The periodic Melnikov vector, the continuation of periodic orbits and a theorem on the non-integrability of perturbed Hamiltonians of n degrees of freedom

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Abstract

The periodic Poincaré - Melnikov vector is defined for a perturbed Hamiltonian system with n degrees of freedom of the form $H = H_0 + \varepsilon H_1$ and its relation to the bifurcation of periodic orbits and the non-integrability of the perturbed system is investigated.

1 The Poincaré - Melnikov vector

Consider the n-degrees-of-freedom nearly integrable Hamiltonian of the form

$$H = H_0 + \varepsilon H_1, \tag{1}$$

where H_0 is integrable, i.e. it possesses $n-1$, in addition to H_0, single-valued, independent integrals in involution and suppose that we can define action-angle variables J_i, w_i $(i = 1, .., n)$, at least in an open domain of phase space, such that

$$H_0 = H_0(J_i). \tag{2}$$

Let $\mathbf{C}$ be the symmetric matrix with elements

$$C_{ij} = \frac{\partial^2 H_0}{\partial J_i \partial J_j}. \tag{3}$$

We also suppose that H_0 is non-degenerate, i.e.

$$\det |\mathbf{C}| \neq 0. \tag{4}$$

According to Liouville - Arnold theorem (e.g. [1], p.271), an open domain of phase space of H_0 is foliated by n-dimensional tori, defined by

$$J_i = a_i = \text{const.} \quad (i = 1, ..., n), \tag{5}$$

which remain invariant under the motion. On any such torus, the unperturbed solution is described by the angle coordinates as

$$w_i = \omega_i t + \vartheta_i \quad (i = 1, ..., n) \tag{6}$$

where w_i are mod(2π) and ϑ_i are arbitrary initial conditions on the torus. Consider a particular torus of H_0, such that the resonant conditions hold

$$\frac{\omega_1}{p_1} = \frac{\omega_2}{p_2} = ... = \frac{\omega_n}{p_n}, \tag{7}$$

where $\omega_i = \frac{\partial H_0}{\partial J_i}$ are the frequencies of the integrable part and p_i are non-zero integers with no common divisor. This resonant torus is foliated by periodic orbits $\mathcal{O}$ with period

$$T = \frac{2\pi p_i}{\omega_i}. \tag{8}$$

The quotient manifold $Q = T^n/\mathcal{O}$ of the resonant torus, with respect to this foliation is a $(n-1)$-dimensional torus T^{n-1} and every periodic orbit is mapped on this manifold to a point with coordinates

$$\varphi_i = p_{i+1}\vartheta_1 - p_1\vartheta_{i+1} \mod(2\pi s_i) \quad (i = 1, ..., n-1), \tag{9}$$

where s_i are the common divisors of the pairs p_1, p_{i+1}.

Following the definition of the homoclinic Melnikov vector given in [2-4], a *periodic* Melnikov vector $\mathbf{M}$ on every periodic orbit of a resonant torus of H_0 may be defined as

$$M_i = \int_0^T [H_1, \Phi_{i0}]\, dt \quad (i = 1, ..., n), \tag{10}$$

where Φ_{i0} is any set of independent involutive integrals of H_0 and the integration is to be carried along the periodic orbit. The above definition is not satisfactory for several reasons:

(i) There is an ambiguity on the selection of the set of integrals Φ_{i0}.

(ii) By definition (10), every M_i depends on the initial conditions ϑ_i only through φ_j, i.e. it is not a function of the torus T^n but of the quotient manifold Q.

(iii) If we choose in particular $\Phi_{n0} = H_0$, then it can be easily shown that $M_n = 0$. If, on the other hand, we select the action variables J_i as n intependent integrals in (10), the projection of $\mathbf{M}$ along the orbit is zero and $\mathbf{M}$ belongs to the tangent space of Q.

