The Speed of Fronts of the Reaction-Difusion Equation

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Abstract

We study the speed of propagation of fronts for the scalar reaction-diffusion equation $u_t = u_{xx} + f(u)$ with $f(0) = f(1) = 0$. We give a new integral variational principle for the speed of the fronts joining the state $u = 1$ to $u = 0$. No assumptions are made on the reaction term $f(u)$ other than those needed to guarantee the existence of the front. Therefore our results apply to the classical case $f > 0$ in $(0, 1)$, to the bistable case and to cases in which f has more than one internal zero in $(0, 1)$.

82.40.Ck, 47.10+g, 3.40Kf, 2.30.Hq

The one dimensional reaction diffusion equation

$$u_t = u_{xx} + f(u) \qquad \text{with} \qquad f(0) = f(1) = 0 \tag{1}$$

with $f(u) \in C^1[0,1]$ has been the subject of much study as it models diverse phenomena in biology, population dynamics, chemical physics, combustion and others [1–5]. Not only is it itself of interest, but based on the rigourous results available for this equation, diverse methods applicable to pattern forming systems have been developed [6–9]. In applications, the reaction term $f(u)$ obeys additional requirements depending on the phenomenon being modelled. Three types of nonlinearities appear to be generic, and are shown in Fig. 1. Type A, for which $f > 0$ in $(0,1)$ is the class to which the classical case of Fisher [10] and Kolmogorov, Petrovskii and Piskounov (KPP) [11] belongs. Type B, usually referred to as the combustion case satisfies $f = 0$ on $(0,a)$ and $f > 0$ on $(a,1)$, while finally, type C, called the bistable case, satisfies $f(u) < 0$ for u in $(0,a)$, $f > 0$ on $(a,1)$ with $\int_0^1 f(u)\, du > 0$. More general cases for f, namely cases in which f has more than one internal zero have also been studied [12,13].

The time evolution of an initial condition $u(x,0)$ has been studied for all the cases mentioned. It was proved [14] that for suitable initial conditions the disturbance evolves into a monotonic travelling front $u = q(x - ct)$ joining the stable state $u = 1$ to $u = 0$. In case A there is continuum of values of c for which a monotonic front exists, the system evolves into the front of minimal speed. In cases B and C there is a single isolated value of the speed for which the front exists. In these two last cases there are threshold effects, necessary conditions for the evolution of the system into the front have been established as well [12,14]. The same is true for reaction terms with more than one internal zero [12]. The problem which interests us here is the determination of the asymptotic speed of the front. There have been numerous studies of this problem, a very complete review is given in [4]. For reaction terms of type A which in addition satisfy $f'(0) > f(u)/u$ the speed is given by [11] $c = c_{KPP} = 2\sqrt{f'(0)}$. Another reaction term of type A, that of a function f approaching a Dirac function at $u = 1$ was studied related to combustion by Zeldovich and Frank–Kamenetskii. They showed [15] that the speed tends to $c_{ZFK} = \sqrt{2\int_0^1 f(u)\,du}$. For an arbitrary reaction function $f(u)$ a local variational principle of the minimax type exists [4,16]. For reaction terms of type A and B we have shown that an integral variational principle of the Rayleigh Ritz type exists [17,18]. Recently an interesting conjecture [19] has been put forward for a restricted class of reaction functions.

The purpose of this article to show that the speed of the front joining the state $u = 1$ to $u = 0$ derives from an integral variational principle without any restriction on f other than those needed to guarantee the existence of the front. The derivation follows an approach similar to the one used to obtain the principle valid only for for positive reaction terms. However, this new principle which is valid for all cases, is not related, nor equivalent, to the previous one.

It is known [14] that for a function of type A, B, or C there exists a strictly decreasing front $u = q(x-ct)$ joining $u = 1$ to $u = 0$ for some $c > 0$. The front satisfies $q_{zz} + cq_z + f(q) = 0$, $\lim u_{z\to-\infty} = 1$, $\lim u_{z\to\infty} = 0$, where $z = x - ct$. Following the usual procedure, since the front is monotonic, we define $p(q) = -dq/dz$, where the minus sign is included so that p is positive. One finds that the monotonic fronts are solutions of

$$p(q)\,\frac{dp}{dq} - c\,p(q) + f(q) = 0, \tag{2a}$$

with

$$p(0) = 0, \qquad p(1) = 0, \qquad p > 0 \quad \text{in} \quad (0,1). \tag{2b}$$

