Absence of renormalization group pathologies near the critical temperature - two examples

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Abstract: We consider real space renormalization group transformations for Ising type systems which are formally defined by

$$e^{-H'(\sigma')} = \sum_{\sigma} T(\sigma,\sigma') e^{-H(\sigma)}$$

where $T(\sigma,\sigma')$ is a probability kernel, i.e., $\sum_{\sigma'} T(\sigma,\sigma') = 1$, for every configuration σ. For each choice of the block spin configuration σ', let $\mu_{\sigma'}$ be the measure on spin configurations σ which is formally given by taking the probability of σ to be proportional to $T(\sigma,\sigma')e^{-H(\sigma)}$. We give a condition which is sufficient to imply that the renormalized Hamiltonian H' is defined. Roughly speaking, the condition is that the collection of measures $\mu_{\sigma'}$ are in the high temperature phase uniformly in the block spin configuration σ'. The proof of this result uses methods of Olivieri and Picco. We use our theorem to prove that the first iteration of the renormalization group transformation is defined in the following two examples: decimation with spacing $b = 2$ on the square lattice with $\beta < 1.36\beta_c$ and the Kadanoff transformation with parameter p on the triangular lattice in a subset of the β, p plane that includes values of β greater than β_c.

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1. Introduction

Many recent papers have shown that position space renormalization group transformations which act on discrete spin systems on a lattice are not defined in a variety of settings. Formally these transformations map a Hamiltonian H into a "renormalized" Hamiltonian H' by the equation

$$e^{-H'(\sigma')} = \sum_{\sigma} T(\sigma,\sigma') e^{-H(\sigma)} \tag{1.1}$$

The spins in the original system are denoted by σ. The "block" spins are denoted by σ'. Here $T(\sigma,\sigma')$ is a probability kernel, i.e., $\sum_{\sigma'} T(\sigma,\sigma') = 1$, for every configuration σ. The spins take on a discrete set of values, e.g., $\{-1,+1\}$. Examples of such transformations include majority rule, decimation, and the Kadanoff transformation. (Introductory discussions of these transformations may be found in [14,20].)

The above equation defines H' in a finite volume, but the existence of the infinite volume limit of H' is nontrivial. Thus the renormalization group map $H \to H'$ may not even be defined. If the temperature is very high or there is a large magnetic field then the transformation is known to be defined [6,8,9]. At low temperatures heuristic arguments were given by Griffiths and Pearce [5,6,7] and by Israel [8] that the transformation is not defined. Rather than using the above equation, one can define the renormalization group transformation as a map on probability measures. Then the hard question is whether or not the renormalized measure is the Gibbs measure of some Hamiltonian. Van Enter, Fernández and Sokal proved that the renormalized measure is not Gibbsian for a variety of models at low temperature [20]. Further examples of renormalization group pathologies may be found in [12,13,18,19].

A key tenet of the renormalization group is that while the correlation length of the original system diverges at a second order transition, the introduction of the block spins should make the correlation length finite. More precisely, if one fixes a choice of the block spin configuration, then the system of original spins conditioned on this block spin configuration will not have a phase transition at the point where the original system does. The introduction of the block spins should shift the location of the critical point to a lower temperature. While this belief may be obvious in momentum space transformations in which one integrates out a slice of momentum, it is not obvious for these position space transformations. For three particular choices of the block spin configuration, including the checkerboard configuration, Kennedy proved that for the majority rule in two dimensions with two by two blocks, the critical temperature is indeed lowered by conditioning on the block spin configuration [10]. Benfatto, Marinari, and Olivieri did a Monte Carlo study of a renormalization group transformation in which the block spin is equal to the sum of the spins in the block [1]. (So the block spins do not just take on two values as the original spins do.) They considered the block spin configuration in which all the block spins are zero. They found that the introduction of this block spin configuration does indeed lower the critical temperature, but only by about 10% . However, preliminary Monte Carlo calculations of Ould-Lemrabott on the two dimensional Ising model with majority rule indicate that the introduction of the block spins lowers the critical temperature by at least a factor of two [17].

To state our main theorem we need some assumptions and definitions. We consider only finite range, translation invariant Hamiltonians. We only consider Ising type systems, i.e., the spin space at each site is $\{-1,+1\}$. The renormalization group kernel $T(\sigma,\sigma')$ is a product over blocks of a local function of the block spin and the spins in the original lattice in that block. We assume that $T(\sigma,\sigma')$ is always greater than zero. At first glance this last assumption appears to rule out many examples, in particular decimation and majority rule. However, it is often possible to reformulate the original system in such a way that $T(\sigma,\sigma')$ is never zero. Consider decimation. Usually one takes $T(\sigma,\sigma')$ to be 1 if for every site i at which there is both an original spin σ_i and a block spin σ'_i we have $\sigma_i=\sigma'_i$. However, we can instead just think of σ as consisting only of the original spins that live at sites without a block spin. All the other original spins are just set equal to the corresponding block spin. With this reduced σ, $T(\sigma,\sigma')$ is always 1. For majority rule we can obtain an equivalent system with $T(\sigma,\sigma')>0$ by first summing out some of the spins in the original system. We do not provide the details of this procedure since we have not been able to verify the hypothesis of our main theorem for majority rule.

Finally, we give the definitions needed for our theorem. Let V be a finite set of sites in the original lattice, and τ a boundary condition for V, i.e., a spin configuration on the sites outside of V. Let σ' be a block spin configuration. Then we define a probability measure which depends on V, τ and σ' by

$$\mu_{\sigma',V,\tau}(F) = \frac{\sum_\sigma F(\sigma) T(\sigma,\sigma') e^{-H(\sigma)}}{\sum_\sigma T(\sigma,\sigma') e^{-H(\sigma)}} \tag{1.2}$$

Here $F(\sigma)$ is a function on the original spin configurations σ and $\mu(F)$ denotes the expectation of such a function with respect to the probability measure μ. The sums over σ are over the spin configurations on V. The Hamiltonian $H(\sigma)$ is the Hamiltonian for the volume V using the boundary condition τ outside of V. The kernel $T(\sigma,\sigma')$ is the product over blocks which intersect the volume V of the kernel for that block. Note that the block spin configuration is fixed throughout the above.

Theorem 1.1: Suppose there exist constants $c < \infty$ and $m > 0$ such that for every finite subset V of the lattice, every two sites $i, j \in V$, every boundary condition τ *and every block spin configuration* σ'

$$|\mu_{\sigma',V,\tau}(\sigma_i\sigma_j) - \mu_{\sigma',V,\tau}(\sigma_i)\,\mu_{\sigma',V,\tau}(\sigma_j)| \leq c \exp(-m|i-j|) \tag{1.3}$$

Then the infinite volume limit of the renormalized Hamiltonian $H'(\sigma')$ exists. It may be written in the form

$$H'(\sigma') = \sum_X H'_X(\sigma') \tag{1.4}$$

where $H'_X(\sigma')$ only depends on the block spins in X and the sum is over finite sets of block spin sites. Furthermore, there is a $\mu > 0$ such that

$$\sum_{X \ni 0} e^{\mu||X||} ||H'_X||_\infty < \infty \tag{1.5}$$

where $||X||$ denotes the cardinality of the smallest connected set of block spin sites which contains X.

