum_{u=1}^{r}\sum_{s=1}^{u} \langle \xi_{\lambda,z} \,|\, \mathcal{W}^s_{\lambda,z}\,\xi_{\lambda,z}\rangle_{\lambda,z}\,, \end{aligned} \tag{4.28}$$

so (4.27) follows. ∎

LEMMA 4.4 *Let $\lambda \in \mathbb{R}$, $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta > 0$. Then*

$$\begin{aligned} J_\lambda(E+i\eta) \;\geq& \\ &\tfrac{(K+1)}{4\eta}\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} + \tfrac{3K(K+1)}{16\eta^2}\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} + \tfrac{K^2(K+1)}{64\eta^3}\frac{\left(\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} - \frac{4\eta}{K}\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}\right)^2}{\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}}\,. \end{aligned} \tag{4.29}$$

Proof: From (4.27) and (4.15) we have

$$\sum_{r=1}^{\infty}\langle \theta_{\lambda,z} \,|\, \mathcal{W}^r_{\lambda,z}\,\theta_{\lambda,z}\rangle_{\lambda,z} \geq \sum_{r=1}^{\infty}\left(\left\{\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} - \tfrac{4\eta}{K}\, r\, \langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}\right\} \vee 0\right) \tag{4.30}$$

$$= \sum_{r=1}^{R_{\lambda,z}}\left(\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} - \tfrac{4\eta}{K}\, r\, \langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}\right) \tag{4.31}$$

$$\geq \frac{K\left(\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z} - \frac{4\eta}{K}\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}\right)^2}{8\eta\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}}\,, \tag{4.32}$$

where

$$R_{\lambda,z} = \left[\frac{K\langle \theta_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}}{4\eta\langle \xi_{\lambda,z} \,|\, \theta_{\lambda,z}\rangle_{\lambda,z}}\right]\,, \tag{4.33}$$

with $x \vee y$ the maximum of x and y and $[x]$ the largest integer $\leq x$.

(4.29) now follows from (4.20) and (4.32). ∎

We are ready to prove Theorem 2.2. If $|E| < \sqrt{K}$, (3.4) gives

$$\theta_{0,E} = \tfrac{2\sqrt{K-E^2}}{K}\,\xi_{0,E} \tag{4.34}$$

and

$$\langle \xi_{0,E} \,|\, \xi_{0,E}\rangle_{0,E} = |G_0(0,0;E)|^2 = \frac{4K}{(K+1)^2 - 4E^2} > \frac{4K}{(K-1)^2}\,. \tag{4.35}$$

Thus

$$\langle \xi_{0,E} \,|\, \theta_{0,E}\rangle_{0,E} = \frac{8\sqrt{K-E^2}}{(K+1)^2 - 4E^2} > 0 \tag{4.36}$$

and

$$\langle\theta_{0,E}\,|\,\theta_{0,E}\rangle_{0,E} \;=\; \frac{16(K-E^2)}{K((K+1)^2-4E^2)} \;>\; 0\,. \tag{4.37}$$

We now use Theorem 3.1, for each E such that $|E| < \sqrt{K}$ we can pick $\lambda_E > 0$ and $\varepsilon_E > 0$ so the map $(\lambda, E', \eta) \to \xi_{\lambda,E'+i\eta} \in \mathcal{K}_\infty$ has a continuous extension to $(-\lambda_E, \lambda_E) \times (E-\varepsilon_E, E+\varepsilon_E) \times [0,\infty)$ satisfying (3.10). It follows that the map $(\lambda, E', \eta) \to \theta_{\lambda,E'+i\eta} \in \mathcal{L}$ also has a continuous extension to $(-\lambda_E, \lambda_E) \times (E-\varepsilon_E, E+\varepsilon_E) \times [0,\infty)$ and (4.7) holds. In addition, the complex valued maps $(\lambda, E', \eta) \to \langle\xi_{\lambda,E'+i\eta}\,|\,\theta_{\lambda,E'+i\eta}\rangle_{\lambda,E'+i\eta}$ and $(\lambda, E', \eta) \to \langle\theta_{\lambda,E'+i\eta}\,|\,\theta_{\lambda,E'+i\eta}\rangle_{\lambda,E'+i\eta}$ have continuous extensions to $(-\lambda_E, \lambda_E) \times (E-\varepsilon_E, E+\varepsilon_E) \times [0,\infty)$ such that

$$\langle\xi_{\lambda,E'+i\eta}\,|\,\theta_{\lambda,E'+i\eta}\rangle_{\lambda,E'+i\eta} \;>\; 0 \quad \text{and} \quad \langle\theta_{\lambda,E'+i\eta}\,|\,\theta_{\lambda,E'+i\eta}\rangle_{\lambda,E'+i\eta} \;>\; 0\,, \tag{4.38}$$

if $\lambda_E > 0$ and $\varepsilon_E > 0$ were chosen sufficiently small.

