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TOPICS IN NONHAMILTONIAN (MAGNETIC–TYPE) INTERACTION OF CLASSICAL HAMILTONIAN DYNAMICAL SYSTEMS. II. GENERALIZED EULER–AMPÈRE EQUATIONS AND BIOT–SAVART OPERATORS.

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Abstract. The paper is devoted to the algebraico–analytic presentation of the Euler–Ampère dynamical equations for two magnetically interacting massive systems (the charged Euler tops) and their generalizations. The interaction dynamics is based on the algebraic structure of isotopic pairs, which underlies the noncanonical version of Poisson brackets depending explicitely on the states of the acting system.

I. Introduction

The Hamiltonian approach lies undoubtly in a core of analytic mechanics elaborated by J.L.Lagrange, A.M.Legendre, S.Poisson, W.R.Hamilton, C.G.J.Jacobi, J.Liouville, their contemporaries and successors. Many important classical Hamiltonian dynamical systems (e.g. the Euler tops [Ar,Fo,LL1]) are somehow related to the Lie algebras [Ar,DNF,Fo,TF] or their nonlinear generalisations [KM]. Really, just the algebraic approach to classical mechanics, which goes back to S.Lie, É.Cartan and H.Poincaré, supplies analytic dynamics by many elegant and physically important examples.

The interaction of Hamiltonian systems may be either Hamiltonian, i.e. defined by a subsidiary interaction term $\mathcal{H}_{\text{int}}$ of the Hamiltonian and, perphaps, by a deformation of initial ("free") Poisson brackets, or nonHamiltonian. The least case, corresponding to certain nonpotential interaction forces, is of an interest from both theoretical and practical points of view.

First, indeed, the Hamiltonian approach to the interaction of systems is no more than a formal abstraction of the prerelativistic mechanics, which ignores a fundamental principle of a short-range action underlying the field–theoretic picture. Zero approximation of the classical mechanics is correct only for small distances between systems and slow velocities of their motion. Though the idea of the short-range action is, in general, incompatible with mechanics of material points and instantly spreading forces, the first most precise field approximation admits a formulation on the language of the nonHamiltonian classical mechanics in terms of nonpotential magnetic–type forces, which depend on velocities. In electrodynamics such forces obey the Ampère-Lorentz law [A,M,L], which is timer than the Hamiltonian Darwin approximation [LL2]. One can formulate more subtle field effects such as induction on the language of the nonHamiltonian classical mechanics, though it claims to introduce the forces explicitely depending on the second time derivatives (the Weber law [W]). Note that a general law of interaction of two charged particles [Fe] may be formulated on the language of the integrodifferential mechanics (the retarded potentials), which goes back to C.Gauss (a letter to W.Weber, 1845) and to B.Riemann [R].

Some analogs of the Ampère-Lorentz law may be written also for particles, interacting via a nonlinear (e.g. nonabelian gauge or gravitational) field. For the gravitational interaction an analog of the Darwin approximation is one of Einstein, Infeld, Hoffman; Eddington, Clark and Fichtenholz [F,LL2]. Certainly, in the nonlinear case the interaction forces are not pairwise (and, moreover, can not be splitted via reduction to the Weber form), though the nonHamiltonian terms themselves are represented as a sum of two-particle ones. Note that one of the most known effects of the general relativity, the shift of the Mercury perihelion (as well as the secular shift of a particle orbit in the gravitational field of the rotating body, the relativistic precession of a spherical top, etc.), is described very precisely by the nonHamiltonian magnetic-type corrections to the Newton force of gravity.

Second, the nonconservative interaction forces appear also in the classical mechanics itself (general nonholonomic constraints [H,Ap,Ch,LCA,NF,P,Wh], gyroscopic systems [TT,Gl,I,LCA,Bo,AKN], Rayleigh–Lienart type interactions [AVK, AKN,P]), in the servomechanics [Ap], the electrotechnics and in the general theory of (natural or artificial) systems with a feedback [B,Bu,St,La,RSC].

Note that the nonHamiltonian corrections, which are caused by a finite speed of interactions (cf. the cited letter of C.Gauss to W.Weber), are not obligatory relativistic. They also appear in the continuum media mechanics, e.g. in the simple model of the interaction of n vortices "freezed" into the ideal fluid [AKN]. In general, the nonHamiltonian terms are the first order corrections related to the retarded potentials and so they are peculiar to a wast class of linear differential equations.

Really, the nonHamiltonian forces of both kinds (magnetic–type and Rayleigh–Lienart type) underlie also the soliton–soliton interactions governed by the n–soliton solutions of nonlinear equations of mathematical physics or their algebrogeometric perturbations. The presence of a "hidden potential action at distance" in the 1–dimensional dynamics of poles of solutions of some nonlinear equations of mathematical physics [Pe] is an exception, which only confirms the general rule (1–dimensional case does not admit magnetic–type forces). The dynamics of zeroes of the polynomial solutions of linear equations is almost always nonHamiltonian [Pe] (with pairwise forces of the Rayleigh–Lienart type).

