On the Growth of Averaged Weyl Sums for Rigid Rotations

S. De Bièvre* G. Forni†

April 26, 1997

* *UFR de Mathématiques et Laboratoire de Physique Théorique et Mathématique Université Paris VII, 2 place Jussieu F-75251 Paris Cedex 05, FRANCE*
† *Dipartimento di Matematica, Università di Bologna, Bologna I-40127, ITALY & Newton Institute, Cambridge, UK*

Abstract

Let $\omega \in \ \setminus\ $ and $f \in L^2(\mathbf{S}^1)$. We study the asymptotic behaviour of the Weyl sums $S(m,\omega)f(x) = \sum_{k=0}^{m-1} f(x+k\omega)$ and their averages $\hat{S}(m,\omega)f(x) = \frac{1}{m}\sum_{j=1}^{m} S(j,\omega)f(x)$, in the L^2-norm. In particular, for a suitable class of Liouville rotation numbers $\omega \in \ \setminus\ $, we are able to construct examples of functions $f \in H^s(\mathbf{S}^1)$, $s > 0$, such that, for all $\epsilon > 0$, $||\hat{S}(m,\omega)f||_2 \geq C_\epsilon m^{\frac{1}{1+s}-\epsilon}$ as $m \to \infty$. In addition, for all $f \in H^s(\mathbf{S}^1)$, $\liminf m^{-\frac{1}{1+s}}(\log m)^{-1/2} \parallel \hat{S}(m,\omega)f \parallel_2 < \infty$, for all $\omega \in \ \setminus\ $.

1 Introduction and statement of the results

We study the asymptotic behaviour of the following skew products on the cylinder:

$$(x_0, v_0) \in \mathbf{S}^1 \times \ \equiv T^*\mathbf{S}^1 \to (x_1 = x_0 + \omega, v_1 = v_0 + f(x_0)). \qquad (1.1)$$

*E-mail: debievre@mathp7.jussieu.fr
†E-mail: forni@dm.unibo.it and forni@dpmms.cam.ac.uk

Here $\omega \in \ \setminus\ $ and $f : \mathbf{S}^1 \to \ ,\quad \mathbf{S}^1 = \ /\ $. After m iterations of the map we have

$$x_m = x_0 + m\omega, \quad v_m = v_0 + S(m,\omega)f(x_0),$$

where the *Weyl sums* $S(m,\omega)f$ of the function f for the rigid rotation by ω are defined to be

$$S(m,\omega)f(x) = \sum_{k=0}^{m-1} f(x + k\omega)\ . \tag{1.2}$$

The asymptotic behaviour of the v-variable is therefore completely determined by the asymptotic behaviour of the Weyl sums.

It is well known [KN] that, if $\{p_k/q_k\}_{k\geq 1}$ denotes the sequence of approximants of the continued fraction expansion of the irrational number ω, then, for any $f \in BV(\mathbf{S}^1)$ of zero mean,

$$\| S(q_k,\omega)f \|_\infty < Var(f) \quad \text{for all } k \in \ . \tag{1.3}$$

The upper bound (1.3), known as the Denjoy-Koksma inequality, rules out the possibility of having growth to infinity of Weyl sums for smooth functions of zero mean. However, the set of "times" where the bound (1.3) holds is rather small since the q_k's grow at least exponentially. Perhaps then $S(m,\omega)f$ still gets to be sufficiently large at intermediate times resulting in growth of the averaged Weyl sums

$$\hat{S}(m,\omega)f(x) = \frac{1}{m}\sum_{j=1}^{m} S(j,\omega)f(x). \tag{1.4}$$

We will show this picture is essentially correct and obtain bounds on the growth of the $\hat{S}(m,\omega)f$. More precisely, we are interested in the L^2 growth, as $m \to +\infty$, of the averaged Weyl sums (1.4) for L^2-functions f. We write f_n for the Fourier coefficients of the function $f \in L^2(\mathbf{S}^1)$, i.e.

$$f(x) = \sum_{n\in} f_n \exp(2\pi i n x)\,,$$

and recall that one can express the smoothness properties of f in terms of its Fourier coefficients using the habitual Sobolev spaces $H^s(\mathbf{S}^1)$:

$$H^s(\mathbf{S}^1) = \{f \in L^2(\mathbf{S}^1)\ |\|\ f\ \|_s^2 := \sum_{n\in} n^{2s}\,|f_n|^2\ <\ +\infty\}. \tag{1.5}$$

To state and interpret our main results, we need some notation. We start by recalling the basic properties of the *continued fraction expansion* of irrational numbers, explained in detail in the book [K]. Let

$$\omega = [a_1, ..., a_k, ...]$$

be the continued fraction expansion of $\omega \in [0,1[\backslash$. Then the sequence of approximants $\{p_k/q_k\}_{k\in}$ (where $p_k/q_k = [a_1, ...a_k]$) is given by the following recursive identities [K,Th.1]: $p_0 = 0, q_0 = 1$, $p_1 = 1, q_1 = a_1$ and, for $k \geq 1$,

$$\begin{aligned} p_{k+1} &= a_k p_k + p_{k-1} \\ q_{k+1} &= a_k q_k + q_{k-1} \,. \end{aligned} \tag{1.6}$$

