On Isospectral Sets of Jacobi Operators

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Abstract

We consider the inverse spectral problem for a class of reflectionless bounded Jacobi operators with empty singularly continuous spectra. Our spectral hypotheses admit countably many accumulation points in the set of eigenvalues as well as in the set of boundary points of intervals of absolutely continuous spectrum. The corresponding isospectral set of Jacobi operators is explicitly determined in terms of Dirichlet-type data.

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1 Introduction

The principal aim of this paper is to study certain bounded self-adjoint Jacobi operators whose inverse spectral theory and isospectral class can be characterized explicitly.

Since the literature on inverse spectral theory for Jacobi operators (especially in the periodic and short-range scattering case) is rather extensive, we confine ourselves to a brief account of those results which are close in spirit to our approach. In this context, the use of auxiliary spectral problems of the Dirichlet-type in connection with either the moment problem or the algebro-geometric approach to (quasi-)periodic finite-gap Jacobi operators, comes to mind first. Dirichlet spectra and the moment problem were first combined in the pioneering work by Kac and van Moerbeke [23], [24], [29]. The Jacobi inversion problem in connection with Dirichlet divisors appeared in Date and Tanaka [12] (see also [36]) and simultaneously in Dubrovin, Matveev, and Novikov [15] with further developments in [28], [30], [31]. (The algebro-geometric method is presented in great detail in [6].) A complete algebro-geometric treatment of Toda and Kac-van Moerbeke hierarchies can be found in [8]; the isospectral torus of quasi-periodic Jacobi operators is explicitly realized in [20]. The next step involved extensions to certain almost periodic and random stationary Jacobi operators with infinitely many gaps in their spectrum. Based on fundamental contributions by Levitan [26], followed by Kotani and Krishna [25] and Craig [11] in the case of Schrödinger operators, Antony and Krishna [2], [3] and especially Sodin and Yuditskiĭ[34], [35] characterized the inverse spectral problem for certain classes of almost periodic Jacobi operators by solving an infinite dimensional Jacobi inversion problem. In the random case Carmona and Kotani [9] provided necessary and sufficient conditions for a Herglotz function to be the expectation of a half-line Weyl m-function for a class of random stationary Jacobi operators. These extensions use elements of harmonic analysis, in particular, Herglotz properties of diagonal Green's functions and their boundary behavior on the real line.

In this paper we consider a different class of bounded Jacobi operators H with infinitely many gaps in their spectrum. More precisely, we assume that the spectrum Σ of H satisfies

$$\Sigma = \mathbb{R} \backslash \bigcup_{j \in J_0 \cup \{\infty\}} \rho_j, \tag{1.1}$$

where $J \subseteq \mathbb{N}$, $J_0 = J \cup \{0\}$,

$$\begin{aligned}
&\rho_0 = (-\infty, E_0), \quad \rho_\infty = (E_\infty, \infty), \\
&E_0 \leq E_{2j-1} < E_{2j} \leq E_\infty, \quad \rho_j = (E_{2j-1}, E_{2j}), \quad j \in J, \\
&-\infty < E_0 < E_\infty < \infty, \rho_j \cap \rho_k = \emptyset \text{ for } j \neq k
\end{aligned} \tag{1.2}$$

such that the set $\mathcal{A}$ of all accumulation points of $\{E_j\}_{j \in J_0 \cup \{\infty\}}$ is countable, that is,

$$\mathcal{A} = \{E_j\}'_{j \in J_0 \cup \{\infty\}} \quad \text{is countable.} \tag{1.3}$$

(Here A' denotes the derived set of $A \subset \mathbb{R}$, i.e., the set of all accumulation points of A.)

Hypotheses (1.1)–(1.3) include situations such as the class of algebro-geometric finite-gap Jacobi operators on one hand and Jacobi operators with pure point spectrum with at most countably many accumulation points on the other hand. Our methods integrate the use of trace formulas and Herglotz functions (as recently outlined by Gesztesy and Simon [17], [18]) and the moment problem. In particular, the isospectral set of all Jacobi operators with spectrum Σ satisfying (1.1)–(1.3) is explicitly determined in Theorem 4.3.

2 Preliminaries

In this section we recall some of the basic facts on Jacobi operators needed in Sections 3 and 4. Detailed accounts of this material can be found, for instance, in [7], Ch. VII, [10], Ch. III, [20], Appendices A–D.

Let $\{a(m) > 0\}_{m \in \mathbb{Z}}$, $\{b(m)\}_{m \in \mathbb{Z}} \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})$ be bounded real-valued sequences and introduce the bounded self-adjoint Jacobi operator H in $\ell^2(\mathbb{Z})$ by

$$(Hf)(m) = (\tau f)(m), \quad f = \{f(m)\}_{m \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \tag{2.1}$$

with the difference expression τ defined by

$$(\tau f)(m) = a(m) f(m+1) + a(m-1) f(m-1) + b(m) f(m), \quad m \in \mathbb{Z}. \tag{2.2}$$