So the theoretical and practical significance of exploration of nonHamiltonian systems is out of any doubts.

An important problem of mathematical physics is to describe various algebraic structures, governing the nonHamiltonian interaction. The reasonableness of such setting has no need in any proofs. Among many arguments the following ones are mostly convincing: (1) the algebraic approach is a new vision of the known differential equations, solutions of which in their turn may encode an important algebraic or geometric information (e.g. instantons or monopoles), (2) often algebraically constructed differential equations possess *a priori* or *a posteriori* important and interesting properties (e.g. some form of integrability), (3) if experimental data are not complete and do not allow to restore the law of an interaction uniquely, the use of the algebraically constructed differential equations gives a correct answer as a rule (cf. with the Ampère-Lorentz law of the elementary current interaction [A,G,Gr,M,L]), (4) the solutions of the Wigner problem (the dynamical inverse problem of the representation theory) [Wi,Ju6] of a derivation of the quantum commutation relations from classical equations of motion are constructed by use of the algebraic structures, which underlie such equations. Some aspects of these topics for the nonHamiltonian interactions were discussed in [Ju7].

The aim of this paper is to imbed the nonHamiltonian magnetic–type interactions into the algebraico–analytic scheme analogous to the well–known one for the Hamiltonian systems.

The author apologizes to the English language reader for the numerous references to the literature in Russian. Several Russian books or papers from the reference list are really translated into English but they are not available to the author himself; however, the main part of the rest Russian publications is essential and the material is not reproduced in the English language literature adequately.

II. Interaction of classical Hamiltonian dynamical systems: general classes

Let us describe the general classes of interactions of Hamiltonian systems.

Definition 1. Let $\dot{X} = \{\mathcal{H}_1(X), X\}_1$ and $\dot{A} = \{\mathcal{H}_2(A), A\}_2$ be two classical Hamiltonian dynamical systems defined by Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ and Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$. Their interaction is called *Hamiltonian* iff it has the Hamiltonian form

$$\begin{cases} \dot{X} = \{\mathcal{H}(X,A), X\} \\ \dot{A} = \{\mathcal{H}(X,A), A\} \end{cases}$$

where $\{\cdot,\cdot\}$ are the new Poisson brackets on both variables A and X, and $\mathcal{H}$ is a new Hamiltonian: $\{\cdot,\cdot\} = \{\cdot,\cdot\}_1 + \{\cdot,\cdot\}_2 + \varepsilon\Phi(\cdot,\cdot)$ and $\mathcal{H}(X,A) = \mathcal{H}_1(X) + \mathcal{H}_2(A) + \varepsilon\mathcal{H}_{int}(X,A)$. Here ε is a coupling constant and $\mathcal{H}_{int}$ is an interaction term.

Often, $\Phi(\cdot,\cdot) = 0$ and $\mathcal{H}_{int}$ is defined by the interaction potential.

The interaction is called *nonHamiltonian* if it can not be defined by the construction above.

It is reasonable to consider certain classes and types of nonHamiltonian interactions.

Classes of nonHamiltonian interaction.

Definition 2. Let $\dot{X} = \{\mathcal{H}_1(X), X\}_1$ and $\dot{A} = \{\mathcal{H}_2(A), A\}_2$ be two classical Hamiltonian dynamical systems defined by Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ and Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$. Their interaction

$$\begin{cases} \dot{X} = \{\mathcal{H}_1(X), X\} + \varepsilon v_1(X, A) \\ \dot{A} = \{\mathcal{H}_2(A), A\} + \varepsilon v_2(X, A) \end{cases}$$

is called *conservative* iff $\dot{\mathcal{H}}_1 + \dot{\mathcal{H}}_2 = \mathcal{L}_{v_1}\mathcal{H}_1 + \mathcal{L}_{v_2}\mathcal{H}_2 = 0$ and *ultraconservative* (or *gyroscopic* [Bo,N,AKN]) iff $\dot{\mathcal{H}}_1 = \mathcal{L}_{v_1}\mathcal{H}_1 = 0$, $\dot{\mathcal{H}}_2 = \mathcal{L}_{v_2}\mathcal{H}_2 = 0$.

Types of nonHamiltonian interaction.

Definition 3. Let $\dot{X} = \{\mathcal{H}_1(X), X\}_1$ and $\dot{A} = \{\mathcal{H}_2(A), A\}_2$ be two classical Hamiltonian dynamical systems defined by Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ and Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$. Their interaction is called *magnetic–type interaction* iff it has the form

$$\begin{cases} \dot{X} = \{\mathcal{H}_1(X), X\}_A \\ \dot{A} = \{\mathcal{H}_2(A), A\}_X \end{cases}$$

with new Poisson brackets explicitely depending on the states of the acting systems. The interaction is called *electromagnetic–type (hybrid) interaction* iff it has the similar form but with the potential interaction term included into the Hamiltonian.