The sequence of denominators $\{q_k\}$ has the following equivalent definition [K, Ths.16 and 17]:

$$q_{k+1} := \min\{j > q_k \ \mid \| j\omega \| < \| q_k\omega \|\} \ , \tag{1.7}$$

where $\| \cdot \|$ denotes the distance from the nearest integer. As a consequence we have that

$$\| j\omega \| \geq \| q_k\omega \| \quad \text{ for all } j < q_{k+1} \, , \tag{1.8}$$

a property which will be systematically exploited. The following well known estimate will also be used:

$$\frac{1}{2q_{k+1}} < \frac{1}{q_k + q_{k+1}} < \| q_k\omega \| < \frac{1}{q_{k+1}} \ . \tag{1.9}$$

We will consider classes $\mathcal{R}_\gamma \subset \ \backslash$ of irrational numbers defined as follows: $\omega \in \mathcal{R}_\gamma$ iff there exists a constant $R > 0$ such that

$$\| q_k\omega \| \leq \frac{R}{q_k^{1+\gamma}} \qquad \forall k \in \, . \tag{1.10}$$

By the definition (1.10), $\mathcal{R}_{\gamma'} \subset \mathcal{R}_\gamma$ if $\gamma' \geq \gamma$ and by (1.9) $\mathcal{R}_0 = \ \backslash$. It is a straigthforward consequence of (1.6) that $\mathcal{R}_\gamma \neq \emptyset$ for any $\gamma > 0$. In fact, a number $\omega \in \mathcal{R}_\gamma$ is easily produced by constructing recursively its continued fraction expansion according to (1.6) and by choosing at each step: $a_k \geq [q_k^\gamma]$, where $[\cdot]$ denotes the *integer part*. A recursive estimate then shows that $\omega = [a_0, ..., a_k, ...] \in \mathcal{R}_\gamma$. This construction produces Diophantine irrational numbers $\omega \in \mathcal{R}_\gamma$ as well as Liouville ones. Actually, it is easy to see that all $\mathcal{R}_\gamma$ are G_δ sets. The intersection $\mathcal{R}_\infty$ of all sets $\mathcal{R}_\gamma$, for $\gamma \geq 0$, is a subset of the set of Liouville irrationals, and is of course still a G_δ dense set. Note that it follows from (1.9)-(1.10) that if $\omega \in \mathcal{R}_\gamma$, then

$$q_{k+1} \geq \frac{1}{2R} q_k^{1+\gamma}\,. \tag{1.11}$$

This clearly shows the sequence q_k is lacunary when $\gamma > 0$ and this all the more so as γ is larger. It will finally be useful to consider, for $\omega \in \mathcal{R}_\gamma$ ($\gamma \geq 0$) and for $r \geq 1$ the set

$$A_r(\omega) = \{m \in \ \mid \exists \ell \in \ \text{ so that } 2q_{\ell+1} \leq m \leq 2rq_{\ell+1}\}. \tag{1.12}$$

The points in $A_r(\omega)$ form a divergent subsequence of . For $\gamma > 0$, they should be thought of as lying "just above" one of the $q_{\ell+1}$ and "far below" the next one.

We can now state our main results. The first one gives lower bounds on the averaged Weyl sums. To simplify the formulations, we will always assume that

$$f_0 = \int_{\mathbf{S}^1} f\,dx = 0. \tag{1.13}$$

Theorem 1.1 *(i) Suppose* $|f_n| \geq c|n|^{-\nu}$ *for some* $\nu > 1/2$. *Let* $\omega \in \mathcal{R}_\gamma$ *for some* $\gamma \geq 0$. *Then for all* $r \geq 1$ *there exists a* $C_r > 0$ *so that for all* $m \in A_r(\omega)$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_r m^{(1-\frac{\nu}{1+\gamma})}. \tag{1.14}$$

Moreover, there exists a constant $C_\gamma > 0$ *so that* $\forall m \in$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_\gamma m^{(1-\frac{\nu}{1+\gamma})\frac{1+\gamma}{1+\gamma+\gamma\nu}}. \tag{1.15}$$

(ii) Let $\omega \in \mathcal{R}_\gamma$ *and* $s \geq 0$. *Then there exists a function* $f \in H^s(\mathbf{S}^1)$ *with the following properties. First, for all* $r \geq 1$ *and for all* $\epsilon > 0$, *there is a constant* $C_{r,\epsilon} > 0$ *so that for all* $m \in A_r(\omega)$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_{r,\epsilon} m^{1-\frac{s}{1+\gamma}-\epsilon}\,. \tag{1.16}$$

Moreover, there exists a $C_{\gamma,\epsilon} > 0$ *so that* for all $m \in$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_{\gamma,\epsilon} m^{(1-\frac{s}{1+\gamma})\frac{1+\gamma}{1+\gamma+\gamma s}-\epsilon}. \tag{1.17}$$

Note that (1.15) is optimised, for $\nu > 1/2$ fixed, as $\gamma \to +\infty$, so that for any Liouville irrational numbers $\omega \in \mathcal{R}_\infty$ you get, $\forall \epsilon > 0$:

$$\| \hat{S}(m,\omega) f \|_2 \geq C_\epsilon m^{\frac{1}{1+\nu}-\epsilon}, \quad \forall m \in . \tag{1.18}$$

Similarly, for $s > 0$ fixed, (1.17) is optimised as $\gamma \to +\infty$. Hence we get the following statement. For all $s \geq 0$ and for any Liouville rotation number $\omega \in \mathcal{R}_\infty$ there exists an $f \in H^s(\mathbf{S}^1)$ such that, for all $\epsilon > 0$,

$$\| \hat{S}(m,\omega) f \|_2 \geq C_\epsilon m^{\frac{1}{1+s}-\epsilon}, \quad \forall m \in . \tag{1.19}$$

Note that the lower bounds get smaller as f is taken smoother. To see that this is not due to a bad choice of rotation number, we prove the following upper bound, which shows that the above lower bounds are close to optimal.