Theorem 2.2 now follows from (4.29). ∎

A TIME EVOLUTION AND GREEN'S FUNCTIONS

In this appendix we prove some useful facts. We consider the Hamiltonian

$$H_\lambda \;=\; \tfrac{1}{2}\Delta + V \quad \text{on} \quad \ell^2(\mathbb{L})\,, \tag{A.1}$$

where V is a fixed potential; we define $r^2(t)$ as in(1.4).

We start with a result of Simon [35].

LEMMA A.1 *For any $t \in \mathbb{R}$ we have*

$$r^2(t) \;\le\; \|\tfrac{1}{2}\Delta\|^2\, t^2\,. \tag{A.2}$$

Proof: Let $|x|$ denote also the operator multiplication by $|x|$. We have

$$r^2(t) \;=\; \||x|e^{itH}\delta_0\|^2 \;=\; \|e^{-itH}|x|e^{itH}\delta_0\|^2\,. \tag{A.3}$$

Using the Fundamental Theorem of Calculus we get

$$r^2(t) \;=\; \|\int_0^t e^{-isH}[H,|x|]e^{isH}\delta_0\,ds\|^2 \;\le\; \|[\tfrac{1}{2}\Delta,|x|]\|^2\,t^2 \;\le\; \|\tfrac{1}{2}\Delta\|^2\,t^2\,. \tag{A.4}$$

∎

The next well known Lemma gives the Laplace transform of $r^2(t)$ in terms of the Green's function $G(x,y;z) \;=\; \langle\delta_x, (H-z)^{-1}\delta_y\rangle$ (e.g., [14, 28]).

LEMMA A.2 *For any $\eta > 0$ we have*

$$\int_0^\infty \mathrm{e}^{-\eta t}\, \mathrm{r}^2(\mathrm{t})\, \mathrm{dt} \;=\; \tfrac{1}{2\pi} \int_{-\infty}^\infty \{\sum_{\mathrm{x}\in\mathbb{L}} |\mathrm{x}|^2\, |\mathrm{G}\,(0,\mathrm{x};\mathrm{E}+\mathrm{i}\tfrac{\eta}{2})|^2\}\, \mathrm{dE}\,. \tag{A.5}$$

Proof: By the spectral theorem,

$$(H - E + i\tfrac{\eta}{2})^{-1} \;=\; \int_0^\infty \mathrm{e}^{\mathrm{itE}} \mathrm{e}^{-\mathrm{it}(\mathrm{H}-\mathrm{i}\frac{\eta}{2})}\, \mathrm{dt}\,. \tag{A.6}$$

Taking matrix elements of both sides and applying Plancherel's Theorem we get

$$\int_0^\infty \mathrm{e}^{-\eta t}\, |\mathrm{e}^{\mathrm{itH}}(0,\mathrm{x})|^2\, \mathrm{dt} \;=\; \tfrac{1}{2\pi} \int_{-\infty}^\infty |\mathrm{G}\,(0,\mathrm{x};\mathrm{E}+\mathrm{i}\tfrac{\eta}{2})|^2\, \mathrm{dE}\,. \tag{A.7}$$

The lemma follows. ∎

¿From (A.7) we obtain

$$\int_{-\infty}^\infty |G\,(0,x;E+i\tfrac{\eta}{2})|^2\, dE \;\le\; \frac{2\pi}{\eta}\,. \tag{A.8}$$

Combining Lemmas A.1 and A.2 we get

$$\int_{-\infty}^\infty \{\sum_{x\in\mathbb{L}} |x|^2\, |G\,(0,x;E+i\tfrac{\eta}{2})|^2\}\, dE \;\le\; \frac{4\pi}{\eta^3}\, \|\tfrac{1}{2}\Delta\|^2\,. \tag{A.9}$$