Lemma 1. *Magnetic–type interaction is ultraconservative.*

Proof. Really, $\dot{\mathcal{H}}_1(X) = \{\mathcal{H}_1(X), \mathcal{H}_1(X)\}_A = 0$ and $\dot{\mathcal{H}}_2(A) = \{\mathcal{H}_2(A), \mathcal{H}_2(A)\}_X = 0$. $\square$

Remark 1. If the interaction is of a magnetic type and $\mathcal{I}(X, A)$ is a magnitude such that $\{\mathcal{H}_i, \mathcal{I}\}_{(\cdot)} = 0$ $(i = 1, 2)$ for all values of the subscript parameters then $\mathcal{I}$ is an integral of motion. However, this construction does not provide us with *all* integrals of motion contrary to the Hamiltonian dynamics.

The simplest case of magnetic-type interaction is one of linear dependence on the states of the opposite system. To describe such dynamics one needs in purely algebraic requisites, which are discussed in the next paragraph.

III. The algebra of isotopic pairs

There exist, at least, two different approaches to the problem of a description of algebraic structures underlying the nonHamiltonian interaction. The first one relates such interaction to certain deformations of the initial algebraic structures (Lie algebras) such as isotopic pairs [Ju1,Ju4,Ju5] or general nonlinear I-pairs [Ju2]. The second one relates the interaction to certain subsidiary algebraic structures (designs) on Lie algebras [Ju3].

The least approach is effective for nonultraconservative interactions of Lienart type, whereas the first one is convenient for nonHamiltonian magnetic-type interactions.

This paragraph is devoted to a systematic introduction to the algebra of the isotopic pairs.

Definition 4 [Ju1,Ju4,Ju5,Ju6]. The pair (V_1, V_2) of linear spaces is called *an (even) isotopic pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X, Y) \mapsto [X, Y]_A = m_1(A, X, Y)$ $(X, Y \in V_1,\ A \in V_2)$ and $(A, B) \mapsto [A, B]_X = m_2(X, A, B)$ $(A, B \in V_2,\ X \in V_1)$ obey the Jacobi identity for all values of a subscript parameter (such operations will be called *isocommutators* and the subscript parameters will be called *isotopic elements*) and are compatible to each other, i.e. the identities

$$\begin{aligned}[X, Y]_{[A,B]_Z} =& \tfrac{1}{2}([[X, Z]_A, Y]_B + [[X, Y]_A, Z]_B + [[Z, Y]_A, X]_B - \\ & [[X, Z]_B, Y]_A - [[X, Y]_B, Z]_A - [[Z, Y]_B, X]_A)\end{aligned}$$

and

$$\begin{aligned}[A, B]_{[X,Y]_C} =& \tfrac{1}{2}([[A, C]_X, B]_Y + [[A, B]_X, C]_Y + [[C, B]_X, A]_Y - \\ & [[A, C]_Y, B]_X - [[A, B]_Y, C]_X - [[C, B]_Y, A]_X)\end{aligned}$$

$(X, Y, Z \in V_1,\ A, B, C \in V_2)$ hold.

Let's discuss this definition.

First, it may be considered as a result of an axiomatization of the following trivial construction: let $\mathcal{A}$ be an associative algebra (f.e. any matrix one) and V_1, V_2 be two linear subspaces in it such that V_1 is closed under the isocommutators $(X, Y) \mapsto [X, Y]_A = XAY - YAX$ with isotopic elements A from V_2, whereas V_2 is closed under the isocommutators $(A, B) \mapsto [A, B]_X = AXB - BXA$ with isotopic elements X from V_1.

Remark 2 [Ju5,Ju6]. Let H be a (finite dimensional) linear space. If $\mathcal{A}$ is a subspace of $\mathrm{End}(H)$ let's put $\mathcal{A}^{\mathfrak{i}} = \{X \in \mathrm{End}(H), \forall A \in \mathcal{A}, \forall B \in \mathcal{A}, AXB - BXA \in \mathcal{A}\}$. Then $\mathcal{A} \subseteq \mathcal{A}^{\mathfrak{i}\mathfrak{i}}$ and $(\mathcal{A}, \mathcal{A}^{\mathfrak{i}})$ is an isotopic pair.

Remark 3 [Ju5,Ju6]. Let $\mathcal{A} \subseteq \mathrm{End}(H)$, $\dim \mathcal{A} = n$, Lie_n be the space of all Lie algebras of dimension n. Then there exists a natural mapping $\mathcal{L} : \mathcal{A}^{\mathfrak{i}} \mapsto \mathrm{Lie}_n$. It should be mentioned that $\mathrm{card}\,\mathcal{L}(\mathcal{A}^{\mathfrak{i}})$ may be not equal to 1 so $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are not the same in general for different X and Y. It means that the isocommutators $[\cdot,\cdot]_X$ and $[\cdot,\cdot]_Y$ determine structures of nonisomorphic Lie algebras on the space $\mathcal{A}$ in general (though they may be isomorphic in particular).