We will show that if the Dobrushin uniqueness condition is satisfied uniformly in the block spins, then the hypothesis of our theorem is satisfied. We verify numerically that the Dobrushin condition is satisfied uniformly in two examples. The first example is decimation for the two dimensional Ising model with scale factor $b = 2$. Van Enter, Fernández and Sokal proved that the renormalized measure is not Gibbsian in this example for $T < T_c/1.73$. We find that the Dobrushin condition is satisfied uniformly, and hence the renormalized Hamiltonian is defined, for $T > T_c/1.36$. In particular it is defined in a neighborhood of the critical point. The second example is the Kadanoff transformation for the triangular lattice in two dimensions. The Kadanoff transformation contains a parameter p. In the limit $p \to \infty$, the Kadanoff transformation becomes the majority rule transformation. We find that there is an interval of values of p for which the Dobrushin condition is satisfied uniformly in the block spins for temperatures slightly below the critical temperature. Thus there are values of p for which the Kadanoff transformation is defined in a neighborhood of the critical point.

The hypothesis of the theorem is similar to one of Dobrushin and Shlosman's many equivalent definitions of completely analytic interactions [4]. Indeed, if for every block spin configuration the interaction (including the renormalization group kernel) is completely analytic with the constants that appear in this property independent of the block spin configuration, then the hypothesis of the theorem holds. We prove theorem 1.1 by developing a convergent polymer expansion. Such an expansion was developed for completely analytic interactions by Olivieri and Picco [15,16]. We do not rely on any of their results, but instead give a different development of the expansion. Our motivation for doing this, besides the maxim that a good theorem deserves more than one proof, is to make this paper as self-contained as possible. We assume the reader knows polymer expansions, and for the examples we assume familiarity with the Dobrushin uniqueness theorem and related results at the level of sections V.1 and V.2 of [11]. We do not assume familiarity with Dobrushin and Shlosman's work on completely analytic potentials [4] or with the work of Olivieri and Picco [15,16].

To prove that an expansion converges one must usually require that some parameter, e.g., the inverse temperature, be small. For an expansion that will work for all completely analytic interactions, one must obtain this smallness from something besides the inverse temperature. Following Olivieri and Picco, our expansion involves a length scale L which will be chosen to be large compared to the correlation length of the system. Correlations between observables separated by at least a distance L are then very small. This is the smallness that drives the expansion.

Our main theorem gives a sufficient condition for the existence of the renormalized

Hamiltonian. It is conceivable that this condition is much stronger than what is needed for existence of the renormalized Hamiltonian. For example, there might be some block spin configurations for which the hypothesis of our theorem is not satisfied, but these block spin configurations could have probability zero in the renormalized measure and so not cause any problems. For a renormalization group transformation which is somewhat different from those considered in this paper, it has been suggested that if the introduction of one particular block spin configuration puts the system in a high temperature phase, then the renormalized Hamiltonian exists [3]. However, there is an example which shows that this is not true in general [18].

In this paper we have only used the Dobrushin uniqueness theorem to show in particular examples that the hypothesis of our theorem is satisfied. This approach does not work in a neighborhood of the critical point in an important example, the majority rule for two by two blocks in two dimensions. One can hope that the more general methods of Dobrushin and Shlosman [4] might work in examples like this.

Our two examples are discussed in section two. Section three contains the proof of the main theorem. The appendix shows how to use the Dobrushin uniqueness condition to verify the hypothesis of the main theorem using only results presented in [11].

2. Examples

In this section we consider two examples in which our theorem may be used to prove that the first iteration of the renormalization group is defined in a neighborhood of the critical point. We verify the hypothesis of the theorem by showing that Dobrushin's condition for uniqueness of the Gibbs state is satisfied uniformly in the block spin configuration. We begin by reviewing Dobrushin's condition. We follow the notation and exposition of section V.1 of [11] closely. For each site j, $\mu_j(\sigma, \sigma_j)$ is the measure on the spin space $\{-1, +1\}$ at the site j. The σ denotes the values of the spins at all the sites other than j. Define

$$\rho_{ij} = \frac{1}{2} \sup\{||\mu_j(\sigma, \cdot) - \mu_j(\omega, \cdot)|| \, : \, \sigma_k = \omega_k \ if \ k \neq i\} \tag{2.1}$$

Then ρ_{ij} measures the amount of change in the distribution of the spin at site j when we flip the spin at site i. Next define

$$\rho_j = \sum_{i \neq j} \rho_{ij}$$
$$\alpha = \sup_j \rho_j \tag{2.2}$$

Then Dobrushin's theorem says that if $\alpha < 1$ then there is a unique Gibbs state. For a finite range Hamiltonian the above condition leads to much more, e.g., exponential decay of the truncated correlations. In our setting α depends on the block spin configuration σ'. If $\alpha < 1$ uniformly in σ', then we get uniform exponential decay of the correlations. This decay can be used to prove hypothesis (1.3) of the main theorem. The argument is given in the appendix. The resulting proposition is as follows.

Proposition 2.1 : Define α as above. If $\sup_{\sigma'} \alpha < 1$, then hypothesis (1.3) of theorem 1.1 follows.

Remark: In our examples, translation invariance will imply that $\sup_{\sigma'} \rho_j$ is independent of j. So $\sup_{\sigma'} \alpha = \sup_{\sigma'} \rho_j$ for any choice of j. We will label the site j that we use by 0 in the examples.

The first example we consider is decimation on the square lattice with $b = 2$, i.e., the spins on a sublattice with spacing 2 are considered as the block spins and the rest of the spins are summed out. We prove that this transformation is defined for $\beta < 1.3645\beta_c$.

Decimation with $b = 2$ is equivalent to two iterations of the decimation transformation with $b = \sqrt{2}$. The first iteration is trivial; the renormalized Hamiltonian, H_1, may be computed explicitly. The four nearest neighbors of each spin that must be summed over are block spins, so the sum may be done explicitly. (See figure 1.)

$$\begin{aligned} &\sum_{\sigma_0} \exp[\beta\sigma_0(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] \\ &\qquad = \exp[a(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_4) + b\sigma_1\sigma_2\sigma_3\sigma_4 + c] \end{aligned} \tag{2.3}$$

with

$$a = \frac{1}{8}\ln(\cosh(4\beta))$$
$$b = \frac{1}{8}\ln(\cosh(4\beta)) - \frac{1}{2}\ln(\cosh(2\beta))$$

There is a similar formula for c, but it plays no role in the renormalized Hamiltonian. The lattice that remains after this first iteration of decimation with $b = \sqrt{2}$ has spacing $\sqrt{2}$.

We rescale it so that it has spacing 1. Then the terms $\sigma_1\sigma_2$, $\sigma_1\sigma_4$, $\sigma_2\sigma_3$ and $\sigma_3\sigma_4$ are nearest neighbor terms in the renormalized Hamiltonian, and the terms $\sigma_1\sigma_3$ and $\sigma_2\sigma_4$ contain lattices sites that are a distance $\sqrt{2}$ apart. The term $\sigma_1\sigma_2\sigma_3\sigma_4$ is the product of the four spins in a plaquette. Thus the renormalized Hamiltonian from the first iteration of the $b = \sqrt{2}$ decimation transformation is

$$H_1 = 2a \sum_{<i,j>:|i-j|=1} \sigma_i\sigma_j + a \sum_{<i,j>:|i-j|=\sqrt{2}} \sigma_i\sigma_j + b \sum_P \prod_{i\in P} \sigma_i \tag{2.4}$$

where the first sum is over nearest neighbor bonds, the second sum is over diagonal bonds and the third sum is over plaquettes P. The factor of 2 in the first term appears because for each bond in this sum there is a contribution from two different sums of the form (2.3). (In all of the above sums, each bond is only summed over once.)