This last estimate should be compared with

$$\sup_{E\in\mathbb{R}} \sum_{x\in\mathbb{L}} |x|^2\, |G\,(0,x;E+i\tfrac{\eta}{2})|^2 \;\le\; \frac{4}{\eta^4}\, \|\tfrac{1}{2}\Delta\|^2\,, \tag{A.10}$$

which follows from Lemma A.1 and the inequality

$$\sum_{x\in\mathbb{L}} |x|^2\, |G\,(0,x;E+i\tfrac{\eta}{2})|^2 \;=\; \||x|(H-(E+i\tfrac{\eta}{2}))^{-1}\delta_0\|^2 \tag{A.11}$$

$$\;=\; \|\int_0^\infty |x| \mathrm{e}^{\mathrm{itE}} \mathrm{e}^{-\mathrm{it}(\mathrm{H}-\mathrm{i}\frac{\eta}{2})}\, \mathrm{dt}\|^2 \tag{A.12}$$

$$\;\le\; \Big(\int_0^\infty \mathrm{e}^{-\frac{\eta}{2}\mathrm{t}} [\mathrm{r}^2(\mathrm{t})]^{\frac{1}{2}}\, \mathrm{dt}\Big)^2\,. \tag{A.13}$$

B THE SUPERSYMMETRIC REPLICA TRICK

In this appendix we derive (4.1) and (4.15), using the supersymmetric replica trick (see [18, 24, 31]).

Let Λ be a finite subset of $\mathbb{B}$, we will use $H_{\lambda,\Lambda}$ to denote the operator H_λ restricted to $\ell^2(\Lambda)$ with Dirichlet boundary conditions. The corresponding Green's function is

$$G_{\lambda,\Lambda}(x,y;z) \;=\; \left\langle \delta_x, (H_{\lambda,\Lambda}-z)^{-1}\delta - y \right\rangle \;, \quad x,y \in \Lambda \text{ and } \; z = E + i\eta \text{ with } E \in \mathbb{R},\ \eta > 0\,. \tag{B.1}$$

The supersymmetric replica trick (see[18]) says that if $x_1, x_2 \in \Lambda$, then for all $z \in \mathbb{C}$ with $\eta = \operatorname{Im} z > 0$, we have

$$G_{\lambda,\Lambda}(x_1,x_2;z) \;=\; i\int \psi(x_1)\bar{\psi}(x_2)\exp\{-i\sum_{x\in\Lambda}\Phi(x)\cdot[(H_{\lambda,\Lambda}-z)\Phi](x)\}D_\Lambda\Phi\;, \tag{B.2}$$

where $\Phi(x) = (\varphi(x), \psi(x), \bar{\psi}(x))$ with $\varphi(x) \in \mathbb{R}^2$ and $\psi(x)$, $\bar{\psi}(x)$ anticommuting "variables" (i.e., elements of a Grassman algebra),

$$\Phi(x)\cdot\Phi(y) \;=\; \varphi(x)\cdot\varphi(y) \;+\; \frac{1}{2}(\bar{\psi}(x)\psi(y) + \bar{\psi}(y)\psi(x))\,.$$

and

$$D_\Lambda\Phi \;=\; \prod_{x\in\Lambda} d\Phi(x) \quad \text{with} \quad d\Phi(x) \;=\; \frac{1}{\pi}d\bar{\psi}(x)d\psi(x)d^2\varphi(x)\,.$$

To compute functions of $\psi, \bar{\psi}$ we expand in a power series that terminates after a finite number of terms due to the anticommutativity. The linear functional denoted by integration against $d\bar{\psi}(x)d\psi(x)$ is defined by

$$\int (a_0 + a_1\psi(x) + a_2\bar{\psi}(x) + a_3\bar{\psi}(x)\psi(x))\,d\bar{\psi}(x)d\psi(x) \;=\; -a_3\,. \tag{B.3}$$

We will use the (bad, but convenient) notation $\Phi(x)^2 = \Phi(x)\cdot\Phi(x)$ and $\varphi(x)^2 = \varphi(x)\cdot\varphi(x)$. We will also denote a generic $\Phi(x) = (\varphi(x), \psi(x), \bar{\psi}(x))$ by $\Phi = (\varphi, \psi, \bar{\psi})$. Notice that if $f\colon\ [0,\infty) \to \mathbb{C}$ is continuously differentiable, then $f(\Phi^2) = f(\varphi^2) + f'(\varphi^2)\bar{\psi}\psi$.

