AXIOMATIC APPROACH TO PERTURBATIVE QUANTUM FIELD THEORY*

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Abstract

A derivation with axiomatic methods of a perturbative expansion for the Wightman functions of a relativistic field theory is described. The method gives also the correlation functions of the fields in KMS states. Using these results, a scattering formalism for QED is introduced, which does not involve any infrared divergent quantities.

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1 INTRODUCTION

I will report on a method of developing the perturbation theory of relativistic quantum fields within the context of axiomatic field theory [1-3]. The central objects of the approach are the Wightman functions

$$W(x_1, \ldots, x_n) = \langle \Phi(x_1) \ldots \Phi(x_n) \rangle \ , \tag{1}$$

where $\langle \cdot\cdot \rangle$ denotes the vacuum expectation value, and $\Phi(x_i)$ is any of the basic fields of the theory under consideration, taken at the point x_i of Minkowski space. Perturbative expansions, in the form of sums over generalized Feynman graphs, are derived, starting from equations of motion as dynamical input and using the Wightman properties [1] as subsidiary conditions for the unambiguous solution of these equations. The W-functions are known to determine the theory completely.

More generally, the method yields the following functions:

$$\mathcal{W}(X_1, s_1| \ldots |X_n, s_n) = \langle T^{S_1}(X_1) \ldots T^{S_n}(X_n) \rangle \ . \tag{2}$$

Here the X_i are finite sets of 4-vectors $x_{i_1}, \ldots, x_{i_{\alpha_i}}$, the s_i are signs, and $T^+(X), T^-(X)$ is a time-ordered or anti-time-ordered product respectively of fields $\Phi(x_1), \ldots, \Phi(x_\alpha)$, $X = \{x_1, \ldots, x_\alpha\}$. These functions include as special cases the Wightman functions (all $\alpha_i = 1$), the completely time ordered Green's functions $\tau(x_1, \ldots, x_\alpha)$ (for $n = 1$), and the functions

$$\langle T^*(x_1, \ldots, x_n) T(y_1, \ldots, y_m) \rangle \tag{3}$$

which occur in unitarity relations and play an important part in the description of particle scattering (see section 4).

The method can also be applied to thermal field theories, in which case the symbol $\langle \ldots \rangle$ stands for the expectation value in a thermal equilibrium state with temperature $T \geq 0$, these states being characterized by the KMS condition [4]. The two cases (vacuum and thermal) will be discussed in parallel.

The interest of being able to calculate the correlation functions W in thermal field theories is obvious. In the vacuum case they are held to be somewhat remoter from the quantities of direct physical interest: S-matrix elements are easier to calculate from τ than from W. But the W are more suitable than the τ for studying some fundamental problems of field theory, because their properties are more directly related to the fundamental assumptions of the theory like locality, spectral properties, and the like. An example of such a problem is finding the exact relation between the physical state space of a gauge theory and its state space in a non-physical gauge, especially a Gupta-Bleuler gauge. This problem has not yet been satisfactorily solved, especially in the Yang-Mills case, where it is connected with the confinement problem. Also, for setting up a convincing scattering formalism for the particles of a gauge theory, for which the customary asymptotic conditions do not hold, the functions (3) turn out to be useful, as will be indicated for QED in section 4. It is therefore an advantage of the present method that it produces directly expressions for the general $\mathcal{W}$-functions.

Of course, these expressions can in principle also be derived from the conventional Feynman rules for the τ-functions [5]. But our method is also of a more fundamental significance, since it avoids some of the doubtful ingredients of the conventional canonical formalism. Using W instead of τ as its basic objects, it avoids the notorious ambiguities of time-ordering. It does not assume asymptotic conditions, which are not satisfied for the charged fields of gauge theories. And it does not need the canonical commutation relations, which have a doubtful status in relativistic field theory, because, according to the available evidence, interacting relativistic fields cannot in general be restricted to sharp times.

And, finally, these considerations show that the methods of axiomatic field theory *can* be used to handle dynamical problems.

For the sake of simplicity I will only discuss the ϕ_4^4-model. But the method can be applied to any relativistic, local, field theory, in particular to gauge theories in covariant, local, gauges. The assumptions underlying our formalism will be stated and briefly discussed in section 2. The results will be described in section 3. In the time available it will be impossible to give proofs, even in a sketchy form. For the proofs I must refer to the original publications [6,7]. Finally, in section 4, an application of these ideas to a proper description of scattering events in QED will be discussed.