The second iteration of the $b = \sqrt{2}$ decimation transformation must be applied to the Hamiltonian H_1, and so it is not trivially computable. We use our main theorem to prove that it is defined, and we use the proposition at the start of this section to verify the hypothesis of the main theorem. Considering the types of terms that appear in H_1, the spins we need to consider to test the Dobrushin condition are shown in figure 2. The site at which we test the condition is 0. The spins at sites $1, 2, 3, 4$ are block spins. Following our convention of denoting block spins by σ' and original spins by σ, these four spins are denoted $\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4$.

The terms in H_1 that involve the spin at site 0 are

$$\begin{aligned} H &= 2a(\sigma'_1 + \sigma'_2 + \sigma'_3 + \sigma'_4)\sigma_0 \\ &\quad + (a + b\sigma'_1\sigma'_2)\sigma_0\sigma_5 + (a + b\sigma'_2\sigma'_3)\sigma_0\sigma_6 + (a + b\sigma'_3\sigma'_4)\sigma_0\sigma_7 + (a + b\sigma'_1\sigma'_4)\sigma_0\sigma_8 \end{aligned}$$

At $\beta = \beta_c$ we find that $\sup_{\sigma'} \alpha = 0.6667$. This implies that $\sup_{\sigma'} \alpha < 1$ in a neighborhood of β_c. In fact we find that $\sup_{\sigma'} \alpha < 1$ for $\beta < 1.3004\beta_c$.

We can extend the interval of β for which we can prove that $b = 2$ decimation is defined with a little more work. In the above we did not test the Dobrushin condition on the original system. Indeed, such a test must fail at β_c. With $b = 2$, some of the sites have two nearest neighbors that are block spins and two that are original spins, but there are also spins that have four nearest neighbors that are original spins. For such sites j the quantity ρ_j is the same as it would be in the Ising model with no decimation, and so ρ_j

cannot be < 1 at β_c. What was crucial in the above was that we first summed out some of the original spins before we tested the Dobrushin condition. ("Original spins" refers to spins in the original Hamiltonian, i.e., non-block spins.) The subset of spins we summed out was a sublattice with spacing $\sqrt{2}$. Consider figure 3. The block spins are indicated by B's and the original spins by circles and X's. We do the sum over the original spins by first summing over the spins indicated by circles, and then over those indicated by X's. The spins in the first category break up into groups of five spins where each spin in the group has a nearest neighbor interaction only with other spins in the group, block spins and spins in the second category. Thus the sum over these five spins is a finite calculation. The result of this calculation is a new Hamiltonian for the spins in the second category and the block spins. Having summed out the spins in the first category, we use our theorem for the sum over the spins in the second category. Computing the effective Hamiltonian that results from the sum over the spins in the first category and then testing that the Dobrushin condition for the spins in the second category holds uniformly in the block spins is a bit of computation. At $\beta = \beta_c$ we find $\sup_{\sigma'} \alpha = 0.3530$. The Dobrushin condition $\sup_{\sigma'} \alpha < 1$ holds for $\beta < 1.3645\beta_c$.

The second example we consider is the Kadanoff transformation on the triangular lattice. For the triangular lattice the blocks are triangles containing three sites. So the block spins live on a lattice with spacing $\sqrt{3}$. The blocking of the triangular lattice is shown in figure 4. If $\sigma_1, \sigma_2, \sigma_3$ are the three spins in a block and σ' is the block spin, then the kernel for the Kadanoff transformation for a single block is

$$t(\sigma_1, \sigma_2, \sigma_3, \sigma') = \frac{\exp[p\sigma'(\sigma_1 + \sigma_2 + \sigma_3)]}{2\cosh[p(\sigma_1 + \sigma_2 + \sigma_3)]}$$

where $p > 0$ is a parameter. As $p \to \infty$ we obtain the majority rule. The Kadanoff transformation may be defined in a much more general setting. Given a blocking of a lattice, the kernel is given by the above formula with $\sigma_1 + \sigma_2 + \sigma_3$ replaced by the sum of the spins in the block. For the hypercubic lattice in two or more dimensions, van Enter, Fernández and Sokal [20] proved that for all $p > 0$, the renormalized measure is non Gibbsian at sufficiently low temperature.

We can rewrite the kernel as

$$t(\sigma_1, \sigma_2, \sigma_3, \sigma') = \exp[p\sigma'(\sigma_1 + \sigma_2 + \sigma_3) + q(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) + c]$$

where q and c are functions of p. They are determined by

$$q(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) + c = -\log(2\cosh[p(\sigma_1 + \sigma_2 + \sigma_3)])$$

which after a little algebra implies

$$q = -\frac{1}{4}[\log(\cosh(3p)) - \log(\cosh(p))]$$

Note that q is negative, so these terms in the kernel have the opposite sign of the nearest neighbor interactions in the Hamiltonian. Thus there is some cancellation between terms in the renormalization group kernel and nearest neighbor interactions for which the two spins are in the same block. One third of the nearest neighbor interactions are between spins in the same block.

We now ask if there are any values of p for which we can prove that the renormalized Hamiltonian is defined in a neighborhood of $\beta = \beta_c$. To apply our theorem we can try testing the Dobrushin condition. Even if we choose p so that $q = -\beta_c$ to get the most possible cancellation between the kernel and terms from the original Hamiltonian, we find that at $\beta = \beta_c$ the Dobrushin condition is not satisfied. Instead we do something similar to what we did for decimation. We sum out a subset of the original spins before we apply our theorem. The subset that we sum out consists of all the original spins that are at the top of the triangular block that they are in. (So the subset contains one third of the original spins.) Each spin being summed out interacts only with spins not being summed out and with block spins. Thus we can do this initial summation explicitly. This initial summation then produces a new Hamiltonian for the remaining original spins. We then check the Dobrushin condition for this new Hamiltonian. Figure 5 shows the spins that are involved in checking the condition. We want to check the condition for the spin at site 0. The spins at sites 1,2 and 3 are in the group that is summed out initially. For example, the sum over the spin at site 1 amounts to computing $f(\sigma_0, \sigma_4, \sigma_7, \sigma_8, \sigma_9, \sigma_5, \sigma_1')$ where

$$\sum_{\sigma_1} \exp[(\beta+q)\sigma_1\sigma_0 + (\beta+q)\sigma_1\sigma_4 + \beta\sigma_1\sigma_7 + \beta\sigma_1\sigma_8 + \beta\sigma_1\sigma_9 + \beta\sigma_1\sigma_5 + p\sigma_1'\sigma_1 + p\sigma_1'\sigma_4 + p\sigma_1'\sigma_0] = \exp[f(\sigma_0, \sigma_4, \sigma_7, \sigma_8, \sigma_9, \sigma_5, \sigma_1')]$$

Here σ_1' is the block spin which is denoted by B1 in figure 5.

The choice of p which yields $q = -\beta_c$ is $p = 0.6585$. For this value of p we find $\sup_{\sigma'} \alpha = 0.9044$ at $\beta = \beta_c$ and $\sup_{\sigma'} \alpha < 1$ for $\beta < 1.0453\beta_c$. With $\beta = \beta_c$ we find that $\sup_{\sigma'} \alpha < 1$ for $0.6128 < p < 0.7839$. Figure 6 shows the region in the β, p plane for which $\sup_{\sigma'} \alpha < 1$, and hence for which the first iteration of the Kadanoff transformation is defined.