The lattice $\mathbb{B}_\ell$ will consist of all sites in $\mathbb{B}$ whose distance from the fixed origin 0 is less than or equal to ℓ; similarly $\mathbb{B}_\ell^{(x|y)}$, $\mathbb{B}_\ell^{(x)}$ will denote all sites in $\mathbb{B}^{(x|y)}$, $\mathbb{B}^{(x)}$, respectively, whose distance from x is less than or equal to ℓ. (For convenience we also allow $\ell = \infty$, in which case it may be omitted from the notation.) We will write $H_{\lambda,\ell}$, $H_{\lambda,\ell}^{(x|y)}$ and $H_{\lambda,\ell}^{(x)}$ for $H_{\lambda,\mathbb{B}_\ell}$, $H_{\lambda,\mathbb{B}_\ell^{(x|y)}}$ and $H_{\lambda,\mathbb{B}_\ell^{(x)}}$, respectively. Similarly, we will use $G_{\lambda,\ell}(x,y;z)$ for $G_{\lambda,\mathbb{B}_\ell}(x,y;z)$ and $G_{\lambda,\ell}(z)$, $G_{\lambda,\ell}^{(x|y)}(z)$, $G_{\lambda,\ell}^{(x)}(z)$ for $G_{\lambda,\ell}(0,0;z)$, $G_{\lambda,\mathbb{B}_\ell^{(x|y)}}(x,x;z)$, $G_{\lambda,\mathbb{B}_\ell^{(x)}}(x,x;z)$, respectively. We have (see [3, Proposition 1.2])

$$\lim_{\ell\to\infty} G_{\lambda,\ell}(x,y;z) \;=\; G_\lambda(x,y;z) \quad \text{for any } x,y \in \mathbb{B},\ E \in \mathbb{R} \text{ and } \eta > 0\;, \tag{B.4}$$

with similar limits for $G_{\lambda,\ell}(z)$, $G_{\lambda,\ell}^{(x|y)}(z)$ and $G_{\lambda,\ell}^{(x)}(z)$.

We now fix $x \in \mathbb{B}$ and set $r = |x| \equiv d(x,0)$. Let $x_0, x_1, \ldots, x_r$ be the shortest path from 0 to x, i.e., $x_0 = 0$, $x_r = x$ and $d(x_n, x_{n+1}) = 1$ for $n = 0, 1, \ldots, r-1$, with $x_n \neq x_{n'}$

for $n \neq n'$. For each $n = 0, 1, \ldots, r-1$ let $J_n = \{1, \ldots, K-1\}$ for $n = 1, \ldots, r-1$ and $J_n = \{0, 1, \ldots, K-1\}$ for $n = 0, r$; we write $\{x_n^{(j)}; \, j \in J_n\}$ for the collection of nearest neighbors sites to x_n which are not in the path $x_0, x_1, \ldots, x_r$. By $\mathbb{B}_{x,\ell}$ we denote the lattice consisting of all sites in $\mathbb{B}$ whose distance from the path $x_0, x_1, \ldots, x_r$ is at most ℓ; notice that as a set

$$\mathbb{B}_{x,\ell} \;=\; \{x_0, x_1, \ldots, x_r\} \cup \left[\cup_{n=0}^{r} \cup_{j \in J_n} \mathbb{B}_\ell^{(x_n^{(j)}|x_n)} \right] .$$

We write $H_{\lambda,x,\ell}$ and $G_{\lambda,x,\ell}(0,x;z)$ for $H_{\lambda,\mathbb{B}_{x,\ell}}$ and $G_{\lambda,\mathbb{B}_{x,\ell}}(0,x;z)$, respectively. As in (B.4), we have

$$\lim_{\ell \to \infty} G_{\lambda,x,\ell}(0,x;z) \;=\; G_\lambda(0,x;z) \,. \tag{B.5}$$

If $\ell < \infty$ are finite and $\operatorname{Im} z > 0$, (B.2) gives

$$G_{\lambda,x,\ell}(0,x;z) \;=\; i \int \psi(0)\bar{\psi}(x) \left\{ \prod_{n=0}^{r-1} \mathrm{e}^{-\mathrm{i}\Phi(\mathrm{x_n})\cdot\Phi(\mathrm{x_{n+1}})} \right\} \left\{ \prod_{n=0}^{r} \Xi_n \right\} D_{\mathbb{B}_{x,\ell}}\Phi \,, \tag{B.6}$$

where

$$\Xi_n \;=\; \mathrm{e}^{\mathrm{i}(\mathrm{z}-\lambda \mathrm{V}(\mathrm{x_n}))\Phi(\mathrm{x_n})^2} \exp\{-\mathrm{i} \sum_{\mathrm{j}\in \mathrm{J_n}} (\Phi(\mathrm{x_n}) \cdot \Phi(\mathrm{x_n^{(j)}}) + \sum_{\mathrm{y}\in\mathbb{B}_\ell^{(\mathrm{x_n^{(j)}}|\mathrm{x_n})}} \Phi(\mathrm{y}) \cdot [(\mathrm{H}_{\lambda,\ell}^{(\mathrm{x_n^{(j)}}|\mathrm{x_n})} - \mathrm{z})\Phi](\mathrm{y}))\} \,. \tag{B.7}$$