2 ASSUMPTIONS

The ϕ_4^4 model is the theory of a scalar, hermitian, Wightman field $\Phi(x)$ on 4-dimensional Minkowski space, satisfying the equation of motion

$$(\Box + m^2)\Phi(x) = -\frac{g}{6}N(\Phi^3(x)) \ . \tag{4}$$

In the vacuum case, m is the physical mass of a stable particle with Φ as interpolating field, g is a coupling constant, and N, standing for "normal product", denotes the renormalization prescription needed to make sense out of the a priori undefined third power of the distribution-valued field $\Phi(x)$. Renormalization, leading to the disappearence of all ultraviolet divergences, can be handled by conventional methods and will not be discussed further.

In thermal field theory we demand that the field equation (4) is exactly the same independently of the temperature T. In other words: the KMS states with any value of T should all generate representations of the same abstract field algebra. This means that the parameters m and g, and the subtraction prescription N are independent of T. In particular, m denotes the mass of the Φ-particle at $T = 0$. It is *not* the mass of any corresponding quasi-particle at positive temperatures, which physical mass is temperature-dependent. Such an independence prescription is necessary to make the temperature dependence of physical quantities like specific heats, transport coefficients, and others [8,9] unambiguous.

The equation of motion (4) implies the following infinite system of partial differential equations for the Wightman functions:

$$(\Box_i + m^2)\langle\Phi(x_1)\dots\Phi(x_i)\dots\Phi(x_n)\rangle = -\frac{g}{6}\langle\dots N(\Phi^3(x_i))\dots\rangle\ , \quad i = 1,\dots,n\ . \tag{5}$$

The right-hand side can be expressed in terms of W-functions, once the normalization prescription N has been fixed.

These equations must be solved, using the Wightman properties of the W as subsidiary conditions. In our perturbative approach we will not need all of these properties. Needed in an essential way are: translation invariance, locality, the cluster property in a weak formulation (see below), and for $T = 0$ the spectral property. In the thermal case the latter is replaced by the KMS condition [4], which we use in its p-space form: let $\tilde{W}$ be the Fourier transform of W, and let $\{p_1,\dots,p_n\}, \{q_1,\dots,q_m\}$, be two sets of 4-momenta. Then

$$\tilde{W}(p_1,\dots,p_n,q_1,\dots,q_m) = e^{\beta P_0}\tilde{W}(q_1,\dots,q_m,p_1,\dots,p_n)\ , \tag{6}$$

where $\beta = (kT)^{-1}$ is the inverse temperature, and P_0 is defined as

$$P_0 = \sum_i p_{i,0}\ . \tag{7}$$

For $\beta \to \infty$ the equation (6) becomes the spectral condition in the vacuum

$$\tilde{W}(p_1, \dots, q_m) = 0 \quad \text{if} \quad P_0 < 0 \; . \tag{8}$$

Lorentz invariance is only used in a marginal manner, for fixing the N-prescription in such a way that the field equation (4) transforms covariantly. This condition is implemented at $t = 0$ and fixes then N also for $T > 0$ because of its required T-independence.

Not used are: positivity, asymptotic conditions, and canonical commutation relations. These are decided advantages of the method. The dubious state of CCR's has already been remarked upon in the Introduction, and positivity and asymptotic conditions are not satisfied in gauge theories in local gauges.

In addition, we also demand the usual renormalization conditions fixing m, g, N, and the field normalization. These conditions, with the exception of the last one, need again be applied at $T = 0$ only, and are then transferred to the thermal case by means of the T-independence of m, g, N.

A perturbative solution of the stated problem is constructed as follows. We insert the perturbative expansion

$$W(x_1, \dots, x_n) = \sum_{\sigma=0}^{\infty} g^\sigma W_\sigma(x_1, \dots, x_n) \tag{9}$$

into the equations (5) and equate the terms of order g^σ on both sides:

$$(\Box_i + m^2) W_\sigma(\dots, x_i, \dots) = -\frac{1}{6} \langle \dots N(\Phi^3(x_i)) \dots \rangle_{\sigma-1} \; . \tag{10}$$