Theorem 1.2 *Let* $\omega \in \ \backslash\ $.

(i) Let $s \geq 0$. *Then there exists a diverging sequence of natural numbers* $\{m_k\}_{k\in}$ *such that, for any* $f \in H^s(\mathbf{S}^1)$

$$\| \hat{S}(m_k,\omega) f \|_2 \leq C \| f \|_s \, m_k^{\frac{1}{1+s}} \sqrt{\log m_k} \,, \tag{1.20}$$

where $\| \cdot \|_s$ *denotes the* H^s *norm.*

In addition, if $\omega \in \ \backslash\ $ *satisfies* $\| q_k \omega \| \geq r/q_k^{1+\gamma}, \quad \forall k \in$ *and for some* $r > 0$, *then for all* $m \in$

$$\| \hat{S}(m,\omega) f \|_2 \leq C \| f \|_s \, m^{1-\frac{s}{1+\gamma}} \sqrt{\log m} \tag{1.21}$$

(ii) Let $\nu > 1/2$ *and* $\epsilon > 0$. *Then there exists a diverging sequence of natural numbers* $\{m_k\}_{k\in}$ *such that, if* $| f_n | \leq C | n |^{-\nu}$, *then*

$$\| \hat{S}(m_k,\omega) f \|_2 \leq C_\epsilon \, m_k^{\frac{1}{1/2+\nu}+\epsilon} \,. \tag{1.22}$$

In addition, if $\omega \in \ \backslash\ $ *satisfies* $\| q_k \omega \| \geq r/q_k^{1+\gamma}, \quad \forall k \in$ *and for some* $r > 0$, *then* $\forall \epsilon$, $\exists C_\epsilon$, *so that for all* $m \in$

$$\| \hat{S}(m,\omega) f \|_2 \leq C_\epsilon \, m^{1-\frac{\nu-1/2}{1+\gamma}+\epsilon} \,. \tag{1.23}$$

Note that (1.20) and (1.22) should be compared to (1.19) and (1.18) respectively, while (1.21) and (1.23) to (1.16) respectively (1.14). We can now explain the general picture emerging from these results. To fix ideas, let $|f_n| = |n|^{-\nu}(\nu > 1/2)$. Then $f \in H^s(\mathbf{S}^1)$ for any $s < \nu - 1/2$. We wish to find irrational ω so that $\| \hat{S}(m,\omega)f \|_2$ goes to infinity as fast as possible. At first, (1.14) suggests that by taking γ large we could achieve a growth exponent arbitrarily close to 1. However (1.22) shows this is illusory since it gives an ω-independent exponent $\frac{1}{\nu+1/2} + \epsilon$, decreasing with ν, in the upper bound. The best lower bound we were able to get is (1.15), which is optimized as $\gamma \to +\infty$, yielding (1.18). Note that there also the exponent tends to zero as ν tends to infinity. Similarly, (1.20) shows that the growth of $\| \hat{S}(m,\omega)f \|_2$ is increasingly slow as f is taken smoother and smoother, which shows (1.19) is close to optimal. In fact, as the proofs will show, the behaviour of $\| \hat{S}(m,\omega)f \|_2$ is controlled by two competing effects. Note first that, if $\omega \in$, then both $\| S(m,\omega)f \|_2$ and $\| \hat{S}(m,\omega)f \|_2$ will typically grow like m. On the other hand, if $\omega \in \ \setminus$, the ergodic theorem tells us that $\| S(m,\omega)f \|_2 = o(m)$, hence $\| \hat{S}(m,\omega)f \|_2 = o(m)$. This suggests that irrationals well approximated by rationals (γ large) are the best candidates for producing growth in $\| \hat{S}(m,\omega)f \|_2$. This is further corroborated by (1.14) and (1.23), which show there is a competing effect between the smoothness of f and the Diophantine properties of the rotation number ω and that for γ sufficiently large the growth exponent approaches 1. However, as γ becomes larger, so do the gaps between successive q_k, since q_{k+1} is at least of the order of $q_k^{1+\gamma}$. This is at the origin of the exponent of (1.15) which gives a weaker growth than expected.

For $\omega \in \mathcal{R}_\infty$, one sees, comparing (1.14) and (1.22), that the averaged Weyl sums continue to fluctuate between $m^{\frac{1}{1/2+\nu}+\epsilon}$ and $m^{1-\epsilon}$, pushing up against the upper bound imposed by the ergodic theorem. So, although averaging did allow us to beat Denjoy-Koksma, the averaged Weyl sums still fluctuate and do not have an asymptotic power law behaviour, for example.

In fact, our lower bounds predict superdiffusive behaviour, i.e.

$$\| \hat{S}(m,\omega)f \|_2 \geq Cm^\tau$$

for some $\tau > 1/2$, for $1/2 < \nu < 1$ and for suitable rotation numbers $\omega \in \mathcal{R}_\gamma$, for γ sufficiently large. On the other hand, the upper bounds in Theorem 1.2, in particular (1.20), rule out diffusive or superdiffusive behaviour for all $f \in H^s(S^1)$ with $s > 1$ and for any choice of an irrational rotation number.