By an explicit computation,

$$\int \exp\{-i(\Phi(x_n)\cdot\Phi(x_n^{(j)}) + \sum_{y \in \mathbb{B}_\ell^{(x_n^{(j)}|x_n)}} \Phi(y) \cdot [(H_{\lambda,\ell}^{(x_n^{(j)}|x_n)} - z)\Phi](y))\} \, D_{\mathbb{B}_\ell^{(x_n^{(j)}|x_n)}}\Phi$$
$$=\; \exp\{\frac{i}{4} G_{\lambda,\ell}^{(x_n^{(j)}|x_n)}(z)\, \Phi(x_n)^2\} \,. \tag{B.8}$$

Thus

$$G_{\lambda,x,\ell}(0,x;z) \;= \tag{B.9}$$
$$i \int \psi(0)\bar{\psi}(x) \left\{ \prod_{n=0}^{r-1} \mathrm{e}^{-\mathrm{i}\Phi(\mathrm{x_n})\cdot\Phi(\mathrm{x_{n+1}})} \right\} \left\{ \prod_{n=0}^{r} \Upsilon_{\ell,n}(\Phi(x_n)^2) \right\} \prod_{n=0}^{r} d\Phi(x_n) \,,$$

where

$$\Upsilon_{\ell,n}(\varphi^2) \;=\; \exp\{i[z - \lambda V(x_n) + \tfrac{1}{4} \sum_{j \in J_n} G_{\lambda,\ell}^{(x_n^{(j)}|x_n)}(z)]\, \varphi^2\} \,, \;\; n = 0, 1, \ldots, r \,. \tag{B.10}$$

We now let $\ell \to \infty$, obtaining

$$G_\lambda(0,x;z) \;= \tag{B.11}$$
$$i \int \psi(0)\bar{\psi}(x) \left\{ \prod_{n=0}^{r-1} \mathrm{e}^{-\mathrm{i}\Phi(\mathrm{x_n})\cdot\Phi(\mathrm{x_{n+1}})} \right\} \left\{ \prod_{n=0}^{r} \Upsilon_n(\Phi(x_n)^2) \right\} \prod_{n=0}^{r} d\Phi(x_n) \,.$$

We can now integrate over the anticommuting variables, getting

$$G_\lambda(0,x;z) \;=\; \tag{B.12}$$
$$\frac{i}{\pi}\int \left\{ \prod_{n=0}^{r-1} \frac{i}{2\pi}\, \mathrm{e}^{-\mathrm{i}\varphi(\mathrm{x_n})\cdot\varphi(\mathrm{x_{n+1}})} \right\} \left\{ \prod_{n=0}^{r} \Upsilon_n\left(\varphi(x_n)^2\right) \right\} \prod_{n=0}^{r} d^2\varphi(x_n)\ .$$

Thus

$$|G_\lambda(0,x;z)|^2 \;=\; \tag{B.13}$$
$$\frac{1}{\pi^2}\int \prod_{n=0}^{r-1} \frac{1}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}(\varphi_+(\mathrm{x_n})\cdot\varphi_+(\mathrm{x_{n+1}})-\varphi_-(\mathrm{x_n})\cdot\varphi_-(\mathrm{x_{n+1}}))} \prod_{\mathrm{n=0}}^{\mathrm{r}} \Gamma_\mathrm{n}(\varphi_+(\mathrm{x_n})^2,\varphi_-(\mathrm{x_n})^2) \prod_{\mathrm{n=0}}^{\mathrm{r}} \mathrm{d}^2\varphi_+(\mathrm{x_n})\mathrm{d}^2\varphi_-(\mathrm{x_n})\ ,$$

where

$$\Gamma_n(\varphi_+^2,\varphi_-^2) \;=\; \Upsilon_n\left(\varphi_+^2\right) \overline{\Upsilon_n\left(\varphi_-^2\right)}\ . \tag{B.14}$$