The derivation follows in a simple way from equation (2a). Let g be any positive function in (0,1) such that $h = -dg/dq > 0$. Multiplying equation (2a) by $g(q)$ and integrating between $q = 0$ and $q = 1$ we obtain, after integration by parts,

$$\int_0^1 f\,g\,dq = c\int_0^1 p\,g\,dq - \int_0^1 \frac{1}{2} h\,p^2\,dq. \tag{3}$$

However, since p, g and h are positive, for fixed q, the function

$$\phi(p) = c\,p\,g - \frac{1}{2} h\,p^2$$

has a maximum at

$$p_{max} = c\frac{g}{h} \tag{4}$$

so

$$\phi(p) \le c^2 \frac{g^2}{2h}$$

at each value of q. It follows then that

$$c^2 \ \ge 2\, \frac{\int_0^1 f\, g\, dq}{\int_0^1 (g^2/h) dq}. \tag{5}$$

This bound on the speed is valid for any f for which a monotonic front exists. To show that this is a variational principle we must show that there exists a function $\hat{g}$ at which the equality holds. From Eq. (4) we see that

$$c\frac{\hat{g}}{\hat{h}} = p$$

which can be integrated. The maximizing g is given by

$$\hat{g} = \exp\left(- \int_{q_0}^{q} \frac{c}{p}\, dq \right) \tag{6}$$

with $0 < q_0 < 1$. Evidently $\hat{g}$ is positive, monotonic decreasing and moreover $\hat{g}(1) = 0$. Near $q = 0$ $\hat{g}$ diverges. We must ensure that the integrals in Eq. (5) exist. To verify this we recall [14] that in the three cases, A, B and C, the front approaches $q = 0$ exponentially, therefore, near zero,

$$p \sim \frac{1}{2}\left(c + \sqrt{c^2 - 4f'(0)} \right)\ q \equiv m\, q.$$

Then we obtain

$$\hat{g}(q) \sim \frac{1}{q^{c/m}}$$

near zero and $f\hat{g}$ and $\hat{g}^2/\hat{h}$ diverge at most as $q^{1-(c/m)}$. The integrals in Eq.(5) exist if $m/c > 1/2$. This condition is always satisfied when $f'(0) < 0$, that is in cases B and C. In case A this condition is satisfied provided that $c > 2\sqrt{f'(0)}$, which is the KPP case. However, choosing as a trial function $g(q) = \alpha(2-\alpha)u^{\alpha-2}$ with $0 < \alpha < 1$ one can check that in the limit $\alpha \to 0$, $c^2 \to 4f'(0)$. Therefore, including all cases our main result is

$$c^2 = \sup \left(2 \frac{\int_0^1 f\, g\, dq}{\int_0^1 (-g^2/g')dq} \right), \tag{7}$$

where the supremum is taken over all positive decreasing functions g in $(0,1)$ for which the integrals exist. Moreover there is always a maximizing g in cases B and C, while for case A there is a maximizing g whenever $c > c_{KPP}$.

While we considered the case of decreasing fronts, similar results can be derived for increasing fronts and for the case of density dependent diffusion following the same approach.

As an example we may apply the above result to the Nagumo equation which corresponds to a reaction term of the form

$$f(u) = u(1-u)(u-a) \qquad \text{with} \qquad 0 < a < 1/2$$

This reaction term is of the bistable type. For this equation the solution to Eq.(2a) is known, it is given by $p(q) = \frac{1}{\sqrt{2}} q(1-q)$ and the speed is given by

$$c = \frac{1}{\sqrt{2}} - a\sqrt{2}.$$

To exhibit in this solvable case that the exact speed can be obtained from the variational principle choose as a trial function

$$g(q) = \left(\frac{1-q}{q} \right)^{1-2a}$$

The integrals can be performed easily. We obtain

$$\int_0^1 (-g^2/g')dq = \frac{\Gamma(1+2a)\Gamma(3-2a)}{(1-2a)\Gamma(4)}$$

and

$$\int_0^1 f\, g\, dq = \frac{(1-2a)\Gamma(1+2a)\Gamma(3-2a)}{4\Gamma(4)}$$

so that $c^2 = (1-2a)^2/2$ which is the exact value. For other non solvable cases, it is a simple matter to obtain accurate values for the speed using standard variational techniques.

I. ACKNOWLEDGMENTS

This work has been partially supported by Fondecyt project 1930559.

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