There are several reasons for our interest in the model (1.1). Note that it is a symplectic transformation of $T^*\mathbf{S}^1$ with symplectic form $dx \wedge dv$, that can be seen as a perturbation of

$$(x, v) \in T^*\mathbf{S}^1 \to (x + \alpha, v) \in T^*\mathbf{S}^1$$

which is completely integrable with invariant tori $v = c$ [G][Be][B]. If f is in $C^k(\mathbf{S}^1)$ and ω sufficiently poorly approximated by the rationals (i.e. Diophantine of sufficiently high order), it is easy to see that those tori are preserved. It suffices to solve

$$f(x) = g(x + \omega) - g(x) \tag{1.24}$$

for a smooth g.

Our results above deal with the opposite extreme, when ω is sufficiently well approximated by the rationals, i.e. weakly Diophantine or Liouville. The tori are then broken and the motion is unbounded. Actually, unboundedness of the Weyl sums $\| S(m, \omega)f \|_2$ (i.e. $\sup_m \| S(m, \omega)f \|_2 = \infty$) is known to be equivalent to the non-solvability of (1.24) for $g \in L^2(, dx)$ [L]. The only previous work that we are aware of discussing the asymptotic behaviour of v_m is [B]. There an argument is proposed claiming to show that the Weyl sums $\| S(m, \omega)f \|_2$ themselves, before averaging, with $f_n \cong |n|^{-\nu}$ satisfy the lower bound (1.14) *for all values of* m and $\nu > 1/2$. As we already pointed out, this can not be true for $\nu > 1$ because of Denjoy-Koksma. As it turns out, Theorem 1.2 shows that the result is not true for the averaged Weyl sums either, as one might have hoped. Only when $\nu < 1$, when Denjoy-Koksma is no longer an obstruction, does the following result indeed give a non-trivial lower bound:

Theorem 1.3 *Suppose $|f_n| \geq c|n|^{-\nu}$ for some $\nu > 1/2$. Then there exists a $C > 0$ so that for all $\omega \in \setminus$ and for all $m \in$*

$$\| S(m, \omega)f \|_2 \geq Cm^{1-\nu}. \tag{1.25}$$

Once the invariant tori are broken, it is legitimate to ask whether the map (1.1) is ergodic on the cylinder $T^*\mathbf{S}^1$ with respect to Lebesgue measure. Note that (1.1) can be viewed as a map on $\mathbf{S}^1 \times \mathbf{S}^1$ as well. The Anzai theorem [P1] shows that it is ergodic on $\mathbf{S}^1 \times \mathbf{S}^1$ as soon as (1.24) is not soluble. Ergodicity on $T^*\mathbf{S}^1$ is more delicate. In [HL], the following result is shown.

If $f \in C^1([0,1])$, $f(0) \neq f(1)$ and if ω is irrational, then the map (1.1) is ergodic on the cylinder $\mathbf{S}^1 \times$, so that $S(m,\omega)f(x)$ is unbounded for almost all x. In this case, the Fourier coefficients of f satisfy $|f_n| > c|n|^{-1}$ so that Theorem 1.1(i) as well as (1.18) apply with $\nu = 1$.

A particular example that has been much studied is

$$f_\beta(x) = \chi_{[0,\beta]}(x) - \beta,$$

for some $0 < \beta < 1$. It is a classical result of Kesten [Ke][P2] that $S(m,\omega)f$ is bounded iff $\beta \in \omega$ (mod 1). The asymptotic behaviour of $\hat{S}(m,\omega)f$ is important in the study of incommensurate one-dimensional structures, where the following model was proposed [AG]. With u_m denoting the m-th atomic position, define the sequence u_m recursively by $(0 < \xi\beta < a)$

$$u_{m+1} - u_m = a + \xi f_\beta(m\omega + x_0),$$

so that

$$u_m = ma + \xi S(m,\omega) f_\beta(x_0) + x_0.$$

The model is said to have no average lattice whenever $S(m,\omega)f_\beta(x_0)$ is unbounded. For $x_0 = 0$, $\beta = 1/2$ and $\omega = \tau^{-2}$, $\tau = \frac{1+\sqrt{5}}{2}$, it was shown in [GLV] and [D] that the averaged Weyl sums diverge logarithmically: this behaviour is expected to persist for a large class of quadratic irrationals ω [CMPS][D]. Since $f_{\beta\ 2k} = 0$, $f_{\beta\ 2k+1} = \frac{1}{i\pi(2k+1)}$ when $\beta = 1/2$, it follows from our work that, if $\omega \in \mathcal{R}_\gamma$ is chosen such that all q_k are odd, then (see (1.14)):

$$\parallel \hat{S}(m,\omega) f_\beta \parallel_2 \geq C_r m^{(1-\frac{1}{1+\gamma})}, \quad \forall m \in A_r(\omega). \tag{1.26}$$

The possibility of having such behaviour for suitable rotation numbers was predicted in a comment at the end of [GLV]. In addition we now have (see (1.15))

$$\parallel \hat{S}(m,\omega) f_\beta \parallel_2 \geq C_\gamma m^{\overline{(1+2\gamma)}^{\gamma}}, \quad \forall m \in . \tag{1.27}$$

In particular, taking $\omega \in \mathcal{R}_\infty$, (1.27) implies that the averaged fluctuations behave almost diffusively, as in the case of a random structure. Indeed, if $\omega \in \mathcal{R}_\infty$, then for all ϵ, there exists a C_ϵ so that

$$\parallel \hat{S}(m,\omega) f_\beta \parallel_2 \geq C_\epsilon m^{\frac{1}{2}-\epsilon}, \quad \forall m \in .$$

2 Proofs

The proofs of our results are based on Fourier series estimates using the basic properties of the continued fraction expansion of irrational numbers recalled in the introduction.