For $\sigma = 0$ the right-hand side is zero. For higher σ we solve the equations by induction with respect to σ. Assuming the problem to have been solved up to order $\sigma - 1$, the right-hand side of (10) is known, and W_σ is determined as solution of the system of n linear differential equations (10), using the Wightman properties and the normalization conditions as subsidiary conditions. All the needed conditions except the cluster property are linear in W, and must therefore be satisfied separately in each order of perturbation theory. The cluster property states that

$$\lim_{a \to \infty} W(x_1, \dots, x_n, y_1 + a, \dots, y_m + a) = W(x_1, \dots, x_n) W(y_1, \dots, y_m) \tag{11}$$

if a tends to infinity in a space-like direction. Since the a-limit need not commute with the derivation of W with respect to g, this condition cannot be easily transformed into a perturbative statement. We will therefore only postulate a rather weak corollary of the condition. As is well known, the perturbative expansion of τ, and therefore also of W, can be considered to be an expansion in powers of $\hbar$ instead of in powers of g. And we demand that the equation (11) hold for each $W(x_1, \ldots, x_n)$ in the lowest non-vanishing order in $\hbar$ contributing to it. This suffices to guarantee the uniqueness of our solution. In my formulas the $\hbar$-dependence is, however, suppressed by setting $\hbar = 1$.

3 RESULTS

The problem described in the previous section is solved in two steps [6,7]. Firstly one proves that the subsidiary conditions single out a unique solution of the system (10) in every order σ. Secondly, a solution of these equations satisfying all subsidiary conditions is written down explicitly as a sum over generalized Feynman graphs.

These graphs, for the general case of the functions $\mathcal{W}$, are defined as follows. First draw an ordinary Feynman graph, called a "scaffolding", of the Φ^4 theory, with σ vertices and with external points corresponding to the arguments of $\mathcal{W}$. This graph is then partitioned into a number of mutually non-overlapping subgraphs, called "sectors". To each factor $T^{s_\alpha}(X_\alpha)$ in $\mathcal{W}$ corresponds an "external sector" containing all the external points of the variables in X_α, but no other external points. This sector carries the number α and is said to be of type s_α (remember $s_\alpha = \pm$). If the adjacent external sectors with numbers α, $\alpha + 1$, are of the same type, there may also exist an "internal sector" not containing any external points, with number $\alpha + 1/2$, and of type $s_{\alpha+1/2} \neq s_\alpha = s_{\alpha+1}$. In the thermal case there may also be an internal sector with number $n + 1/2$ if the extremal external sectors 1 and n are of the same type. In this case we have $s_{n+1/2} \neq s_n = s_1$. Such extremal internal sectors are not present in the vacuum case.

To such a partitioned graph we assign an integrand as follows. Each external point carries a variable x_i, each vertex an integration variable u_j. Within a sector of positive type the usual Feynman rules apply, with vertex factors -ig and propagators

$$D_F(\xi) = \Delta_F(\xi) + C_T(\xi) \ , \tag{12}$$

where Δ_F is the familiar Feynman propagator and C_T is the thermal correction

$$C_T(\xi) = i(2\pi)^{-3} \int d^4p [e^{\beta|p_0|} - 1]^{-1} \delta(p^2 - m^2) e^{-ip\xi} \ , \tag{13}$$

which is only present for $T > 0$. In sectors of negative type the complex-conjugate Feynman rules apply. A line connecting two points (external or internal) in different sectors, with variables z_i, z_j, carries the propagator $-iD_+(z_i - z_j)$. Here z_i is the variable in the lower- numbered sector, and

$$D_+(\xi) = \Delta_+(\xi) + C_T(\xi) \ , \tag{14}$$

where again Δ_+ is the familiar invariant function and C_T is only present for $T > 0$. The graph is then integrated over the internal variables u_j, and $\mathcal{W}$ is obtained as a formal sum over all partitioned graphs satisfying the above rules.

Primitively UV divergent subgraphs exist only within sectors, and the corresponding divergences are removed by any of the conventional renormalization procedures. The individual renormalized graphs give then finite contributions if $T = 0$ and $m > 0$. For $m = 0$ and $T = 0$ the individual graphs may be infrared divergent, but these divergences cancel in the sum over all partitioned graphs with the same scaffolding. For $T > 0$ the existence problem is open even in finite orders of perturbation theory.