Indeed, in this case

$$M_i = T \frac{\partial < H_1 >}{\partial \vartheta_i}$$

where

$$< H_1 >= \frac{1}{T} \int_0^T H_1 dt \tag{11}$$

is the average value of H_1 along the periodic orbit, which depends on ϑ_i through φ_j, so that

$$M_1 = \sum_{j=1}^{n-1} p_{j+1} \frac{\partial < H_1 >}{\partial \varphi_j}$$

$$M_{j+1} = -p_1 \frac{\partial < H_1 >}{\partial \varphi_j} \quad (j = 1, ..., n-1).$$

Let ω be the frequency vector, tangent to the orbit. Then, by taking into account (7), it is easy to show that

$$\mathbf{M}^T \omega = 0. \tag{12}$$

We define the *Poincaré - Melnikov* vector $\mathbf{P}$ on $T_\varphi Q$ as

$$P_i = \frac{\partial < H_1 >}{\partial \varphi_i} \quad (i = 1, ..., n-1). \tag{13}$$

For the above definition, any regular linear combination of φ_i can be used. Poincaré [5,6] actually used as coordinates in Q the ϑ_2,...,ϑ_n when $\vartheta_1 = 0$. Let $\Phi_{10}, \Phi_{20}, ..., \Phi_{(n-1)0}, H_0$ be a complete set of integrals for H_0, independent on the particular resonant torus. Then the corresponding Melnikov vector $\mathbf{M}$ is related to $\mathbf{P}$ by

$$M_i = T \sum_{j=1}^{n-1} D_j^{(i)} P_j \quad (i = 1, ..., n-1), \tag{14}$$

where

$$D_j^{(i)}(J_1, ..., J_n) = p_{j+1} \frac{\partial \Phi_{i0}}{\partial J_1} - p_1 \frac{\partial \Phi_{i0}}{\partial J_{j+1}} \tag{15}$$

while $M_n = 0$. Note that for every integral Φ_{i0} of H_0 there exists a corresponding vector $\mathbf{D}^{(i)}$, defined by equation (15). Suppose now that s integrals Φ_{k0} $(k = 1, .., s)$

produce linearly dependent vectors $\mathbf{D}^{(k)}$, i.e.

$$\sum_{k=1}^{s} \lambda_k D_j^{(k)} = 0 \quad (j = 1, ..., n-1). \tag{16}$$

Then the gradients of the integrals $\Phi_{10}, .., \Phi_{s0}, H_0$ are also linearly dependent, i.e.

$$\sum_{k=1}^{s} \mu_k \frac{\partial \Phi_{k0}}{\partial J_r} + \mu_0 \frac{\partial H_0}{\partial J_r} = 0 \quad (r = 1, ..., n), \tag{17}$$

with

$$\mu_k = -\lambda_k \frac{\partial H_0}{\partial J_1}, \quad \mu_0 = \sum_{k=1}^{s} \lambda_k \frac{\partial \Phi_{k0}}{\partial J_1}.$$

If, on the contrary, the selected integrals are linearly independent, the transformation (14) is regular. The Poincaré - Melnikov vector, as defined in (13), is strongly related to the non-integrability of the perturbed system and to the continuation of the non-isolated periodic orbits of H_0 with respect to ε.

2 The continuation of periodic orbits

In this section we will present a theorem by Poincaré, relating the Poincaré - Melnikov vector to the continuation of periodic orbits to the perturbed system.

Consider a Hamiltonian system possessing a T-periodic solution. Let $\mathbf{G}(t)$ be a fundamental matrix of solutions of the system of first order variational equations around this periodic solution, such that $\mathbf{G}(0) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The eigenvalues λ_j of the *monodromy matrix* $\mathbf{G}(T)$ are the *characteristic multipliers* and the quantities

$$\alpha_j = \frac{1}{T} \ln \lambda_j \mod \frac{i 2\pi}{T} \quad i = \sqrt{-1}, \ (j = 1, ..., 2n) \tag{18}$$

are the *characteristic exponents* of the aforesaid periodic solution. Because of the symplecticity of $\mathbf{G}(T)$, the characteristic exponents in a Hamiltonian system appear in opposite pairs. It is known ([6], §64) that, in an autonomous system, a pair of zero exponents always exists. The periodic orbit is linearly stable, if all other pairs are purely imaginary. If a periodic orbit possesses at least one pair of non-zero α_j, it is called *isolated*. If the system possesses p integrals in involution whose gradients are linearly independent on the periodic solution, there exist $2p$ zero characteristic exponents ([6], §71).

Consider the integrable Hamiltonian H_0. If the non-degeneracy condition (4) holds, the frequencies ω_i vary continuously with the actions J_i. In this case, the set of resonant tori is dense in the phase space of H_0. The periodic orbits on a resonant torus where eqs. (7) hold, are non-isolated, i.e. they possess $2n$ zero characteristic exponents.