Remark 4 [Ju6]. The construction of the remark 1 may be immediately generalized. Namely, let (V_1, V_2) be an isotopic pair, $\mathcal{A} \subseteq V_1$, $\mathcal{A}^{\mathfrak{i}} := \{X \in V_2 : (\forall A, B \in \mathcal{A})[A, B]_X = 0\}$. Then $(\mathcal{A}, \mathcal{A}^{\mathfrak{i}})$ is an isotopic pair.

Remark 5 [Ju4]. One may also consider the following construction. Let (V_1, V_2) be an isotopic pair, A and X be elements of V_1 and V_2, respectively. Put $U_1 = \{B \in V_1 : [B, A]_X = 0\}$ and $U_2 = \{Y \in V_2 : [Y, X]_A = 0\}$, then (U_1, U_2) is an isotopic pair.

Numerous examples of isotopic pairs were considered in [Ju1,Ju4,Ju6]. Very intriguing related topics (isotopic pairs and Lie superalgebras, isotopic pairs and Lie $\mathfrak{g}$-bunches; isotopic pairs, Lie hybrids and L.V.Sabinin's nonlinear geometric algebra, etc.) were discussed in [Ju5,Ju6]; the knowledge of these topics is not necessary for the understanding of the following material.

Let us concentrate an attention on some new aspects of the algebraic theory.

Let (V_1, V_2) be an isotopic pair and e_1, e_2 be certain fixed elements of V_1, V_2, respectively. It is convenient to define four operators $\mathrm{ad}^{(1)}(x) : V_1 \mapsto V_1$, $\mathrm{ad}^{(2)}(u) : V_2 \mapsto V_2$, $\mathrm{coad}^{(1)}(x) : V_2 \mapsto V_2$, $\mathrm{coad}^{(2)}(u) : V_1 \mapsto V_1$:

$$\mathrm{ad}^{(1)}(x) = [x, \cdot]_{e_2},\ \mathrm{ad}^{(2)}(u) = [u, \cdot]_{e_1}, \quad \mathrm{coad}^{(1)}(x) = [e_2, \cdot]_x,\ \mathrm{coad}^{(2)}(u) = [e_1, \cdot]_u,$$

where $x \in V_1$, $u \in V_2$.

Lemma 2. *Operators* $\mathrm{ad}^{(i)}$ *supply* V_1 *and* V_2 *by the structures of Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$*:*

$$\begin{aligned}
[\mathrm{ad}^{(1)}(x), \mathrm{ad}^{(1)}(y)] &= \mathrm{ad}^{(1)}([x, y]_{e_2}),\\
[\mathrm{ad}^{(2)}(u), \mathrm{ad}^{(2)}(v)] &= \mathrm{ad}^{(1)}([u, v]_{e_1}).
\end{aligned}$$

Operators $\mathrm{coad}^{(i)}$ *supply* V_1 *and* V_2 *by the structures of* $\mathfrak{g}_2$*– and* $\mathfrak{g}_1$*–modules, respectively:*

$$\begin{aligned}
[\mathrm{coad}^{(1)}(x), \mathrm{coad}^{(1)}(y)] &= \mathrm{coad}^{(1)}([x, y]_{e_2}),\\
[\mathrm{coad}^{(2)}(u), \mathrm{coad}^{(2)}(v)] &= \mathrm{coad}^{(2)}([u, v]_{e_1}),
\end{aligned}$$

where $x, y \in V_1$, $u, v \in V_2$.

Proof. The statement of the Lemma is a sequence of the Jacobi and Faulkner–Ferrar identities

$$[X, Y]_{[A,B]_Z} = [[X, Z]_A, Y]_B + [[Z, Y]_A, X]_B - [[X, Y]_B, Z]_A,$$

which hold in any isotopic pair.

Definition 5 [Ju6]**.** An isotopic pair (V_1, V_2) is called *contragredient*, if $V_1 = V_2^*$, $V_2 = V_1^*$ and for all $A, B \in V_2$, $X, Y \in V_1$ the following sequence holds:

$$\langle [X, Y]_B, A\rangle = -\langle [X, Y]_A, B\rangle = \langle [A, B]_X, Y\rangle = -\langle [A, B]_Y, X\rangle,$$

where $\langle\cdot,\cdot\rangle$ is a natural pairing of V_1 and V_2, $X, Y \in V_1$, $A, B \in V_2$.

Remark 6. In the contragredient isotopic pair (V_1, V_2) the operators $\mathrm{coad}^{(i)}$ are conjugate to the operators $\mathrm{ad}^{(i)}$.

Definition 6. The pair of mappings $L_i : V_i \mapsto V_i$ is called *a derivative* of the isotopic pair (V_1, V_2) iff

$$\begin{aligned}
L_1([x, y]_u) &= [L_1 x, y]_u + [x, L_1 y]_u + [x, y]_{L_2 u},\\
L_2([u, v]_x) &= [L_2 u, v]_x + [u, L_2 v]_x + [u, v]_{L_1 x},
\end{aligned}$$

where $x, y \in V_1$, $u, v \in V_2$.