4 ASYMPTOTIC CONDITIONS IN QED

Experimentalists usually observe particles, not fields. In order to make contact with experiment, a field theory must therefore be able to describe particles. Traditionally this is achieved by means of "asymptotic conditions" stating that the interacting fields of the theory, or appropriate local functions of them, converge for large negative or positive times to free fields, whose connections with a particle picture are well understood. In axiomatic field theory there are essentially two different versions of such conditions: the Haag-Ruelle condition [2,3] involving strong convergence of time dependent states, and the LSZ condition [3] involving weak convergence of suitably averaged field operators. Both these conditions can be proved in theories possessing discrete mass hyperboloids in their momentum spectrum. Unfortunately, gauge theories do not fall in this class. Indeed, all available evidence shows that neither of the two conditions is satisfied for fields carrying gauge charges. (An exception are theories like the electroweak model with spontaneously broken gauge symmetry). This raises the problem of a proper description of particle scattering in such theories. And a concomitant problem is that of formulating asymptotic completeness in such theories, i.e. the statement that the scattering states span the full state space.

In the following I propose a solution of these problems in the case of QED. The solution is based on a theorem which can be proved in perturbation theory with the same kind of methods as used for what was explained in the previous sections. Again, the results will be stated without proofs. The proofs can be found in refs. [10]. In the first reference a particularly suitable Gupta-Bleuler gauge is used, in the second it is shown how the results obtained can be transferred to physical gauges like the Coulomb gauge, and how to use them for establishing a scattering formalism.

I shall use a rather condensed notation, not explicitly distinguishing between the various fundamental fields $\psi, \bar{\psi}, A_\mu$, of the theory. The symbol Φ will denote any of these fields, as the case may be. We define, for suitably normalized Φ's (the normalization condition is non-trivial):

$$\begin{aligned} & S(\ldots, \mathbf{p}_i, \ldots, t) \\ & \quad = \int \Pi_i dp_{i0} \exp\{-it(p_{i0} - (\mathbf{p}_i^2 + m_i^2)^{1/2})\} T(\Pi_i \Phi(p_i)) \ . \end{aligned} \tag{15}$$

Here $T(\ldots)$ is the Fourier transform of the time ordered product of the fields $\Phi(x_i)$. Let α be the number of ψ in this product, β that of $\bar{\psi}, \gamma + \gamma'$ that of A's. Let $\mathcal{A} \subset R^{3\alpha}, \mathcal{B} \subset R^{3\beta}, \mathcal{C} \subset R^{3\gamma}$ be smooth sets and $\mathcal{C}_s \subset R^3$ a smooth set containing the origin in its interior. Then the following statements hold to every finite order of perturbation theory.

<u>Theorem</u>

a) *The limit*

$$\begin{aligned} \Pi_{\mathcal{ABC}} &= \lim_{t\to\infty} \sum_{\gamma'=0}^{\infty} \frac{1}{\gamma'!} \int_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}\times\mathcal{C}_s^{\times\gamma'}} \prod_{i=1}^{\alpha+\beta+\gamma+\gamma'} d^3p_i \times \\ & \quad \times S^*(\ldots, \mathbf{p}_i, \ldots, t)|\rangle K_i(\mathbf{p}_i) \langle| S(\ldots, \mathbf{p}_i, \ldots, t) \end{aligned} \tag{16}$$

exists and is a projection operator. The K_i are kernels whose exact form depends on the type of the i^{th} field.

b) *For $\mathcal{A}_\alpha = R^{3\alpha}$, $\mathcal{B}_\beta = R^{3\beta}$, $\mathcal{C} = \emptyset$, $\mathcal{C}_s = R^3$ we have*

$$\sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} \Pi_{\mathcal{A}_\alpha \mathcal{B}_\beta \emptyset} = \mathbf{1} \ . \tag{17}$$

$|\rangle$ is the vacuum ket. The same results hold, of course, also for $t \to -\infty$.

These results compare as follows with the traditional formulation. $S^*|\rangle$ is the kind of state considered in the Haag-Ruelle condition and can be proved under favorable conditions to converge strongly to a state of free particles. Under these conditions the limit (16) without the summation over γ' exists in the strong operator topology, and is a projection. The expression on the left-hand side of (17) exists then also, if all summations, including the one over γ', are carried out *after* taking the t-limit. In an asymptotically complete theory the result is the identity. Our result is thus distinguished from the usual formulation by summing *first* over soft photons and taking the t-limit afterwards. A price to be paid for this change is this: the limit is now not attained in the strong operator topology, but only in the sense of sesquilinear forms. What converges are the matrix elements of the expression (16) between smooth states, i.e. states obtained by applying polynomials in the fields $\Phi(x)$, averaged over sufficiently smooth test functions, to the vacuum $|\rangle$.