Let $\mathbf{A}$ be the $(n-1) \times (n-1)$ matrix

$$A_{ij} = \frac{\partial^2 < H_1 >}{\partial \varphi_i \partial \varphi_j}. \tag{19}$$

Concerning the continuation of the non-isolated periodic orbits of H_0 to the perturbed system, the following Theorem is known

Theorem 1 (Poincaré, [5], [6] §42,79): If on a resonant torus of H_0, for a particular periodic orbit defined on Q by the point φ_i^*, the Poincaré - Melnikov vector has a simple zero in the vector sense, i.e.

$$\mathbf{P}(\varphi_i^*) = 0, \quad \det |\mathbf{A}(\varphi_i^*)| \neq 0, \tag{20}$$

then, for ε in an open interval around zero,

a) the perturbed Hamiltonian (1) has a T-periodic solution, which depends analytically on ε and, for $\varepsilon = 0$, coincides with the unperturbed solution φ_i^*.

b) The characteristic exponents of this solution can be expanded in powers of $\sqrt{\varepsilon}$,

$$\alpha_i = \sqrt{\varepsilon}\, \alpha_{i1} + O(\varepsilon). \tag{21}$$

The α_{i1}^2 are eigenvalues of the $n \times n$ matrix

$$\mathbf{E} = \mathbf{BC}, \tag{22}$$

where $\mathbf{C}$ is defined in (3) and $\mathbf{B}$ is the $n \times n$ matrix

$$B_{ij} = \frac{\partial^2 < H_1 >}{\partial \vartheta_i \partial \vartheta_j}, \tag{23}$$

evaluated at the periodic solution φ_i^* of H_0.

Differentiating (12) with respect to ϑ_j, we obtain

$$\mathbf{B}\, \omega = 0 \tag{24}$$

and since ω is non-zero, $\det |\mathbf{B}| = 0$, so that a pair of zero α_{i1} always exists. Note that by (9), B_{ij} and thus α_{i1} can be expressed as functions of A_{ij}.

An attempt to prove non-integrability of H by using this theorem is made in [5], but is demolished by the fact that one cannot prove the existence of an infinity of isolated periodic orbits, for small, but fixed ε [7,8].

3 A theorem on the non-integrability of the perturbed system

Assume that the perturbed Hamiltonian H possesses n independent integrals $\Phi_1, ..,$ $\Phi_{(n-1)}$, H, analytic in an open interval around $\varepsilon = 0$, i.e. Φ_i are expandable as

$$\Phi_i = \Phi_{i0} + \varepsilon \Phi_{i1} + O(\varepsilon^2) \quad (i = 1, ..., n-1). \tag{25}$$

Since Φ_i are integrals of H, it must hold $[H, \Phi_i] = 0 \;\; (i = 1, .., n-1)$ and equating terms of the same order in ε, we obtain

$$[H_0, \Phi_{i0}] = 0, \tag{26}$$

$$[H_0, \Phi_{i1}] + [H_1, \Phi_{i0}] = 0. \tag{27}$$

Equations (26), (27), which have been derived by Poincaré, hold identically in phase space. Equations (26) indicate that Φ_{i0} are integrals of H_0. It is known ([6], §81,82) that

(i) if H possesses $k \leq n-1$ independent integrals Φ_i, they can always be selected such that $\Phi_{10}, ..., \Phi_{k0}$, H_0 are independent and also,

(ii) if the non-degeneracy condition (4) holds, then Φ_{i0} do not depend on the angles.

We are interested in disproving the existence of any possible integrals of H, so we will only use the above two properties and the identity (27), which is a necessary condition for all possible independent integrals Φ_i.