First of all, it is easy to establish that

$$\| S(m,\omega)f \|_2^2 = \sum_{n\in} |f_n|^2 \frac{\sin^2 \pi m n\omega}{\sin^2 \pi n\omega},$$

and

$$\| \hat{S}(m,\omega)f \|_2^2 = \sum_{n\in} |f_n|^2 G_m(n\omega)\,,$$

where the function $G_m : \to {}^+$ is defined as

$$G_m(x) := \frac{1}{4\sin^2 \pi x} |1 - e^{\pi i(m+1)x} \frac{\sin \pi m x}{m \sin \pi x}|^2 \,. \tag{2.1}$$

We need the following estimates on the functions G_m.

Lemma 2.1 *The functions $G_m : \to {}^+$ are 1-periodic and satisfy the symmetry property $G_m(1-x) = G_m(x)$ for $0 \le x \le 1$, hence*

$$\| \hat{S}(m,\omega)f \|_2^2 = \sum_{n\in} |f_n|^2 G_m(\| n\omega \|)\,, \tag{2.2}$$

where $\| \cdot \|$ denotes the distance from the nearest integer. Moreover, there exist constants $C > c > 0$ such that the following inequalities hold:

$$c/\sin^2 \pi x \le G_m(x) \le C/\sin^2 \pi x \qquad \forall\ 1/m \le x \le 1/2. \tag{2.3}$$

and

$$cm^2 \le G_m(x) \le Cm^2 \qquad \forall\ 0 \le x \le 1/m \tag{2.4}$$

In addition, the upper bounds in (2.3) and (2.4) hold for all $x \in$.

Proof. We know from basic calculus that

$$\frac{2}{\pi} \le \frac{\sin \pi x}{\pi x} \le 1 \quad \text{for all } 0 \le x \le 1/2\,, \tag{2.5}$$

where the upper bound holds for all $x \in$. It follows from (2.5) that

$$\frac{2}{\pi} \le |\frac{\sin \pi m x}{m \sin \pi x}| = |\frac{\sin \pi m x}{m x} \frac{x}{\sin \pi x}| \le \frac{\pi}{2} \ , \tag{2.6}$$

where the lower inequality holds for $0 \le x \le 1/2m$ and the upper inequality for all $x \in$, by evenness and periodicity.

Thus the inequality

$$G_m(x) \le C_1 / \sin^2 \pi x \quad \forall\, x \in$$

holds by choosing C_1 large enough:

$$\frac{1}{4}\Big(1 + \max_{x \in \mathbf{S}^1} |\frac{\sin \pi m x}{m \sin \pi x}|\Big)^2 \le \frac{1}{4}\Big(1 + \frac{\pi}{2}\Big)^2 \le C_1.$$

Secondly, the inequality

$$G_m(x) \ge c_1 / \sin^2 \pi x \quad \forall 1/2 \ge x \ge 1/m \tag{2.7}$$

holds since by (2.5),

$$|\frac{\sin \pi m x}{m \sin \pi x}| \le \frac{1}{m \sin \pi x} \le \frac{1}{2} \quad \forall\, \frac{1}{m} \le x \le 1/2 \ ,$$

which implies

$$|1 - e^{\pi i (m+1) x} \frac{\sin \pi m x}{m \sin \pi x}|^2 \ge \Big(1 - |\frac{\sin \pi m x}{m \sin \pi x}|\Big)^2 \ge 1/4 \ ,$$

so that we can take $c_1 \le 1/16$ in (2.7). This proves (2.3). In order to prove (2.4), write

$$u_m(x) := 1 - \frac{\sin \pi m x}{m \sin \pi x} \, e^{\pi i (m+1) x} \, .$$

There exist $C > c > 0$ such that,

$$c m x \le |u_m(x)| \le C m x \qquad \forall 0 \le x \le 1/m.$$

In fact, the estimate from above can be obtained as follows. Since

$$u_m(x) = 1 - \frac{\sin \pi m x}{\pi m x} + \frac{\sin \pi m x}{\pi m x}\,(1 - \frac{\pi x}{\sin \pi x}) + \frac{\sin \pi m x}{m \sin \pi x}\,(1 - e^{\pi i (m+1) x}) \, ,$$

the conclusion follows by basic calculus, (2.5) and (2.6). As to the estimate from below, we will consider separately the two intervals $0 \leq x \leq 1/2m$ and $1/2m \leq x \leq 1/m$. In the first case, since $u_m(0) = 0$ and, by a straigthforward computation,

$$u'_m(x) = \Big(\pi \frac{\cos \pi x}{\sin \pi x} - \pi m \frac{\cos \pi m x}{\sin \pi m x} - i\pi(m+1)\Big) \frac{\sin \pi m x}{m \sin \pi x}\, e^{\pi i(m+1)x} \;,$$

the conclusion follows by the mean value theorem and (2.6). In the second case, it can be noticed that the above argument proving (2.7) holds true for $x \geq 1/2m$. Since $G_m(x) = |u_m(x)|^2/4\sin^2 \pi x$, (2.4) follows. □

The functions $\frac{\sin^2 \pi m x}{\sin^2 \pi x}$ and $G_m(x)$ are traced in Fig. 2.1. Note that the behaviour of both functions is similar in the region $0 \leq x \leq \frac{1}{4m}$, where they are both of order m^2. For bigger values of x the difference manifests itself in that unlike $\frac{\sin^2 \pi m n\omega}{\sin^2 \pi n\omega}$, $G_m(x)$ has no zeros (see (2.3)). This eventually explains the lower bounds of Theorem 1.1 which we can now start proving.