3. Proof of the main result

Throughout this section we will use a Fourier series representation of functions of Ising spins. If V is a finite set of sites, then every function $F(\sigma)$ of the spins in V may be written in the form

$$F(\sigma) = \sum_X c_X \sigma^X \tag{3.1}$$

where X is summed over all subsets of V (including the empty set), $\sigma^X = \prod_{i \in X} \sigma_i$, and the constants c_X are given by

$$c_X = \sum_\sigma \sigma^X F(\sigma) \tag{3.2}$$

The sum is over the spin configurations σ on V. This sum is normalized so that $\sum_\sigma 1 = 1$, i.e., we include a factor of $2^{-|V|}$ in the definition of the sum. Throughout this section all sums over spin configurations will be normalized so that the sum of 1 is 1.

As is standard for expansion methods, we work in a finite volume V, but all our estimates will be uniform in the volume. The existence of the infinite volume limit will follow by the usual arguments. In the following most quantities depend on the finite volume V, the choice of boundary condition outside of it and the block spin configuration, but we do not make this dependence explicit. In the following the partition function we want to compute is of the form

$$Z = \sum_\sigma T(\sigma, \sigma') e^{-H(\sigma)}$$

where $T(\sigma, \sigma')$ is the renormalization group kernel. This kernel is a product over blocks of a local kernel, and in the above we only include the terms corresponding to the blocks in V. Recall that we are assuming that the kernel $T(\sigma, \sigma')$ is always greater than zero. Thus we may take its logarithm and simply include it in the Hamiltonian. So in the following, $e^{-H(\sigma)}$ will actually stand for $T(\sigma, \sigma')e^{-H(\sigma)}$. We will usually just denote this by e^{-H}.

(Since the kernel is a product over blocks of a function of the spins in that block, the logarithm is a finite range interaction.)

The first step in the proof is to show that hypothesis (1.3) of the theorem implies a condition on free energies. By free energy we will always mean minus the logarithm of the partition function. Of course this differs from the usual definition by a factor of β.

Lemma 3.1: Suppose that hypothesis (1.3) holds. For a finite volume V, a boundary condition τ outside of V and a block spin configuration σ', let $F_{\sigma',V,\tau}$ be the free energy, i.e., minus the logarithm of the partition function. Then there is a constant c such that for every finite volume V, every boundary condition τ, every block spin configuration σ' and every two sites $i, j \notin V$ we have

$$|\sum_{\tau_i,\tau_j} \tau_i \tau_j \, F_{\sigma',V,\tau}| \leq c e^{-m|i-j|} \tag{3.3}$$

Here m is the same constant that appears in (1.3), but c is a new constant.

Proof: When we compute $F_{\sigma',V,\tau}$, the Hamiltonian we use consists of those terms in the infinite volume Hamiltonian whose support intersects V. To make a connection with the correlation functions that appear in hypothesis (1.3), we now define a slightly different free energy for the volume V. Let W be $V \cup \{i,j\}$. Let $F'_{\sigma',V,\tau}$ be the free energy for the volume V computed using the Hamiltonian that consists of all the terms in the infinite volume Hamiltonian whose support intersects W. Note that in both $F_{\sigma',V,\tau}$ and $F'_{\sigma',V,\tau}$ we sum over the spins in V. The only difference is that in the latter we include some additional terms in the Hamiltonian. But since these terms do not involve any spins in V, the relationship between $F_{\sigma',V,\tau}$ and $F'_{\sigma',V,\tau}$ is trivial. Their difference equals the sum of the terms in the Hamiltonian whose support intersects W but not V. In particular

$$\sum_{\tau_i,\tau_j} \tau_i \tau_j \, F'_{\sigma',V,\tau} = \sum_{\tau_i,\tau_j} \tau_i \tau_j \, F_{\sigma',V,\tau}$$

if $|i-j|$ is large enough (depending on the range of the Hamiltonian). So it suffices to prove (3.3) with $F_{\sigma',V,\tau}$ replaced by $F'_{\sigma',V,\tau}$.

In (3.3), V, σ' and τ do not change except for τ_i and τ_j. So we will denote $F'_{\sigma',V,\tau}$ by simply $F(\tau_i,\tau_j)$. Define

$$A = \exp[-F(+1,+1)]$$

$$B = \exp[-F(-1,-1)]$$

$$C = \exp[-F(+1,-1)]$$

$$D = \exp[-F(-1,+1)]$$

Then

$$\sum_{\tau_i,\tau_j} \tau_i \tau_j \, F_{\sigma',V,\tau} = -\ln(AB/CD)$$

τ gives a boundary condition for the volume W in an obvious way; we simply drop τ_i and τ_j. We will apply (1.3) to the volume W. A little computation shows that

$$\mu_{\sigma',W,\tau}(\sigma_i\sigma_j) - \mu_{\sigma',W,\tau}(\sigma_i)\, \mu_{\sigma',W,\tau}(\sigma_j) = \frac{4(AB-CD)}{(A+B+C+D)^2}$$

We will show $\ln(AB/CD)$ is small by showing that AB/CD is close to 1. We start with

$$\begin{aligned} \left| \frac{AB}{CD} - 1 \right| &= \frac{|AB-CD|}{CD} \\ &= \frac{|AB-CD|}{(A+B+C+D)^2} \frac{(A+B+C+D)^2}{CD} \end{aligned}$$

Our assumptions on the Hamiltonian and $T(\sigma,\sigma')$ easily imply that the change in the free energy when any single boundary spin is flipped is bounded by a constant. Hence there is a constant M such that the ratio of any two of A, B, C and D is bounded above by M. This implies

$$\frac{(A+B+C+D)^2}{CD} \leq 16M$$

So we have

$$\left| \frac{AB}{CD} - 1 \right| \leq \frac{|AB-CD|}{(A+B+C+D)^2} 16M = 4M |\mu_{\sigma',W,\tau}(\sigma_i\sigma_j) - \mu_{\sigma',W,\tau}(\sigma_i)\mu_{\sigma',W,\tau}(\sigma_j)|$$

and so (1.3) implies (3.3). ∎

We divide the lattice into blocks which are L sites long on each side and so contain L^ν sites. These blocks are not the blocks used by the renormalization group transformation. When we wish to emphasize this fact we will refer to these blocks as L-blocks. Whenever we refer to L-blocks, we mean only those L-blocks that appear in this partitioning of the lattice. There are other blocks with side L that are not part of this partition, but they will never appear in our proof. L will be chosen large and we also choose it so that our L-blocks are commensurate with the blocks in the renormalization group transformation, i.e., each renormalization group block is a subset of an L-block. To keep the notation under control, we now restrict our attention to two dimensions; the generalization to higher dimensions is straightforward. (In a few places where the dependence of a quantity on the number of dimensions is significant, we will state the result for an arbitrary number of dimensions, denoting the number of dimensions by ν.) We divide the L-blocks into four types labelled by $i = 1, 2, 3, 4$ as shown in figure 7. (In ν dimensions there would be 2^ν types of blocks.) The crucial property is that the distance between any two L-blocks of the same type is at least L. Let $\sum_i$ denote the summation over the spins in V which are in a type i L-block. Then we trivially have

$$Z = \sum_4 \sum_3 \sum_2 \sum_1 \exp(-H) \tag{3.4}$$

We will use b to denote an L-block. The notation $b : i$ means that b is a type i block. For example, $\prod_{b:i}$ is the product over all blocks b of type i. Let σ_b denote the spin configuration on b. Then we have

$$\sum_i = \prod_{b:i} \sum_{\sigma_b}$$

We start by considering $\sum_1 e^{-H}$. Write the Hamiltonian in its Fourier representation

$$H = \sum_X c_X \sigma^X$$

Given a set of sites X we define $\bar{X}$ to be the union of all the L-blocks that contain at least one site in X. We think of $\bar{X}$ as the support of the term $c_X \sigma^X$ viewed on the scale L. We use B to denote a union of L-blocks. In everything that follows, *the only B's that appear are those that are small enough that they are contained in some $3L$ by $3L$ block.* For such a B we define

$$H_B = \sum_{X:\bar{X}=B} c_X \sigma^X$$

Since H is finite range, if L is chosen large enough, then every term in H will be in exactly one H_B and so $H = \sum_B H_B$. Also, no term in H can contain sites from two different type