Parametrizing equations (27) along the orbits of the unperturbed motion, we derive that

$$\frac{d\Phi_{i1}}{dt} = [H_1, \Phi_{i0}], \tag{28}$$

where both sides are evaluated along a particular solution of H_0. By integrating equations (28) along any periodic orbit on the resonant torus obeying the conditions (7) for one period T and demanding Φ_{i1} to be single-valued functions, we deduce the equations

$$M_i = 0 \quad (i = 1, ..., n-1), \tag{29}$$

where M_i is the component of $\mathbf{M}$ corresponding to Φ_{i0}, which must hold on every solution on this resonant torus. Since Φ_{i0} are unknown, we need further investigate the above equations, which, by (14), may be written in the form

$$\sum_{j=1}^{n-1} D_j^{(i)} P_j = 0 \quad (i = 1, ..., n-1). \tag{30}$$

If, on at least one orbit, the Poincaré - Melnikov vector $\mathbf{P}$ is non-zero, then from equation (30) we get

$$\det \mid D_j^{(i)} \mid = 0 \tag{31}$$

and, as shown in section 1, this leads to

$$\frac{D(H_0, \Phi_{10}, ..., \Phi_{(n-1)0})}{D(J_1, J_2, ..., J_n)} = 0. \tag{32}$$

If the vector P is different from zero on a dense set of resonant tori of H_0, on at least one orbit on each torus, equation (31), according to (i), contradicts the fact that Φ_i are independent integrals of H, i.e. H does not possess a complete set of integrals, analytic in an open interval around zero.

In what follows, we will obtain conditions for the non-existence of a certain number $s < n-1$ of independent integrals Φ_i of H. Every such integral will supply a relation of the form (30). Differentiating this relation with respect to φ_m, we form the homogeneous linear system

$$\sum_{j=1}^{n-1} A_{mj} D_j^{(i)} = 0. \tag{33}$$

The matrix $\mathbf{A}$ depends on the particular periodic orbit, while $\mathbf{D}^{(i)}$, according to (15), depend only on the selected resonant torus. The maximum allowable number of independent non-zero solutions of (33) equals $(n-1) - \mathrm{rank}\mathbf{A}$. Let us suppose that on the set of periodic orbits of this particular torus

$$\max \{\mathrm{rank}\mathbf{A}\} = r_i,$$

where the subscript i denotes the particular torus. This means that there exists at least one orbit with $\mathrm{rank}\mathbf{A} = r_i$ and therefore allows the least number of independent solutions of (33). Then system (33) admits at most $(n-1-r_i)$ independent non-zero solutions. On the other hand, equations (16) and (17) show that, if H possesses s independent integrals, in addition to H, then equation (33), evaluated on any periodic orbit, possesses s independent non-zero solutions of the form (15), which means that

$$s \leq n - 1 - r_i$$

and thus Hamiltonian H cannot possess more than $(n - r_i)$ independent integrals, including H, on this particular resonant torus. In order to find the maximum number of allowed independent integrals, in the open domain of phase space where action-angle variables are defined, we need consider the rank of $\mathbf{A}$ on a dense set of resonant tori in this domain. If on this dense set

$$\min r_i = r,$$

it is certain that $s \leq n-1-r$, since for each of these tori $n-1-r_i \leq n-1-r$. Thus we may state the following

Theorem 2 [9]: Consider the n-degrees-of-freedom Hamiltonian $H = H_0 + \varepsilon H_1$ where H_0 is a non-degenerate integrable part for which action-angle variables can be defined in an open domain of phase space. Then, in on open interval of ε around zero:

a) If a dense set of resonant tori of H_0 is found, such that the Poincaré - Melnikov vector $\mathbf{P}$ is not identically zero, then H does not possess a complete set of analytic integrals.

b) If, similarly, on a dense set of resonant tori rank$\mathbf{A}(\varphi_i^*) \geq r$, where φ_i^* correspond to the orbit on each torus, for which rank$\mathbf{A}$ is maximum, then H cannot possess more than $(n-r)$ independent integrals, including H, analytic in ε.

An application of the above theorem to a system of weakly coupled unharmonic oscillators can be found in [9].

4 Conclusions

By combining the two theorems presented in the preceding sections, the following conclusions can be drawn:

(i) If the integral Φ_{i0} of H_0 is condinued analytically for $\varepsilon \neq 0$, the corresponding component of the Melnikov vector M_i is identically zero, as can be seen by equation (29).

(ii) The continuation of periodic orbits can be proved by Theorem 1, only if H possesses no additional integral, analytic in ε. In the opposite case, according to (33), det $|\mathbf{A}| = 0$ and the Poincaré - Melnikov vector has no simple zeroes, with the possible exception of a nowhere dense set of resonant tori.

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