Since the $\{V(x);\ x\in\mathbb{B}\}$ are independent random variables, we have

$$\mathbb{E}(|G_\lambda(0,x;z)|^2) \;=\; \tag{B.15}$$
$$\frac{1}{\pi^2}\int \prod_{n=0}^{r-1} \frac{1}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}(\varphi_+(\mathrm{x_n})\cdot\varphi_+(\mathrm{x_{n+1}})-\varphi_-(\mathrm{x_n})\cdot\varphi_-(\mathrm{x_{n+1}}))} \prod_{\mathrm{n=0}}^{\mathrm{r}} \mathbb{E}(\Gamma_\mathrm{n}(\varphi_+^2,\varphi_-^2)) \prod_{\mathrm{n=0}}^{\mathrm{r}} \mathrm{d}^2\varphi_+(\mathrm{x_n})\mathrm{d}^2\varphi_-(\mathrm{x_n})\ ,$$

with

$$\mathbb{E}(\Gamma_n(\varphi_+^2,\varphi_-^2)) \;=\; \mathrm{e}^{\mathrm{iE}(\varphi_+^2-\varphi_-^2)-\eta(\varphi_+^2+\varphi_-^2)}\mathrm{h}(\lambda(\varphi_+^2-\varphi_-^2))\,(\xi_{\lambda,z}(\varphi_+^2,\varphi_-^2))^{|\mathrm{J_n}|} \tag{B.16}$$

where $|J_n| = K-1$ if $n = 1,\ldots,r-1$ and $|J_n| = K$ for $n = 0, r$, where $\xi_{\lambda,z}$ is defined in (3.1).

A moment of reflexion now reveals that (4.1) is simply an elegant (and useful) way of rewriting (B.15). Moreover, we have from (3.1) and (4.7) that

$$\theta_{\lambda,z} \;=\; \mathbb{E}\left(\mathcal{I}_\lambda^{(0)}(z)\,\exp\left\{\frac{i}{4}\left(G_\lambda^{(0)}(z)\,\varphi_+^2 \;-\; \overline{G_\lambda^{(0)}(z)}\,\varphi_-^2\right)\right\}\right)\ . \tag{B.17}$$

Proceeding as in the derivation of (B.15), we get

$$\langle\theta_{\lambda,z}\,|\,\mathcal{V}_{\lambda,z}^{|x|}\,\xi_{\lambda,z}\rangle_{\lambda,z} \;=\; \mathbb{E}(\mathcal{I}_\lambda^{(0^{(0)}|0)}(z)\,|G_\lambda(0,x;z)|^2) \;>\; 0 \tag{B.18}$$

and

$$\langle\theta_{\lambda,z}\,|\,\mathcal{V}_{\lambda,z}^{r}\,\theta_{\lambda,z}\rangle_{\lambda,z} \;=\; \mathbb{E}(\mathcal{I}_\lambda^{(0^{(0)}|0)}(z)\,\mathcal{I}_\lambda^{(x^{(0)}|x)}(z)\,|G_\lambda(0,x;z)|^2) \;>\; 0\ , \tag{B.19}$$

proving (4.15).

C SOME SPECTRAL ANALYSIS

Let $\mathcal{X}$ be a Banach space. A bounded linear operator A on $\mathcal{X}$ will be said to be of *type K* if A has discrete spectrum outside some closed set contained in the open unit disk in the complex plane, i.e., we have

$$\sigma(A) \;=\; \Lambda \cup \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \;, \tag{C.1}$$

where $\Lambda \subset D = \{z \in \mathbb{C};\ |z| < 1\}$ is closed in $\mathbb{C}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset \mathbb{C} \backslash D$ is a finite collection of isolated eigenvalues with finite multiplicity. Powers of compact operators and, more generally, Riesz operators are of type K.

THEOREM C.1 *Let A be a bounded linear operator of type K on the Banach space $\mathcal{X}$ and let us fix $x \in \mathcal{X}$ and $\ell \in \mathcal{X}^*$. We set*

$$\alpha(n) \;=\; \ell(A^n x) \quad \text{for} \;\; n = 1, 2, \ldots \;. \tag{C.2}$$

If $\alpha(n)$ is not constant in n and $\lim_{n\to\infty} \alpha(n)$ exists in $\mathbb{C}$, then $\alpha(n) \to 0$ exponentially fast as $n \to \infty$, i.e., there exist $C < \infty$ and $\gamma > 0$ such that

$$|\alpha(n)| \;\leq\; C\, \mathrm{e}^{-\gamma \mathrm{n}} \quad \text{for all} \;\; \mathrm{n} = 1, 2, \ldots \;. \tag{C.3}$$