Remark 7. The derivatives of an isotopic pair (V_1, V_2) form a Lie algebra, which will be denoted by $\mathrm{Der}(V_1, V_2)$.

Lemma 3. *The pairs of mappings* $(\mathrm{ad}^{(i)}, \mathrm{coad}^{(i)})$ *realize homomorphisms* $\mathfrak{g}_i \mapsto \mathrm{Der}(V_1, V_2)$ *of Lie algebras* $\mathfrak{g}_i$ *into the Lie algebra* $\mathrm{Der}(V_1, V_2)$.

Proof. One should use the Faulkner–Ferrar identities.

Definition 6. Two elements e_1 and e_2 of the isotopic pair (V_1, V_2) $(e_i \in V_i)$ are called *polar* if

$$[e_1, \cdot]_{e_2} = 0, \qquad [e_2, \cdot]_{e_1} = 0.$$

Remark 8. Let (V_1, V_2) be an isotopic pair, e_1 and e_2 be two arbitrary elements of V_1 and V_2, respectively, (U_1, U_2) be a subpair of (V_1, V_2) of the remark 4. Then e_1, e_2 are polar elements in (U_1, U_2).

Note that two elements A and B of the isotopic pair $(\mathrm{End}(H), \mathrm{End}(H))$ are polar iff $AB = BA = C$, where C is the scalar matrix.

Lemma 4. *The operators* $\mathrm{ad}^{(i)}$ *and* $\mathrm{coad}^{(i)}$ *realize the Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *as subalgebras of* $\mathrm{Der}(\mathfrak{g}_2)$ *and* $\mathrm{Der}(\mathfrak{g}_1)$*, respectively, iff* e_1 *and* e_2 *are polar elements.*

Proof. The statement of the Lemma is a sequence of the compatibility identities between isocommutators in the isotopic pairs and the definition of the polar elements.

So the most important case is one of $\mathfrak{g}_1 \simeq \mathfrak{g}_2$.

Definition 7 [Ju4]**.** A Lie algebra $\mathfrak{g}$ is called *a magnetic Lie algebra* iff the pair of spaces $(\mathfrak{g}, \mathfrak{g})$ is supplied by the $\mathfrak{g}$–equivariant structure of isotopic pair.

Note that any isotopic pair (V_1, V_2) such that $\dim V_1 = \dim V_2 = n$ is a magnetic Lie algebra with the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^n$. Various notrivial examples of the magnetic Lie algebras were considered in [Ju4].

Lemma 5. *Let* $\mathfrak{g}$ *be a magnetic Lie algebra, then* (V_1, V_2) $(V_1 = \mathfrak{g} \oplus \mathbb{R} \cdot e_1,\ V_2 = \mathfrak{g} \oplus \mathbb{R} \cdot e_2)$ *admits a natural structure of the isotopic pair with polar elements* e_1 *and* e_2*. Operators* $\mathrm{ad}^{(i)}$ *and* $\mathrm{coad}^{(i)}$ *are essentially the operators of the adjoint representation of* $\mathfrak{g}$*.*

The proof of the Lemma is straightforward.

Thus, the necessary algebraic requisites are collected so it is reasonable now to enter upon a construction of the generalized Euler–Ampère dynamics.

IV. Biot-Savart operators and the generalized Euler–Ampère equations

In the paper [Ju1] there were proposed certain versions of nonHamiltonian dynamical equations associated with isotopic pairs. However, the considered toy equations are initially more typical for problems of classical mechanics (gyroscopic systems) and electromechanics (electromotors) than for general ones of Ampère-Lorentz electrodynamics and its analogues (the first nonHamiltonian approximations of nonlinear field interactions), though they may be useful for other interesting physical problems, which were discussed in [Ju1,Ju5].

Here the generalizations of the dynamical equations of [Ju1] will be constructed and certain their properties will be explored. The crucial role will be played by the (generalized) Biot-Savart operators in the construction.

First, note that the isocommutators in an isotopic pair (V_1, V_2) define families of compatible Poisson brackets $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_X$ ($A \in V_2$, $X \in V_1$) in the spaces $S^{\cdot}(V_1) \subset C^{\infty}(V_1^*)$ and $S^{\cdot}(V_2) \subset C^{\infty}(V_2^*)$, respectively. Here $S^{\cdot}(H)$ is the sum of all symmetric degrees of the space H and $C^{\infty}(H)$ is the space of all smooth functions on H. The compatibility means that a linear combination of any two Poisson brackets is also a Poisson bracket.

If $\mathfrak{g}$ is a magnetic Lie algebra then one may define composite Poisson brackets in $S^{\cdot}(\mathfrak{g}) \subset C^{\infty}(\mathfrak{g}^*)$ as a sum of the Poisson brackets related to the Lie algebra $\mathfrak{g}$ itself (Lie–Berezin brackets) and to the isotopic pair structure in $(\mathfrak{g}, \mathfrak{g})$. Such composite brackets will be also denoted by $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_X$, where X belongs to the first copy and A belongs to the second copy of $\mathfrak{g}$.