Proof of Theorem 1.1. (i) Let $\omega \in \mathcal{R}_\gamma$ and denote as before by $\{p_k/q_k\}_{k\in}$ the sequence of truncations of its continued fraction expansion. We can assume that $1 + \gamma - \nu > 0$, since otherwise the bounds in (1.14)-(1.15) are trivially satisfied. By Lemma 2.1 a lower bound for the averaged Weyl sum $\hat{S}(m,\omega)f$ can be obtained as follows:

$$\| \hat{S}(m,\omega)f \|_2^2 \geq c\Big(\sum_{\|n\omega\| \geq 1/m} |f_n|^2 \, \| n\omega \|^{-2} \, + m^2 \sum_{\|n\omega\| < 1/m} |f_n|^2 \Big) \,. \tag{2.8}$$

Suppose now that $m \in A_r(\omega)$, so that, for some $\ell \in$

$$\frac{1}{2rq_{\ell+1}} \leq \frac{1}{m} \leq \frac{1}{2q_{\ell+1}}.$$

Hence (1.9) implies

$$\frac{1}{m} \leq \| q_\ell \omega \| \leq \frac{2r}{m}.$$

Using (2.8) we therefore get

$$\| \hat{S}(m,\omega)f \|_2^2 \geq c|f_{q_\ell}|^2 \, \| q_\ell \omega \|^{-2},$$

from which (1.14) follows upon using (1.11).

To prove (1.15), we proceed as follows. Let $(k_m)_{m\in}$ be the sequence of natural numbers defined in the following way:

$$k_m = \max\{k \in \ | \ \| q_k\omega \| \geq 1/m\}\,. \tag{2.9}$$

Then

$$\| \hat{S}(m,\omega) f \|_2^2 \geq c\Big(|f_{q_{k_m}}|^2 q_{k_m+1}^2 \ + \ m^2 |f_{q_{k_m}+1}|^2\Big) \tag{2.10}$$

and, by the hypothesis on the Fourier coefficients of f,

$$\| \hat{S}(m,\omega) f \|_2 \geq c\Big(q_{k_m}^{-\nu} q_{k_m+1} \ + \ m q_{k_m+1}^{-\nu}\Big)\,. \tag{2.11}$$

Consider now for all $k \in$ the functions

$$S_k(x) = q_k^{-\nu} q_{k+1} + x q_{k+1}^{-\nu} \quad \text{and} \quad S_{k,\alpha}(x) = x^{-\alpha} S_k(x)\,, \quad x \in {}^{+}\,. \tag{2.12}$$

The idea is to find the exponent $\alpha \in (0,1)$ so that all the functions $S_{k,\alpha}$ have a common strictly positive lower bound on ${}^{+}$. In fact, basic calculus shows that $S_{k,\alpha}$ has a unique minimum on ${}^{+}$ and that

$$\min S_{k,\alpha} = c_\alpha q_k^{-\nu(1-\alpha)} q_{k+1}^{1-(1+\nu)\alpha}\,. \tag{2.13}$$

where $c_\alpha > 0$ and depends only on $\alpha \in (0,1)$. Since $\omega \in \mathcal{R}_\gamma$, by (1.11),

$$\min S_{k,\alpha} \geq c_\alpha q_{k+1}^{(1-\alpha)(1-\frac{\nu}{1+\gamma})-\nu\alpha}\,. \tag{2.14}$$

Thus, we may choose $\alpha \in (0,1)$ such that

$$(1-\alpha)(1-\frac{\nu}{1+\gamma}) - \nu\alpha = 0 \tag{2.15}$$

thereby obtaining

$$S_{k,\alpha}(x) \geq c_\alpha > 0 \ \ \forall x \in {}^{+}\,. \tag{2.16}$$

A short computation of the exponent given by (2.15) yields (1.15) since, by (2.11) and (2.16),

$$\| \hat{S}(m,\omega) f \|_2 \geq c S_{k_m}(m) \geq c m^\alpha S_{k_m,\alpha}(m) \geq c'_\alpha m^\alpha\,. \tag{2.17}$$

(ii) Let $\omega \in \mathcal{R}_\gamma$. Consider

$$f(x) = \sum_{k\in} \frac{1}{k} q_k^{-s} \exp 2\pi i q_k x. \tag{2.18}$$

Clearly $f \in H^s(\mathbf{S}^1)$. Also, for all $s' > s$, $\exists c_{s'} > 0$ so that $|f_{q_k}| \geq c_{s'}\, {q_k}^{-s'}$ for all k. The reasoning is now identical to the one in (i), with s' replacing ν. $\Box$

We now turn to the proof of Theorem 1.2 which deals with upper bounds along subsequences for the averaged Weyl sums. We start with a lemma.