1 blocks. So $H_B = 0$ if B contains more than one type 1 block. Thus we have

$$H = \sum_{B:no\,1} H_B + \sum_{b:1} \sum_{B:b\subset B} H_B \tag{3.5}$$

where $B : no\,1$ means that B does not contain any type 1 blocks. Define F^1 by

$$\exp(-F^1) = \sum_1 \exp(-H) \tag{3.6}$$

So F^1 is a function of the spins in blocks of types $2, 3$ and 4. Equation (3.5) implies that the sum in (3.6) factors into a product over type 1 blocks of the sum over the spins in that block. Thus we have

$$F^1 = \sum_{B:no\,1} H_B + \sum_{b:1} F^{1,b} \tag{3.7}$$

where for type 1 blocks b we define

$$F^{1,b} = -\ln[\sum_{\sigma_b} \exp(-\sum_{B:b\subset B} H_B)] \tag{3.8}$$

Next we try to compute $\sum_2 \exp(-F^1)$. However, F^1 can contain terms which involve spins in more than one type 2 block. Thus this sum does not factor into a product of independent sums over the type 2 blocks. Note that the terms in F^1 which prevent the factorization are supported on sets of sites with diameter greater than L. To proceed we need to distinguish these long range terms that prevent the factorization from the short range terms in F^1 that do not. We do this by looking at the supports of the terms in F^1 on scale L. Write F^1 in its Fourier representation.

$$F^1 = \sum_X c^1_X \sigma^X$$

and define for sets B which are a union of L-blocks,

$$F^1_B = \sum_{X:\bar{X}=B} c^1_X \sigma^X \tag{3.9}$$

So $F^1 = \sum_B F^1_B$. The definition of $F^{1,b}$ implies that $F^{1,b}$ is supported in a neighborhood of b, and so is supported in the $3L$ by $3L$ square centered about b. Thus F^1_B is nonzero only if B is a subset of some $3L$ by $3L$ block. We say B is long range (LR) if it contains two L-blocks which are separated by a distance of at least L. Otherwise we say B is short range (SR). The terms that prevent the factorization are the F^1_B's for B's that contain at least two type 2 blocks. Such B's are long range. So if we define

$$\begin{aligned} F^1_{SR} &= \sum_{B:SR} F^1_B \\ F^1_{LR} &= \sum_{B:LR} F^1_B \end{aligned} \tag{3.10}$$

then $\sum_2 \exp(-F^1_{SR})$ will factor into a product over the type 2 blocks. If F^1_{LR} is small, then we can hope to use the factorization that occurs when $F^1_{LR} = 0$ to develop a polymer expansion.

We define F^2 by

$$\exp(-F^2) = \sum_2 \exp(-F^1_{SR})$$

We will show that the computation of F^2 is a local operation, just as the computation of F^1 was. The short range B contain at most one type 2 block. Thus we may write F^1_{SR} as

$$F^1_{SR} = \sum_{B:SR} F^1_B = \sum_{B:SR,\, no2} F^1_B + \sum_{b:2} \sum_{B:SR,\, b\subset B} F^1_B$$

and so

$$F^2 = \sum_{B:SR,\, no2} F^1_B + \sum_{b:2} F^{2,b}$$

where for type 2 blocks b we define

$$F^{2,b} = -\ln[\sum_{\sigma_b} \exp(-\sum_{B:SR,\, b\subset B} F^1_B)]$$

We continue the above definitions inductively. Given F^i we decompose it as $F^i = \sum_B F^i_B$ where F^i_B contains the terms in F^i whose support X satisfies $\bar{X} = B$. Then $F^i = F^i_{SR} + F^i_{LR}$ with

$$\begin{aligned} F^i_{SR} &= \sum_{B:SR} F^i_B \\ F^i_{LR} &= \sum_{B:LR} F^i_B \end{aligned} \tag{3.11}$$

The terms in F^i_{LR} prevent the sum over spins in type $i+1$ blocks from factoring, so we define F^{i+1} by

$$\exp(-F^{i+1}) = \sum_{i+1} \exp(-F^i_{SR}) \tag{3.12}$$

As before the computation of F^{i+1} is local:

$$F^{i+1} = \sum_{B:SR,\, no\,(i+1)} F^i_B + \sum_{b:i+1} F^{i+1,b}$$

where for type $i+1$ blocks b we define

$$F^{i+1,b} = -\ln[\sum_{\sigma_b} \exp(-\sum_{B:SR,\, b\subset B} F^i_B)]$$

In this construction F^i is a function of the spins in type j blocks for $j > i$. In particular, F^4 will not depend on the spins inside the finite volume V, only on the boundary spins outside of V. However, F^4 is not the free energy for this volume since we have dropped all the long range terms in the above. To get the true free energy we proceed as follows. The expectation that is associated with the Hamiltonian H is

$$< f >= Z^{-1} \sum_4 \sum_3 \sum_2 \sum_1 \exp(-H)\, f$$

We define a modified expectation E by

$$E\, f = \exp(F^4) \sum_4 \sum_3 \sum_2 \sum_1 \exp(-H + F^1_{LR} + F^2_{LR} + F^3_{LR})\, f \tag{3.13}$$

It is straightforward to use our definitions to check that $E1 = 1$, and that the partition function is given by $Z = \exp(-F^4)\bar{Z}$ where

$$\bar{Z} = E \exp(-F^1_{LR} - F^2_{LR} - F^3_{LR}) \tag{3.14}$$

Now we develop an expansion for $\bar{Z}$. For each allowable long range B we define

$$K(B) = \exp(-F^1_B - F^2_B - F^3_B) - 1 \tag{3.15}$$

("Allowable" means that B is a union of L-blocks and is small enough to fit inside some $3L$ by $3L$ square.) Then

$$\exp(-F^1_{LR} - F^2_{LR} - F^3_{LR}) = \prod_B [K(B) + 1]$$

where the product is over all allowable long range B. This equals

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{B_1,\cdots,B_n:distinct} K(B_1)\cdots K(B_n)$$

The sum is over distinct $B_1, \cdots, B_n$. This does not mean they must be disjoint, only different. Inserting the above in (3.14) we have

$$\bar{Z} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{B_1,\cdots,B_n : distinct} EK(B_1) \cdots K(B_n) \tag{3.16}$$

The next ingredient we need for our expansion is a factorization property for the expectation E. More precisely, we seek a condition on functions f and g of the spins which implies

$$E\, f\, g = E\, f\;\; E\, g \tag{3.17}$$

Since E is not simply the measure in which all the spins are independent, this property is not trivial. We claim that there is a constant c which depends only on the number of dimensions such that if $dist(supp\, f, supp\, g) > cL$, then (3.17) holds. To prove this we first consider the computation of Ef. It begins with $\sum_1 f e^{-H}$. The support of f may involve more than one type 1 block. So the inclusion of f in this sum messes up the factorization, but only in a region near the support of f. More precisely, it is easy to see that there is a function f^1 such that

$$\sum_1 f e^{-H} = f^1 e^{-F^1}$$

and the support of f^1 is contained in the set of sites within a distance $c_1 L$ of the support of f, where c_1 is a constant. In general,

$$\sum_i f e^{-F_{SR}^{i-1}} = f^i e^{-F^i}$$

where f^i is supported on the sites within a distance $c_i L$ of the support of f. The analogous statement holds for g. Thus if the supports of f and g are sufficiently well separated, then the supports of f^i and g^i will not overlap and (3.17) follows.