Let us define the generalized Euler–Ampère dynamics now.

Definition 8. Let $\mathfrak{g}$ be a magnetic Lie algebra. Let us consider two elements $\mathcal{H}_1$ and $\mathcal{H}_2$ (*Hamiltonians*) in $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively. Let us also consider two symmetric mutually anticonjugate operators $R_1 \in \mathrm{Hom}(V_2^*, V_2) = V_2 \otimes V_2$ and $R_2 \in \mathrm{Hom}(V_1^*, V_1) = V_1 \otimes V_2$ (the symmetricity means that $R_i \in S^2(V_i)$), $R_1 + R_2^* = 0$ (*the generalized Biot-Savart operators*). The equations

$$(\forall X \in V_1 \subset C^{\infty}(V_1^*)) \quad \dot{X}(P_t) = \{\mathcal{H}_1, X\}_{R_2(Q_t)}(P_t),$$
$$(\forall A \in V_2 \subset C^{\infty}(V_2^*)) \quad \dot{A}(Q_t) = \{\mathcal{H}_2, A\}_{R_1(P_t)}(P_t),$$

where $P_t \in V_1^*$ and $Q_t \in V_2^*$ are called *the generalized Euler–Ampère equations associated with the magnetic Lie algebra $\mathfrak{g}$, Biot-Savart operators R_i and Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Certainly, the conditions $X \in V_1 \subset C^{\infty}(V_1^*)$, $A \in V_2 \subset C^{\infty}(V_2^*)$ may be changed to the general ones $X \in C^{\infty}(V_1^*)$, $A \in C^{\infty}(V_2^*)$.

Remark 9. If V_i are somehow identified with V_i^* than one may consider Biot-Savart operators R_i as elements of $\mathrm{End}(\mathfrak{g}_i)$. For example, such situation is realized if $\mathfrak{g}$ is a compact magnetic Lie algebra (i.e. a compact Lie algebra, which is a magnetic one). Note that all semisimple Lie algebras $\mathfrak{g}$ admit a $\mathfrak{g}$–equivariant structures of the contragredient isotopic pairs [Ju4]. Some dynamical equations of the papers [Ju1,Ju5] are particular case of the generalized Euler–Ampère equations with R_i equal to identical operators.

Remark 10. If $\mathfrak{g}$ is a magnetic Lie algebra $\mathfrak{so}(3)$ [Ju4], $\mathcal{H}_i$ are defined by the quadratic Casimir functions, operators R_i are traceless and have two equal eigenvalues, than one receives the equations of motion of two spherical charged particles with fixed centers (the Euler tops), interacting via Ampère-Biot-Savart forces. The quantities X and A are identified with angular velocities and the absolute coordinate system is used.

The situation is the same after an account of the first order corrections produced by the Bernett and Einstein-de Haas gyromagnetic effects [V]. However, the dynamical self–induction processes are neglected (see the remarks at the Introduction).

The same data but with $\operatorname{tr} R_i \neq 0$ describes the nonHamiltonian approximations for the gravitational and nuclear [Da] interactions.

Remark 11. Certainly, generalized Euler–Ampère dynamics realizes a nonHamiltonian (magnetic–type) interaction and, therefore, it is ultraconservative (gyroscopic).

Remark 12 (hystorical notes). Note that a geometric formalism for systems under *external* magnetic or gyroscopic forces is really well-known (see e.g. [N,DKN,AKN, Bo]). There exists an equivalent canonical realization in this case, so the non-Hamiltonian approach is excessive. The crucial point of the algebraic description of the *interaction* forces of those types is played by the compatibility conditions in the definition of the isotopic pair. Such conditions were written in a slightly weaker form in the paper by Faulkner J.R. and Ferrar J.C. [FF]. The present form was introduced in [Ju1,Ju4], where its supergeneralization was also formulated. The classical essentially nonHamiltonian dynamics, associated with isotopic pairs, was introduced in [Ju1]. Its quantum counterpart was considered in [Ju5], where the Wigner problem (the dynamical inverse problem of representation theory) was solved for the discussed model. Note that the generalized Euler–Ampère dynamics is also essentially nonHamiltonian.

Remark 13. It seems that some kind of dynamical equations, associated with isotopic pairs, may help to construct their *globalizations*. Such globalizations and their constructions are undoubtly important for the nonlinear geometric algebra [S1-S3]. In its turn methods of the nonlinear geometric algebra may be useful for the global analysis of the long time behaviour of solutions of the dynamical equations. The simple examples of isotopic pairs indicate a relation of this topic with the theory of classical r–matrices [STS,KM] and its quantum counterpart [D,RTF].

Note that the methods of the nonlinear geometric algebra (pseudo(quasi)groups of transformations) are effectively used in the classical theory of nonlinear Poisson brackets and their asymptotic quantization [KM]. So their penetration into the nonHamiltonian mechanics would be very natural. The unified geometroalgebraic approach to the nonlinear Poisson brackets and nonHamiltonian systems is really welcome.