Proof: By redefining Λ we can assume $|\lambda_i| \geq 1$ in (C.1) for all $i = 1, 2, \ldots, k$. It follows from (C.1) that (see [17, section III.6])

$$A \;=\; R \;+\; \sum_{i=1}^{k} (\lambda_i P_i + D_i) \;, \tag{C.4}$$

where R and P_i, D_i, $i = 1, 2, \ldots, k$, are bounded linear operators on $\mathcal{X}$ such that:

(i)

$$P_i P_j \;=\; \delta_{ij} P_i \quad \text{for all} \;\; i, j = 1, 2, \ldots, k \;. \tag{C.5}$$

(ii) For each $i = 1, 2, \ldots, k$

$$P_i D_i \;=\; D_i P_i \;=\; D_i \tag{C.6}$$

and

$$D_i^{h_i} \;=\; 0 \quad \text{for some positive integer} \;\; h_i \;. \tag{C.7}$$

(iii)

$$\sigma(R) \;=\; \Lambda \tag{C.8}$$

and

$$P_i R \;=\; R P_i \;=\; 0 \quad \text{for each} \;\; i = 1, 2, \ldots, k \;. \tag{C.9}$$

It follows that

$$A^n \;=\; R^n \;+\; \sum_{i=1}^{k} (\lambda_i P_i + D_i)^n \quad \text{for all} \;\; n = 1, 2, \ldots . \tag{C.10}$$

Since the spectral radius of R is < 1, $\ell(R^n x)$ can be bounded as in (C.3). Thus it suffices to consider the case when $R = 0$, in which case we will show that $\alpha(n)$ does not have a limit in $\mathbb{C}$ as $n \to \infty$, unless it is constant in n.

¿From (C.6) and(C.7) we have

$$(\lambda_i P_i + D_i)^n \;=\; \lambda_i^n P_i \;+\; \sum_{s=1}^{h_i - 1} \binom{n}{s} \lambda_i^{n-s} D_i^s \; . \tag{C.11}$$

for any $n \geq h_i - 1$. It then follows that

$$\ell((\lambda_i P_i + D_i)^n x) \;=\; \sum_{s=0}^{h_i - 1} \binom{n}{s} \lambda_i^{n-s} a_{i,s} \tag{C.12}$$

for some complex numbers $a_{i,s}$, $s = 0, 1, \ldots, h_i - 1$. If they are not all zero, we can choose $s_i \in \{0, 1, \ldots, h_i - 1\}$ such that

$$\ell((\lambda_i P_i + D_i)^n x) \;=\; \binom{n}{s_i} \lambda_i^n (b_i \;+\; \beta_i(n)) \tag{C.13}$$

for some complex numbers $b_i \neq 0$ and $\beta_i(n)$ with $\lim_{n\to\infty} \beta_i(n) = 0$.

Thus, if $R = 0$ and $\alpha(n) \not\equiv 0$, we have, for all $n \geq \sup_{i=1,2,\ldots,k}(h_i - 1)$, that

$$\alpha(n) \;=\; \binom{n}{s} r^n \left(\sum_{j=1}^{t} c_j \mathrm{e}^{\mathrm{i}2\pi\theta_j \mathrm{n}} \;+\; \beta(\mathrm{n}) \right) \; , \tag{C.14}$$

for some $r \geq 1$, nonnegative integer s, positive integer t, distinct $\theta_j \in [0, 1)$, $j = 1, 2, \ldots, t$, and $\beta(n)$ with $\lim_{n\to\infty} \beta(n) = 0$. If $t = 1$, we have $\lim_{n\to\infty} \alpha(n) = \infty$, unless $r = 1$ and $s = 0$, in which case we also have $\beta(n) = 0$, so $\alpha(n)$ is either constant or has no limit as $n \to \infty$. If $t > 1$, we use Lemma C.2 below to conclude that $\limsup_{n\to\infty} |\alpha(n)| = \infty$, unless $r = 1$ and $s = 0$, in which case $|\alpha(n)|$ is bounded but $\alpha(n)$ has no limit as $n \to \infty$. ∎

LEMMA C.2 *Let $\theta_j \in [0, 1)$, $j = 1, 2, \ldots, t$ be distinct ($t > 1$) and c_j, $j = 1, 2, \ldots, t$ be nonzero complex numbers. Then $\Gamma(n) \equiv \sum_{j=1}^{t} c_j \mathrm{e}^{\mathrm{i}2\pi\theta_j \mathrm{n}}$ has no limit as $n \to \infty$.*

Proof: If $\theta_j \in [0, 1)$, $j = 1, 2, \ldots, t$ are all rationals, then the range of $\Gamma(n)$ consists of a finite number (at least 2, since $t > 1$) of values, each taken infinitely many times. It thus suffices to prove that if $\theta_j \in [0, 1)$, $j = 1, 2, \ldots, t$ are all irrationals, then $\Gamma(n)$ has an infinite number of distinct limit points as $n \to \infty$.