We define a collection $B_1, B_2, \cdots, B_n$ to be connected if for every B_i and B_j we can find $B_{k_1}, B_{k_2}, \cdots, B_{k_l}$ in the list $B_1, \cdots, B_n$ such that $B_{k_1} = B_i$, $B_{k_l} = B_j$, and for $m = 1, \cdots, l-1$, the distance between B_{k_m} and $B_{k_{m+1}}$ is at most $c_1 L$. Geometrically, $B_1, \cdots, B_n$ are connected if when we "fatten" each set up by a boundary of width $c_1 L/2$, then the union of the fattened sets is a connected set. A connected collection $\{B_1, B_2, \cdots, B_n\}$ will be called a *polymer* and denoted typically by P. Two polymers P_1 and P_2 are said to be connected if $P_1 \cup P_2$ is connected; otherwise they are said to be disconnected. The weight of a polymer $P = \{B_1, \cdots, B_n\}$ is defined to be

$$W(P) = EK(B_1) \cdots K(B_n) \tag{3.18}$$

If P and P' are disconnected polymers, then the factorization property (3.17) and the definition of disconnectedness imply that $W(P \cup P') = W(P)\; W(P')$. Thus we have

$$\bar{Z} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{P_1, \cdots, P_n : disconnected} W(P_1) \cdots W(P_n) \tag{3.19}$$

where the sum is over collections of polymers $P_1, \cdots, P_n$ such that each pair P_i, P_j is disconnected when $i \neq j$.

To obtain a convergent expansion for $\log \bar{Z}$, we need to show that $W(P)$ is small. We bound it by

$$|W(P)| \leq \prod_{i=1}^{n} ||K(B_i)||_\infty \tag{3.20}$$

If $||F_B^i||_\infty$ is small, then $||K(B)||_\infty$ will be small. The smallness of the former should come from condition (3.3) in lemma 3.1. However, there is a counting problem that must be overcome. Fix a finite volume and consider the free energy F associated with it. So F is a function of the boundary spins. Write F in its Fourier representation

$$F = \sum_X c_X \sigma^X$$

Here the sets X range over subsets of the boundary spins. We have

$$c_X = \sum_\sigma \sigma^X F$$

where σ is summed over boundary spin configurations. (Recall that sums over spins configurations are normalized so that $\sum_\sigma 1 = 1$.) If X is long range then there are two sites i and j in X with $|i-j| \geq L$. So condition (3.3) implies that $|c_X| \leq ce^{-mL}$. However, the number of subsets of even a single L-block grows as 2^{L^2} and so overwhelms the smallness of e^{-mL}. Thus we cannot hope to bound F by using $|F| \leq \sum_X |c_X|$. A second problem that we must deal with is that condition (3.3) applies to free energies and F^i is not exactly a free energy since we dropped long range terms. The following lemma handles the counting problem.

Lemma 3.2 : Suppose that condition (3.3) holds. Then there is a constant c_0 such that for any finite volume V and any boundary condition σ_∂ and any long range B,

$$||F_B(V,\sigma_\partial)||_\infty \leq c_0 L^{\nu|B|} e^{-mL}$$

where $F(V,\sigma_\partial)$ is the free energy for the volume V with boundary condition σ_∂. $|B|$ is the number of L-blocks in B. The constant c_0 depends only on the constant c in condition (3.3) and the number of dimensions. The constant m is the same constant m that appears in condition (3.3) (The definition of $F_B(V,\sigma_\partial)$, which is probably obvious at this point, may be found at the start of the proof.)

Remark: The lemma applies to any volume V. The volumes we apply it to are rather unusual, in particular they are not connected. For example, to obtain bounds on F^1 we would use a volume which consists of the L-blocks of type 1 that are contained in our original finite volume.

Proof: Write $F(V,\sigma_\partial)$ in its Fourier representation

$$F(V,\sigma_\partial) = \sum_X c_X \sigma^X$$

$F_B(V,\sigma_\partial)$ contains those terms $c_X\sigma^X$ such that $X \subset B$ and X contains at least one site from each L-block in B. We can extract precisely these terms from $F(V,\sigma_\partial)$ by the following rather complicated operation. Let $b_1, b_2, \cdots, b_l$ be the L-blocks in B. (So $l = |B|$.) Since B is long range we can order them so that b_1 and b_2 are at least a distance L apart. Label the spins in b_k by σ_i^k with $i = 1, 2, \cdots, L^2$. Let $i_1, i_2, \cdots, i_l$ be integers between 1 and L^2 and consider

$$\sum_{\sigma_{B^c}} \sum_{\sigma_1^1,\sigma_2^1,\cdots,\sigma_{i_1}^1} \sum_{\sigma_1^2,\sigma_2^2,\cdots,\sigma_{i_2}^2} \cdots \sum_{\sigma_1^l,\sigma_2^l,\cdots,\sigma_{i_l}^l} \sigma_{i_1}^1 \sigma_{i_2}^2 \cdots \sigma_{i_l}^l \, F(V,\sigma_\partial) \tag{3.21}$$

This operation is designed to wipe out lots of the terms in $F(V,\sigma_\partial)$. The first sum is over all the spin configurations outside of B. This wipes out all the terms $c_X\sigma^X$ for which X is not a subset of B. Keeping in mind that there is a factor of $\sigma_{i_1}^1$ in the above, the second sum wipes out a term unless σ^X does not contain σ^1_j for $j < i_1$ and does contain $\sigma^1_{i_1}$. Together with the remaining sums we see that the only terms that survive the summation in (3.21) are those that have $X \subset B$ and X contains at least one site in each of $b_1, \cdots, b_l$ with $i_1, ..., i_l$ being the first such site in each of the respective blocks. The L-blocks b_1 and b_2 are at least a distance L apart, so the spins $\sigma^1_{i_1}$ and $\sigma^2_{i_2}$ are at least a distance L apart. Since (3.21) contains

$$\sum_{\sigma^1_{i_1}, \sigma^2_{i_2}} \sigma^1_{i_1}, \sigma^2_{i_2} F(V, \sigma_\partial)$$

and all of the sums are normalized, the $|| \quad ||_\infty$ of (3.21) is bounded by $c \exp(-mL)$ by (3.3). To obtain all the terms in $F_B(V, \sigma_\partial)$ we must sum each of $i_1, i_2, \cdots, i_l$ from 1 to L^2, which produces a factor of L^{2l}. ∎

Remark: Since B must be a subset of a hypercube with side $3L$, $|B|$ is bounded by a dimension dependent constant. For the purposes of this paper the value of this constant does not really matter because of the factor e^{-mL}. However, if one wants to extend these methods to infinite range Hamiltonians with a power law decay, then the value of this constant becomes very important. In the analogous estimate in [16], Olivieri and Picco have $L^{2\nu}$, a better bound than our lemma. However, this would still require that the Hamiltonian decay faster than $1/r^{2\nu}$, while decay that is faster than $1/r^\nu$ should be sufficient. It appears that there is no hope of proving this optimal sort of result with the methods of this paper.