Let us discuss some properties of the generalized Euler–Ampère equations now.

Let us supply a compact Lie algebra $\mathfrak{g}$ by a canonical structure of the magnetic Lie algebra, defined via the Killing form [Ju2]. The related isotopic pair is just the Okubo pair [Ju2].

Theorem 1. *If Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by quadratic Casimir functions then the Euler–Ampère dynamics on the compact magnetic Lie algebra $\mathfrak{g}$ has the solution*

$$X_t = \frac{\mathcal{E}_1}{||Y_t||} Y_t,$$

$$A_t = \frac{\mathcal{E}_2}{||B_t||} B_t,$$

where $\mathcal{E}_1^2 = \mathcal{H}_1(X_0)$, $\mathcal{E}_2^2 = \mathcal{H}_2(A_0)$, and

$$\begin{pmatrix} Y_t \\ B_t \end{pmatrix} = \exp\left\{ \begin{pmatrix} 0 & R \\ -R^* & 0 \end{pmatrix} t \right\} \begin{pmatrix} Y_0 \\ B_0 \end{pmatrix},$$

where R is the Biot–Savart operator.

Proof. First, note that $\|Y_t\| = \|B_t\|$ for all t. The Euler–Ampère equations

$$\dot{X}_t = (R_1(A_t), X_t)X_t - \|X_t\|^2 R_1(A_t), \quad \dot{A}_t = -((R_2(X_t), A_t)A_t - \|A_t\|^2 R_2(X_t))$$

follows from this fact immediately (here $(\cdot,\cdot)$ is the Killing form, $\|Z\| = \sqrt{(Z,Z)}$) $\square$

Corollary. *A motion of a pair of the magnetically interacting generalized Euler–Ampère tops is almost periodic with almost periods $\frac{2\pi}{\omega_i}$, where ω_i are the eigenvalues of the Biot–Savart operator.*

Remark 14. If the Biot–Savart operator is traceless and $\mathrm{SO}(n-1)$–invariant ($n = \dim \mathfrak{g}$), then the generalized Euler–Ampère dynamics is periodic. In particular, the dynamics of a pair of two magnetically interacting spherical tops is periodic.

V. The generalized Euler–Ampère dynamics on the magnetic Lie groups

Let us consider a certain generalization of dynamics discussed earlier.

Definition 9. *A magnetic Lie group* is a Lie group G, which Lie algebra $\mathfrak{g}$ is a magnetic Lie algebra.

Let us consider two copies G_1 and G_2 of a magnetic Lie group G and the cotangent fiber bundles T^*G_1, T^*G_2 over them. These bundles are supplied by canonical symplectic structures, which define Poisson brackets in the spaces $C^\infty(T^*(G_i))$ of smooth functions over T^*G_i. On the other hand, the structure of isotopic pair in $\mathfrak{g} \oplus \mathfrak{g}$ perturbes these Poisson brackets, so that the new brackets in $C^\infty(T_1^*)$ and $C^\infty(T^*2)$ become to depend on the points of T^*G_2 and T^*G_1, respectively.

Definition 10. Let G be a magnetic Lie group, $\mathcal{H}_1$, $\mathcal{H}_2$ be two smooth G_i–invariant functions on the cotangent bundles T^*G_i over G_i, R_1 and R_2 be two mutually anticonjugate $G_1 \times G_2$–invariant sections of the bundles $\mathrm{Hom}_{G_1\times G_2}(T^*G_1, TG_2)$ and $\mathrm{Hom}_{G_2\times G_1}(T^*G_2, TG_1)$, respectively (*the generalized Biot–Savart operators*). The equations

$$\begin{aligned}
(\forall X \in C^\infty(T^*G_1)) \quad \dot{X}(P_t) &= \{\mathcal{H}_1, X\}_{R_2(P_t,Q_t)}(P_t),\\
(\forall A \in C^\infty(T^*G_2)) \quad \dot{A}(Q_t) &= \{\mathcal{H}_2, A\}_{R_1(P_t,Q_t)}(P_t),
\end{aligned}$$

where $P_t \in T^*G_1$ and $Q_t \in T^*G_2$ are called *the generalized Euler–Ampère equations associated with the magnetic Lie group G, Biot-Savart operators R_i and Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Remark 15. Note that now the Biot–Savart operators depend on both P_t and Q_t. However, the P_t in the first equation and Q_t in the second equation appears in the Biot–Savart operators only via its projection on the base G_1 (G_2) of the cotangent bundle T^*G_1 (T^*G_2).

Remark 16. Iff $G = \mathrm{SO}(3)$ and $\mathcal{H}_i$ are the Euler hamiltonians, then the generalized Euler–Ampère equations describe a motion of two asymmetric magnetically interacting charged particles with fixed centers.

Remark 17. The generalized Euler–Ampère dynamics of Def.10 is ultraconservative (gyroscopic).

The main topic of this paragraph is a nonHamiltonian reduction ([AKN]) in the generalized Euler–Ampère equations associated with magnetic Lie groups.