So suppose $\theta_j \in [0,1)$, $j = 1,2,\ldots,t$ are all irrationals. Relabeling the indices if necessary, we can find $u \in \{1,2,\ldots,t\}$ such that $1,\theta_1,\theta_2,\ldots,\theta_u$ are linearly independent over the integers, and there exist linear functions f_j on $\mathbb{R}^{(u+1)}$ with integer coefficients, $j = u+1, u+2, \ldots, t$, such that $\theta_j = f_j(1,\theta_1,\theta_2,\ldots,\theta_u)$ for $j = u+1, u+2, \ldots, t$. We define the complex valued function F on the u-dimensional torus $\mathbb{T}^u$ by

$$F(\varphi_1,\varphi_2,\ldots,\varphi_u) \;=\; \sum_{j=1}^{t} c_j e^{i2\pi\varphi_j} \;\text{ with }\; \varphi_{\rm j} = {\rm f_j}(1,\varphi_1,\varphi_2,\ldots,\varphi_{\rm u}) \;\text{ for }\; {\rm j = u+1, u+2, \ldots, t}\ .$$

We also let τ be the rotation on $\mathbb{T}^u$ given by

$$\tau(\varphi_1,\varphi_2,\ldots,\varphi_u) \;=\; (\varphi_1+\theta_1, \varphi_2+\theta_2, \ldots, \varphi_u+\theta_u)\ ,$$

all sums being mod 1. As the f_j, $j = u+1, u+2, \ldots, t$, are linear functions, we have

$$\Gamma(n) \;=\; F(\tau^n(0,0,\ldots,0))\ .$$

Since $1,\theta_1,\theta_2,\ldots,\theta_u$ are linearly independent over the integers, τ is a minimal ergodic transformation on $\mathbb{T}^u$, and hence the orbit of any point in $\mathbb{T}^u$ visits each neighborhood infinitely many times. F being continuous, it follows that any point on its range is a limit point for $\Gamma(n)$ as $n \to \infty$. But the the range of F is a connected compact subset of $\mathbb{C}$, which is clearly not a single point, and hence has infinite cardinality. ∎

D SOME ELEMENTARY PROBABILITY

LEMMA D.1 *Let X_n be a sequence of random variables such that:*

1. *With probability one*

$$0 \;\le\; X_n \;\le\; M \;<\; \infty \quad \textit{for all } n. \tag{D.1}$$

2.

$$\limsup_{n\to\infty} \mathbb{E}(X_n) \;>\; 0\ . \tag{D.2}$$

Then

$$\mathbb{P}(\limsup_{n\to\infty} X_n \;>\; 0) \;>\; 0\ . \tag{D.3}$$

Proof: By passing to a subsequence we assume $\mathbb{E}(X_n) > a$ for all n and some $a > 0$. We have

$$a \;<\; \mathbb{E}(X_n;\, X_n > \tfrac{a}{2}) \;+\; \mathbb{E}(X_n;\, X_n \le \tfrac{a}{2}) \tag{D.4}$$

$$\le\; M\mathbb{P}(X_n > \tfrac{a}{2}) \;+\; \tfrac{a}{2}\mathbb{P}(X_n \le \tfrac{a}{2}) \tag{D.5}$$

$$=\; (M - \tfrac{a}{2})\mathbb{P}(X_n > \tfrac{a}{2}) \;+\; \tfrac{a}{2}\ , \tag{D.6}$$

so

$$\mathbb{P}(X_n > \tfrac{a}{2}) \;>\; \frac{a}{2M-a} \;>\; 0\,. \tag{D.7}$$

We now define the events

$$A_n \;=\; \{X_n > \tfrac{a}{2}\}$$

and

$$A \;=\; \limsup_{n\to\infty} A_n\,.$$

It now follows from (D.7) that

$$\mathbb{P}(A) \;\geq\; \limsup_{n\to\infty} \mathbb{P}(A_n) \;\geq\; \frac{a}{2M-a} \;>\; 0\,. \tag{D.8}$$

Since

$$A \;\subset\; \{\limsup_{n\to\infty} X_n \;>\; 0\}\,,$$

the lemma is proven. ∎

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