Lemma 2.2 *Suppose that $\omega \in \setminus$ satisfies the following property for some $\gamma > 0$: there exist a constant $R > 0$ and an infinite set $\mathcal{K} \subset$ such that, for any $k \in \mathcal{K}$,*

$$\| q_k\omega \| < \frac{R}{q_k^{1+\gamma}} \, . \tag{2.19}$$

Define, for any $0 < \gamma^- \leq \gamma$, $m_k = [q_k^{1+\gamma^-}/R]$ for $k \in \mathcal{K}$. Then:

(i) If $f \in H^s(\mathbf{S}^1)$ for some $s \geq 0$, then

$$\| \hat{S}(m_k, \omega) f \|_2 \leq C \| f \|_s \left(m_k^{1-\frac{s}{1+\gamma^-}} + m_k^{\frac{1}{1+\gamma^-}} \right) ; \tag{2.20}$$

(ii) If, in particular, $| f_n | \leq C \mid n \mid^{-\nu}$, then

$$\| \hat{S}(m_k, \omega) f \|_2 \leq C \left(m_k^{1-\frac{\nu-1/2}{1+\gamma^-}} + m_k^{\frac{1}{1+\gamma^-}} \right) . \tag{2.21}$$

Proof. We split the series (2.2) as follows:

$$\| \hat{S}(m,\omega) f \|_2^2 = \sum_{n < q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) + \sum_{n \geq q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) \, , \tag{2.22}$$

By the definition (2.9) of k_m and by property (1.8),

$$\| n\omega \| \geq \| q_{k_m}\omega \| \geq 1/m \quad \text{for all } n < q_{k_m+1} \, . \tag{2.23}$$

Hence, (2.3) and (1.9) imply

$$\sum_{n < q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) \leq C \| f \|_0^2 \, q_{k_m+1}^2 \, . \tag{2.24}$$

Clearly, the estimate obtained in Lemma 2.1 and the bound on the Fourier coefficients of any $f \in H^s(\mathbf{S}^1)$ given by (1.5) yield

$$\sum_{n \geq q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) \leq C \| f \|_s^2 \; m^2 {q_{k_m+1}}^{-2s} \, . \tag{2.25}$$

Suppose $\ell \in \mathcal{K}$ is sufficiently large and choose $m_\ell = [q_\ell^{1+\gamma^-}/R]$ with $0 < \gamma^- \leq \gamma$. Then it follows from (1.9) and (2.19) that

$$\| q_\ell \omega \| < \frac{1}{m_\ell} \leq \| q_{\ell-1}\omega \| \ .$$

Hence, by (2.22)-(2.25), and since $k_{m_\ell} = \ell - 1$ (see (2.9)), it follows that

$$\| \hat{S}(m_\ell, \omega) f \|_2 \leq C \| f \|_s \left(m_\ell q_\ell^{-s} + q_\ell \right) ,$$

which yields (2.20). The proof of (2.21) follows by noticing that if the Fourier coefficients of f satisfy the estimates $|f_n| \leq C n^{-\nu}$, $\nu > 1/2$, then $f \in H^{\nu^-}(S^1)$, for all $\nu^- < \nu - 1/2$. $\square$

Proof of Theorem 1.2. Lemma 2.2 already contains the main part of the proof of Theorem 1.2. In fact, suppose $f \in H^s(\mathbf{S}^1)$. Then, Lemma 2.1 and (1.8) imply

$$\| \hat{S}(m,\omega) f \|_2^2 \leq C \Big(\sum_{j < q_{k_m+1}} |f_j|^2 \| j\omega \|^{-2} + m^2 \sum_{j \geq q_{k_m+1}} |f_j|^2 \Big) , \tag{2.26}$$

where k_m is defined as in (2.9). The first summand on the right of (2.26) can be estimated, using (1.8), (1.9) and the defining property (1.5) of the Fourier coefficients of $f \in H^s(S^1)$, by

$$\sum_{k \leq k_m} \| q_k \omega \|^{-2} \sum_{q_k \leq j < q_{k+1}} |f_j|^2 \leq C \| f \|_s \sum_{k \leq k_m} \frac{q_{k+1}^2}{q_k^{2s}} . \tag{2.27}$$

Assume that the irrational number ω is such that for some $\gamma \geq 0$,

$$\| q_k \omega \| \geq 1/q_k^{1+\gamma} , \tag{2.28}$$

for all but a finite number of $k \in$. Hence by (1.9)

$$\sum_{k \leq k_m} \frac{q_{k+1}^2}{q_k^{2s}} \leq \sum_{k \leq k_m} q_{k+1}^{2(1-\frac{s}{1+\gamma})} \leq k_m q_{k_m+1}^{2(1-\frac{s}{1+\gamma})} , \tag{2.29}$$

provided $1 + \gamma \geq s$, since the sequence $(q_k)_{k\in}$ is increasing. We recall also that, since by (1.6) $q_k \geq 2^{k/2}$ for all $k \in$, by its definition (2.9), $k_m \leq C \log m$.