The method amounts to introducing the time development as a natural infrared regularization. For finite t the terms in the γ'-sum exist individually, but the t-limit exists only for the sum, not for the individual terms. The time t thus takes over the role usually played by ad hoc regularization parameters like a positive photon mass or a IR momentum cutoff.

The statement b) in the Theorem is the promised new formulation of asymptotic completeness.

For establishing a scattering formalism we use part a) of the Theorem for describing the outgoing state. In terms of particles we can interpret $\Pi_{\mathcal{ABC}}$ as the projection onto the set of states with α observed electrons with momenta in $\mathcal{A}$, β observed positrons in $\mathcal{B}$, γ observed photons in $\mathcal{C}$, and any number γ' of non-observed soft photons with momenta in $\mathcal{C}_s \cdot \mathcal{C}_s$ need not be small: no "small" terms are neglected in (16).

The inclusive cross section for finding such a final state is then given by

$$\sigma_{incl}(\mathcal{A} \times \mathcal{B} \times \mathcal{C}) = (\Psi_{in}, \Pi_{\mathcal{ABC}} \Psi_{in}) \ . \tag{18}$$

For the description of the initial state Ψ_{in} we must use a different method. Let me just briefly indicate this for a two-particle initial state prepared at time $t = 0$, with particles localized at that time in neighbourhoods of the points $\mathbf{0}$ and $\mathbf{x}$ with a macroscopic distance $|\mathbf{x}|$. We define provisionally

$$\Psi_{in} = \int d^4p \tilde{\varphi}_1(p) \Phi(-p) \int d^4q \tilde{\varphi}_2(q) e^{-i(\mathbf{q},\mathbf{x})} \Phi(-q) |\rangle \ , \tag{19}$$

where the wave functions $\tilde{\varphi}_i$ are sufficiently smooth test functions with compact supports in small neighbourhoods of two linearly independent points P_1, P_2, on the respective mass shells, and whose Fourier transforms $\varphi_i(u)$ in configuration space are negligibly small (relative to a given experimental accuracy) outside of a small neighbourhood of the origin. The asymptotic behaviour of the expression (18) for $|\mathbf{x}| \to \infty$ can then be determined. It is found that the dominant contribution $D(\mathbf{x})$ decreases like $|\mathbf{x}|^{-2}$, provided that the initial configuration corresponds to a classically possible scattering event. This means that straight world lines through $u_1 = 0$ and $u_2 = (0, \mathbf{x})$ in the directions $P_{1,2}$ meet approximately in a point, the point where the interaction actually takes place, and that a final configuration in $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ is compatible with momentum conservation. For sufficiently large $|\mathbf{x}|$ this dominant contribution is a sufficient approximation to (18), and we can define

$$\sigma_{incl} = |\mathbf{x}|^2 D(\mathbf{x}) \ . \tag{20}$$

A detailed evaluation leads to an expression which can be considered as a generalization to inclusive cross sections of the LSZ reduction formula. No detour through a non-existent S-matrix is needed in this derivation.

Consider a process with two initial particles with 4-momenta $P_{1,2}$, and n observed final particles with momenta $Q_1, \ldots, Q_n$. If P is an electron momentum we define

$$N(P) = -2\pi i(\not{P} - m)\tau'(P) \ , \tag{21}$$

whre τ' is the clothed electron propagator, and similarly for positrons. For photons we set $N(P) = 1$. Observed photons must of course be in physical states (spinor and vector indices have been suppressed in my formulas). The inclusive cross section for the process in question is then given by

$$\begin{aligned} \sigma(P_i \to Q_j) &= [\Pi N(-P_i)\Pi N(Q_j)]^{-1}(P_1^2 - m_1^2)^2 \ldots (Q_n^2 - m_n^2)^2 \times \\ &\quad \times \langle |T^*(Q_1, \ldots, Q_n, -P_1, -P_2)T(Q_1, \ldots, -P_2)| \rangle' \end{aligned} \tag{22}$$

where the prime in $\langle | \ldots | \rangle'$ denotes omission of the δ^4 factor from momentum conservation. All momenta P_i, Q_j, lie on the respective mass shell. At the mass shell $N(\pm P)$ is IR divergent in the electron case, and so is the amputated function $\langle |T^*T| \rangle$. The expression (22) must therefore be calculated as a limit taken from momenta with $P_i^2 < m_i^2, Q_j^2 < m_j^2$. The function $\langle |T^*T| \rangle$ is one of the $\mathcal{W}$ discussed before, and can be calculated from the rules given in section 3.

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