The above lemma only applies to free energies, and the quantities F^i are not quite free energies. To address this problem we first need the following technical lemma which says that the operation of extracting the long range part is continuous in some sense.

Lemma 3.3 : For any function $F(\sigma)$ and any B which is a union of L-blocks,

$$||F_B||_\infty \leq c(|B|) ||F||_\infty$$

where $c(|B|)$ is a constant which depends only on $|B|$, the number of L-blocks in B.

Proof: Write $F(\sigma)$ in its Fourier representation

$$F(\sigma) = \sum_X c_X \sigma^X$$

Let

$$\bar{F}(\sigma) = \sum_{\sigma_{B^c}} F(\sigma)$$

where the sum is over all spin configurations on the complement of B. We have

$$||\bar{F}||_\infty \leq ||F||_\infty \tag{3.22}$$

This sum over σ_{B^c} kills any term in the Fourier representation that is not supported inside B. So

$$\bar{F}(\sigma) = \sum_{X:X\subset B} c_X \sigma^X$$

Thus $\bar{F}(\sigma)$ contains all the terms that go into F_B. Unfortunately, it contains some additional terms, those whose support is a proper subset of B. In fact, we have

$$\bar{F}(\sigma) = \sum_{B':B'\subset B} F_{B'}$$

and so

$$F_B(\sigma) = \bar{F}(\sigma) - \sum_{B':B'\subset B, B'\neq B} F_{B'}$$

Together with (3.22) this implies

$$||F_B||_\infty \leq ||F||_\infty + \sum_{B':B'\subset B, B'\neq B} ||F_{B'}||_\infty$$

The lemma now follows by induction on $|B|$. ∎

Lemma 3.4 : Suppose that condition (3.3) holds. Then there are constants c', p, L_0 such that

$$||F_B^i||_\infty \leq c' L^p e^{-mL}$$

if $L \geq L_0$ and B is LR. Here m is the same constant that appears in (3.3)

Proof: Fix a B_0 that is LR. $F^{1,b}$ only depends on terms H_X in the Hamiltonian for which $X \cap b \neq \emptyset$. $F^{i,b}$ only depends on $F^{i-1,B}$ for B with $b \subset B$. Thus there is a constant c such that F_B^i depends on a term in H_X only if X is within a distance cL of B. This implies that we can change the volume V and the boundary condition outside of the set

$$\Lambda(B_0) = \{i : dist(i, B_0) \leq cL\}$$

and $F^i_{B_0}$ will be unchanged. In particular, when computing $F^i_{B_0}$ we can replace V by $V \cap \Lambda(B_0)$, and replace the boundary condition by any boundary condition that agrees with the original one inside $\Lambda(B_0)$. Thus we can just assume $V \subset \Lambda(B_0)$.

We define $\tilde{F}^i$ by

$$\begin{aligned} \exp(-\tilde{F}^i) &= \sum_i \sum_{i-1} \cdots \sum_1 \exp(-H) \\ &= \sum_i \exp(-\tilde{F}^{i-1}) \end{aligned} \tag{3.23}$$

Note that each $\tilde{F}^i$ is the free energy of some volume, albeit a rather strange one. In the following, when we say that a quantity is $O(L^p e^{-mL})$ we will mean that there is a constant c' and an integer L' such that the $\| \quad \|_\infty$ of the quantity is bounded by $c' L^p e^{-mL}$ for $L \geq L'$. We now argue by induction on i. The inductive assumption is that

$$\tilde{F}^i_B - F^i_B = O(L^p e^{-mL})$$

for all B, both SR and LR. Since $\tilde{F}^i$ is the free energy of some volume, lemma 3.2 implies

$$\tilde{F}^i_B = O(L^p e^{-mL})$$

if B is LR. Thus proving the inductive assumption will prove the lemma. The inductive assumption is trivially true for $i = 1$ since $\tilde{F}^1 = F^1$.

Assume the inductive claim is true for $i-1$. The number of L-blocks in $\Lambda(B_0)$ is bounded by a constant that only depends on the number of dimensions. Since $V \subset \Lambda(B_0)$, the number of B such that $F^i_B \neq 0$ is also bounded by a constant which depends only on the number of dimensions. By the inductive assumption this implies

$$\sum_{B:SR} (F^{i-1}_B - \tilde{F}^{i-1}_B) = O(L^p e^{-mL})$$

Lemma 3.2 implies

$$\sum_{B:LR} \tilde{F}^{i-1}_B = O(L^p e^{-mL})$$

Thus

$$\left(\sum_{B:SR} F^{i-1}_B \right) - \tilde{F}^{i-1} = \sum_{B:SR} (F^{i-1}_B - \tilde{F}^{i-1}_B) - \sum_{B:LR} \tilde{F}^{i-1}_B = O(L^p e^{-mL})$$

And so

$$\begin{aligned} F^i &= -\ln\left(\sum_i \exp[-\sum_{B:SR} F_B^{i-1}]\right) \\ &= -\ln\left(\sum_i \exp[-\tilde{F}^{i-1} + O(L^p e^{-mL})]\right) \\ &= -\ln\left(\sum_i \exp(-\tilde{F}^{i-1})\right) + O(L^p e^{-mL}) \\ &= \tilde{F}^i + O(L^p e^{-mL}) \end{aligned}$$

By lemma 3.3 this implies $F^i_{B_0} - \tilde{F}^i_{B_0} = O(L^p e^{-mL})$. ∎

Lemma 3.4 implies that there is a function $\epsilon(L)$ such that

$$|W(P)| \leq \epsilon(L)^{|P|}$$

with $\epsilon(L) \to 0$ as $L \to \infty$. (Here $|P|$ denotes the number of B's in the polymer P.) The weight of a polymer $W(P)$ is a function of the block spins σ'. All of the above estimates are uniform in the block spin configuration σ', so we have in fact shown that

$$\sup_{\sigma'} |W(P)| \leq \epsilon(L)^{|P|} \tag{3.24}$$

The representation (3.19) says that $\bar{Z}$ is a gas of polymers with a two body interaction - each pair of polymers must be "disconnected." Standard results on the polymer expansion [2] then imply that if we choose L sufficiently large, then we have a convergent expansion for $\ln \bar{Z}$.

$$\ln \bar{Z} = \sum_{P_1,\cdots,P_n} \psi_c(P_1,\cdots,P_n) W(P_1)\cdots W(P_n) \tag{3.25}$$

where ψ_c is the connected part of our two body interaction. In particular $\psi_c(P_1,\cdots,P_n)$ vanishes whenever $\cup_i P_i$ is not connected in the sense of connectedness that we defined for the polymers

The weight $W(P)$ will depend on the block spins in P. It can also depend on some of the block spins outside of P, but we will now argue that there is a constant a such that $W(P)$ depends only on the block spins σ'_i with i within a distance aL of P. The following statements follow from the definitions of the various quantities. $F^{1,b}$ only depends on the block spins in b. $F^{2,b}$ only depends on the block spins in b and the two type 1 L-blocks adjacent to b. By induction we then see that there is a constant a_i such that $F^{i,b}$ only depends on the block spins within a distance $a_i L$ of b. Thus $K(B)$ only depends on the block spins within a distance $a_0 L$ of B for some constant a_0. Recall that if $P = B_1, \cdots, B_n$, then the weight of P is $E\, K(B_1) \cdots K(B_n)$. The quantity $K(B_1) \cdots K(B_n)$ only depends on the block spins within a distance $a_0 L$ of P. However, the expectation E also depends on the block spins. So $W(P)$ may depend on more block spins than $K(B_1) \cdots K(B_n)$ did. However, it follows by an argument similar to that which proved (3.17) that the expectation E can only extend the range of dependence on the block spins by a finite amount, i.e., there is a constant a such that $W(P)$ only depends on the block spins within a distance aL of P.