The second summand on the right of (2.26) can be estimated by

$$\| f \|_s \, m^2 q_{k_m+1}^{-2s} \, . \tag{2.30}$$

Finally,

$$\| \hat{S}(m,\omega) f \|_2 \leq C \| f \|_s \left((\log m)^{1/2} q_{k_m+1}^{1-\frac{s}{1+\gamma}} + m q_{k_m+1}^{-s} \right) . \tag{2.31}$$

Take $m_\ell = q_{\ell+1} + q_\ell$. Then it is easily checked that $k_{m_\ell} = \ell$ so that

$$\| \hat{S}(m_\ell,\omega) f \|_2 \leq C \| f \|_s \left((\log m_\ell)^{1/2} m_\ell^{1-\frac{s}{1+\gamma}} + m_\ell^{1-s} \right) . \tag{2.32}$$

Now choose $\gamma = s$. Then

$$1 - \frac{s}{1+\gamma} = \frac{1}{1+s} \quad \text{and} \quad 1 - s < \frac{1}{1+s} \, . \tag{2.33}$$

Hence,

$$\| \hat{S}(m_k,\omega) f \|_2 \leq C \| f \|_s \left((\log m_k)^{1/2} m_k^{\frac{1}{1+s}} \right) . \tag{2.34}$$

The above agument settles the case of a rotation number satisfying (2.28) for $\gamma = s$, i.e. sufficiently strongly Diophantine compared to the degree of smoothness of the function $f \in H^s(S^1)$.

Finally, we are left with the case that

$$\| q_k \omega \| < \frac{1}{q_k^{1+s}} \, , \tag{2.35}$$

for $k \in \mathcal{K}$ an infinite set of natural numbers. We are therefore under the hypothesis of Lemma 2.2 (for $\gamma = s$), which gives, taking $\gamma^- = s$,

$$\| \hat{S}(m_k,\omega) f \|_2 \leq C \| f \|_s \left(m_k^{1-\frac{s}{1+s}} + m_k^{\frac{1}{1+s}} \right) \, , \tag{2.36}$$

for a suitable choice of the sequence $(m_k)_{k \in \mathbf{N}}$. ¿From (2.34) and (2.36) equation (1.20) now follows. The proof of (1.22) follows from (1.20) by noticing that if the Fourier coefficients of f satisfy the estimates $|f_n| \leq C n^{-\nu}$, $\nu > 1/2$, then $f \in H^{\nu^-}(S^1)$, for all $\nu^- < \nu - 1/2$. Finally, to prove (1.21) and (1.23), it is enough to use the hypothesis on ω in (2.31). $\square$

Proof of Theorem 1.3. We have

$$\begin{aligned} \| S(m,\omega)f \|_2^2 &\geq c \sum_{n\in} \frac{1}{|n|^{2\nu}} \frac{\sin^2 \pi m n \omega}{\sin^2 \pi n \omega} \\ &\geq c \sum_k \frac{1}{{q_k}^{2\nu}} \frac{\sin^2 \pi m \| q_k \omega \|}{\sin^2 \pi \| q_k \omega \|} \ . \end{aligned}$$

Let $k'_m = \min\{k \ | \| q_k \omega \| < \frac{1}{2m}\}$, then

$$\| S(m,\omega)f \|_2 \geq C \ m \ q_{k'_m}^{-\nu}.$$

But, since $\| q_{k'_m - 1}\omega \| \geq \frac{1}{2m}$, (1.9) implies $2m \geq q_{k'_m}$, so that

$$\| S(m,\omega)f \|_2 \geq C_\nu \ m^{1-\nu},$$

proving the result. $\square$

Acknowledgements: Stephan De Bièvre would like to thank the University and the Research Centre of Crete, where part of this work was performed, for their hospitality. Giovanni Forni is grateful to the UFR de Mathématiques et LPTM, Université de Paris VII, for their very kind hospitality during Spring 1995 while most of this work was performed.

References

[AG] S. Aubry, C. Godrèche, *Incommensurate structure with no average lattice: an example of one-dimensional quasi-crystal*, J. Physique C3 (1986), 187-196.

[B] M. Berry, *Incommensurability in an exactly soluble quantal and classical model for a kicked rotator*, Physica 10D (1984), 369-378.

[Be] J. Bellissard, *Stability and instability in quantum mechanics*, in "Trends and developments in the 80's", Eds. S. Albeverio and Ph. Blanchard, Bielefeld (1985).

[CMPS] D. Crisp, W. Moran, A. Pollington, P. Shiue, *Substitution invariant cutting sequences*, Journal de Théorie des nombres de Bordeaux 5 (1993), 123-137.

[D] J.M. Dumont, *Summation formulae for substitutions on a finite alphabet*, Number Theory and Physics, Eds. J.M. Luck, P. Moursa, M. Waldschmidt, Springer Lecture Notes in Physics 47 (1990), 185-194.

[G] G. Gallavotti, *Elements of classical mechanics,* Springer Texts and Monographs in Physics (1983).

[GLV] C. Godrèche, J.M. Luck, F. Vallet, *Quasiperiodicity and types of order: a study in one dimension*, J. Phys. A 20 (1987), 4483-4499.

[HL] P. Hellekalek, G. Larcher, *On the ergodicity of a class of skew products*, Isr. J. Math. (4) 3 (1986), 301-306.

[K] A.Khinchin, *Continued Fractions,* The University of Chicago Press, Chicago and London (1964).

[Ke] H. Kesten, *On a conjecture by Erdösz and Szüsz related to uniform distribution modulo 1,* Acta Arithm. 12 (1966), 193-212.

[KN] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Wiley, New York (1974).

[L] P. Liardet, *Regularities of distribution*, Compositio Mathematica 61 (1987), 267-293.

[P1] K. Petersen, *Ergodic theory*, Cambridge Univ. Press, Cambridge (1983).

[P2] K. Petersen, *On a series of cosecants related to a problem in ergodic theory*, Compositio Mathematica 26 3 (1973), 313-317.