To perform the nonHamiltonian reduction by $G_1 \times G_2$ means to consider the motion of Euler tops interacting via Ampère–Biot–Savart forces in their proper coordinates. The result is rather straightforward.

Theorem 2. *After the nonHamiltonian reduction the generalized Euler–Ampère equations will have the form*

$$\begin{aligned}
(\forall X \in C^\infty(T^*G_1)) \quad \dot{X}(P_t) &= \{\mathcal{H}_1, X\}_{R_2(Q_t)}(P_t), \\
(\forall A \in C^\infty(T^*G_2)) \quad \dot{A}(Q_t) &= \{\mathcal{H}_2, A\}_{R_1(P_t)}(Q_t), \\
\dot{R}_1 &= \operatorname{ad}(Q_t) \circ R_1 + R_1 \circ \operatorname{ad}^*(P_t), \\
\dot{R}_2 &= \operatorname{ad}(P_t) \circ R_2 + R_2 \circ \operatorname{ad}^*(Q_t),
\end{aligned}$$

where R_i are dynamical variables from $\operatorname{Hom}(V_2^*, V_1)$ *and* $\operatorname{Hom}(V_1^*, V_2)$*, respectively. Operators* ad *and* ad* *are the adjoint and coadjoint operators in the Lie algebra* $\mathfrak{g}$*.*

The variables R_i will be called *the dynamical Biot–Savart operators.* Note that R_i are mutually conjugate. So the reduced Euler–Ampère equations on the magnetic Lie groups may be considered as certain analogs of the Euler–Ampère equations on the magnetic Lie algebras with dynamical Biot–Savart operators. So it is reasonable to explicate the algebraic structure of the dynamical Biot–Savart operators.

Let $x \in V_1$, $u \in V_2$, then one may define an element $R_{x,u}$ of $\operatorname{Der}(V_1, V_2)$ in the following way:

$$R_{x,u}(y) = [x, y]_u \quad (u \in V_1), \qquad R_{x,u}(v) = [u, v]_x \quad (v \in V_2).$$

Such derivations will be called *internal.*

Lemma 6. *The internal derivations of an isotopic pair (V_1, V_2) form the Lie algebra* $\operatorname{Inn}(V_1, V_2)$*, which is an ideal of* $\operatorname{Der}(V_1, V_2)$*.*

Proof. One should use the Faulkner–Ferrar identities once more to prove that $\operatorname{Inn}(V_1, V_2)$ is the Lie algebra and the definition of the derivation to prove the second part of the Lemma.

If (V_1, V_2) is the contragredient isotopic pair then the elements of $\operatorname{Inn}(V_1, V_2)$ define pairs of mappings from $\operatorname{Hom}(V_2^*, V_1) = \operatorname{End}(V_1)$ and $\operatorname{Hom}(V_1^*, V_2) = \operatorname{End}(V_2)$. So it is rather natural to suppose that the dynamical Biot–Savart operators are corresponded to certain elements of $\operatorname{Inn}(\mathfrak{g}_1, \mathfrak{g}_2)$ ($\mathfrak{g}_1 \simeq \mathfrak{g}_2 \simeq \mathfrak{g}$) or more generally to elements of $\operatorname{Der}(\mathfrak{g}_1, \mathfrak{g}_2)$. Such setting explicates an algebraic structure of the reduced Euler–Ampère dynamics of Theorem 2.

Remark 18. Note that the dependence of the Poisson brackets of any system on the states of the same system is hidden into the dynamical Biot–Savart operators so that the reduced Euler–Ampère equations receive "visually" a form claimed by Def.3.

VI. Perspectives: Quantum Euler–Ampère dynamics

The performed algebraisation of the nonHamiltonian magnetic–type interactions is sufficient for the quantization of the Euler–Ampère dynamics, i.e. a certain solution of the dynamical inverse problem of the representation theory in sense of [Ju6].

Remark 19. Quantum analogs of the magnetically interacting Euler tops may be used as models for the magnetic interactions of atomic nuclei, nucleons and nucleonic pairs, Cooper pairs, charged solitons, quantum vortices, etc. They may be also adopted for a description of the first quantum gauge field (wionic) approximations of the electroweak interactions of particles (the analog of the quantized Maxwell electromagnetic theory is the gauge Weinberg–Salam electroweak theory, and the analog of the Hamiltonian Darwin approximation is the Fermi theory or the V-A current theory of Gell–Mann, Feinmann, Marshak and Sudarshan [O1,MR,O2]). It seems that the quantum nonHamiltonian constructions may be important for many concrete applications, e.g. the quantum gyroscopes [PS,PRS:Ch.XI], the high T_c superconductive technologies, the spin generators and amplifiers [PRS:Ch.VI], quantum processors and quantum ROM/RAM media, etc.

The quantum mechanical version of the Euler–Ampère dynamics is proposed to be discussed in the forthcoming paper, where the general questions of the quantum nonHamiltonian interactions will be considered.

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