The renormalized Hamiltonian H' is equal to $-\ln Z = F^4 - \ln \bar{Z}$. F^4 is a local function of the block spins. Define *supp*$(P_1 \cup \cdots \cup P_n)$ to be the set of block spin sites that are within a distance aL of $P_1 \cup \cdots \cup P_n$. Since $\psi_c(P_1, \cdots, P_n)$ vanishes if $P_1 \cup \cdots \cup P_n$ is not connected, *supp*$(P_1 \cup \cdots \cup P_n)$ will be a connected set of block spins. The results of the previous paragraph show that a term $-\psi_c(P_1, \cdots, P_n) W(P_1) \cdots W(P_n)$ in the expansion of $-\ln \bar{Z}$ only depends on the block spins in *supp*$(P_1 \cup \cdots \cup P_n)$. For each finite set of block spin sites, define $H'_X(\sigma')$ to be the sum of the terms with *supp*$(P_1 \cup \cdots \cup P_n) = X$. F^4 is a sum of terms, each of which only depends on a finite number of block spins. We include each of these terms in the appropriate H'_X. We have shown that the renormalized Hamiltonian may be written in the form

$$H'(\sigma') = \sum_X H'_X(\sigma') \tag{3.26}$$

where $H'_X(\sigma')$ only depends on the block spins in X. Our expansion shows that each H'_X has an infinite volume limit and they satisfy

$$\sum_{X \ni 0} ||H'_X||_\infty < \infty \tag{3.27}$$

Let $|X|$ denote the number of block spin sites in X. Then there is an L dependent constant M such that for every term in (3.25) with *supp*$(P_1 \cup \cdots \cup P_n) = X$ we have $|X| \leq M \sum_{i=1}^n |P_i|$. It follows that we can even include a factor of $\exp(\mu |X|)$ in (3.27), and the sum will still be finite if $\mu > 0$ is small enough. We organized things above so that the sets X are connected. Thus we have completed the proof of theorem 1.1.

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Appendix

In this appendix we prove proposition 2.1. We will show that if $\sup_{\sigma'} \alpha < 1$, then hypothesis (1.3) of theorem 1.1 holds. Recall that the original Hamiltonian is finite range and translation invariant. Since H is finite range, for the terms with $\rho_{ij} \neq 0$, the distances $|i-j|$ are bounded by a constant. Thus we can choose $\gamma < 1$ and $\epsilon > 0$ sufficiently small, so that

$$\sum_{i:i\neq j} e^{\epsilon|i-j|} \rho_{ij} \leq \gamma < 1 \tag{A.1}$$

for all σ'.

Fix a block spin configuration σ'. The finite volume measures $\mu_{\sigma',V}$ were defined in eq. (1.2). Since the hypothesis of the Dobrushin uniqueness theorem is satisfied, for every σ' we get a unique infinite volume Gibbs state $\mu_{\sigma'}$. By condition (A.1) and theorem V.2.1 of [11], there are constants $c > 0, m < \infty$ such that for every block spin configuration σ'

$$|\mu_{\sigma'}(\sigma_i\sigma_j) - \mu_{\sigma'}(\sigma_i)\mu_{\sigma'}(\sigma_j)| \leq ce^{-m|i-j|} \tag{A.2}$$

This tells us that for every block spin configuration, the infinite volume measure of the constrained system has exponentially decaying correlations, and the decay is uniform in the block spin configuration. However, for condition (1.3) to hold, we need the same decay for all finite volumes and boundary conditions. This follows easily as we now show.

The finite volume measure $\mu_{\sigma',V}$ may be thought of as an infinite volume measure which is supported entirely on spin configurations that agree with the boundary condition τ outside of V. Fix a V, τ and σ'. For each site j, we define a measure ν_j on the spin space $\{-1, 1\}$. For $j \in V$, $\nu_j = \mu_{\sigma',\{j\}}$. For $j \notin V$, ν_j is the measure which assigns probability 1 to τ_j and probability 0 to $-\tau_j$. For a function f on spins configurations, $\nu_j(\omega, f)$ denotes the expectation of f with respect to ν_j. It is a function of ω, the spin configuration on the set of sites different from j. Clearly then,

$$\mu_{\sigma',V}(\tau, \nu_j(\omega, f)) = \mu_{\sigma',V}(\tau, f)$$

ρ_{ij} is defined by (2.1). We define $\tilde{\rho}_{ij}$ by the same equation with μ_j replaced by ν_j. It follows easily from the definition of ν_j that

$$\tilde{\rho}_{ij} = \begin{cases} \rho_{ij} & \text{if } j \in V \\ 0 & \text{if } j \notin V \end{cases}$$

So $\tilde{\rho}_{ij} \leq \rho_{ij}$. Hence

$$\sup_j \sum_{i:i\neq j} e^{\epsilon|i-j|}\, \tilde{\rho}_{ij} \leq \gamma < 1$$

The infinite volume measure corresponding to the ν_j's is the meausure $\mu_{\sigma',V}$. So theorem V.2.1 of [11] implies (A.2) holds with $\mu_{\sigma'}$ replaced by $\mu_{\sigma',V}$. Thus the hypothesis of the main theorem is satisfied.

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Figure Captions

Fig. 1. For decimation with $b=\sqrt{2}$, each spin in the original lattice that is summed out has only block spins as nearest neighbors. The spins at sites 1,2,3 and 4 are block spins and hence fixed. Thus the sum over the spin at site 0 may be done explicitly.

Fig. 2. After one iteration of the $b=\sqrt{2}$ decimation transformation, we check the Dobrushin condition for site 0. The effective Hamiltonian (2.4) couples the spin at this site to all the spins shown in the figure. The spins at sites 1,2,3 and 4 are block spins.

Fig. 3. The original spins are denoted by circles and X's, the block spins by B's. The circles form clusters of five sites with no nearest neighbor interactions between two such clusters. So the sum over the spins at the circles may be done explicitly.

Fig. 4. The blocking of the triangular lattice is shown with the block spins denoted by B. Each block contains the three sites in the original lattice that are adjacent to the block spin. The block spins live on a triangular lattice indicated with dashed lines.

Fig. 5. We test the Dobrushin condition at site 0. The spins at sites 1,2 and 3 are first summed out explicitly. The resulting effective Hamiltonian couples the spin at 0 to all of the spins shown.

Fig. 6. p is the parameter in the Kadanoff transformation for the triangular lattice. The Dobrushin condition is satisfied uniformly in the block spins, and hence the renormalized Hamiltonian exists, in the region to the left of the curve.

Fig. 7. The division of the lattice into large L by L blocks is shown. These blocks are then grouped into four types as indicated by 1,2,3,4. The spins in the blocks are then summed over in this order.