The Green's function $G(z,m,n)$ associated with the resolvent $(H-z)^{-1}$ of H then can be represented by

$$\begin{aligned} G(z,n,n') &= (\delta(n),(H-z)^{-1}\delta(n')) \\ &= W(u_-(z),u_+(z))^{-1} \left\{ \begin{array}{ll} u_-(z,n)u_+(z,n'), & n \le n' \\ u_+(z,n)u_-(z,n'), & n \ge n' \end{array} \right. , \\ & \qquad\qquad z \in \mathbb{C}\backslash\sigma(H), \quad n,n' \in \mathbb{Z}. \end{aligned} \tag{2.3}$$

Here $\delta(n) = \{\delta_{m,n}\}_{m\in\mathbb{Z}}$, $\sigma(.)$ abbreviates the spectrum, $u_\pm(z,.)$ are Weyl solutions satisfying

$$\tau u_\pm(z) = z u_\pm(z), \quad u_\pm(z,.) \in \ell^2((m_0,\pm\infty)\cap\mathbb{Z}), \quad m_0 \in \mathbb{Z}, \quad z \in \mathbb{C}\backslash(H), \tag{2.4}$$

and $W(f,g)(n)$ denotes the Wronskian

$$W(f,g)(m) = a(m)[f(m)g(m+1) - f(m+1)g(m)], \quad m \in \mathbb{Z}. \tag{2.5}$$

Since H is in the limit point case at $\pm\infty$, $u_\pm(z,.)$ are unique up to constant multiples. They can be chosen to be holomorphic for $z \in \mathbb{C}\backslash\sigma_{ess}(H)$ ($\sigma_{ess}(.)$ denoting the essential spectrum).

Next, denote by $H_{\pm,n}$, $n \in \mathbb{Z}$ the restrictions of H to $\ell^2([n\pm 1,\pm\infty)\cap\mathbb{Z})$ with a Dirichlet boundary condition at the point $n \in \mathbb{Z}$, that is,

$$(H_{\pm,n}f)(m) = (\tau f)(m), \quad f \in \{g \in \ell^2([n\pm 1,\pm\infty)\cap\mathbb{Z}) | g(n) = 0\}. \tag{2.6}$$

The Weyl m-functions $m_{\pm,n}(z)$ associated with $H_{\pm,n}$ are then given by

$$\begin{aligned} m_{\pm,n}(z) &= (\delta(n\pm 1),(H_{\pm,n}-z)^{-1}\delta(n\pm 1)) \\ &= \left\{ \begin{array}{l} -u_+(z,n+1)/[a(n)u_+(z,n)] \\ -u_-(z,n-1)/[a(n-1)u_-(z,n)] \end{array} \right. . \end{aligned} \tag{2.7}$$

Introducing the abbreviations,

$$g(z,n) = G(z,n,n), \tag{2.8}$$

$$h(z,n) = 2a(n)G(z,n,n+1) - 1, \tag{2.9}$$

$$M_+(z,n) = a(n)^2 m_{+,n}(z), \tag{2.10}$$

$$M_-(z,n) = a(n-1)^2 m_{-,n}(z) + z - b(n), \tag{2.11}$$

one infers

$$g(z,n) = -[M_+(z,n) + M_-(z,n)]^{-1}, \tag{2.12}$$
$$g(z,n+1) = M_+(z,n)M_-(z,n)[M_+(z,n) + M_-(z,n)]^{-1}, \tag{2.13}$$
$$h(z,n) = [M_+(z,n) - M_-(z,n)][M_+(z,n) + M_-(z,n)]^{-1}. \tag{2.14}$$

We recall that for all $n \in \mathbb{Z}$, $g(.,n)$ and $M_\pm(.,n)$ are Herglotz functions (in contrast to $h(.,n)$). Finally, if $\sigma_p(.)$ abbreviates the point spectrum (i.e., the set of eigenvalues) one obtains

$$E \in \sigma_p(H) \text{ if and only if } -\lim_{\epsilon\downarrow 0} i\epsilon[g(E+i\epsilon,n) + g(E+i\epsilon,n+1)] > 0, \tag{2.15}$$

$$\mu \in \sigma_p(H_{\pm,n}) \text{ if and only if } -\lim_{\epsilon\downarrow 0} i\epsilon M_\pm(\mu+i\epsilon,n) > 0. \tag{2.16}$$

3 The Direct Spectral Problem

In this section we discuss the direct spectral problem for a certain class of reflectionless bounded Jacobi operators.

In order to set the stage we first recall that $g(z,n)$ admits an exponential Herglotz representation [5] of the form

$$g(z,n) = |g(i,n)| \exp\left\{\int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right] \xi(\lambda,n) d\lambda\right\}, \tag{3.1}$$

where, for all $n \in \mathbb{Z}$,

$$0 \le \xi(\lambda,n) \le 1 \text{ for a.e. } \lambda \in \mathbb{R}, \tag{3.2}$$
$$\xi(\lambda,n) = \lim_{\epsilon\downarrow 0} \mathrm{Im}\{\ln[g(\lambda+i\epsilon,n)]\} \text{ for a.e. } \lambda \in \mathbb{R}. \tag{3.3}$$

One can normalize $\xi(\lambda,n)$ by demanding

$$\xi(\lambda,n) = 0 \text{ for } \lambda < \inf\{\sigma(H)\}. \tag{3.4}$$

Our principal spectral hypothesis on H then reads as follows.

Hypothesis 3.1. *(i). H is a bounded self-adjoint Jacobi operator. Hence its spectrum can be written as*

$$\sigma(H) = \mathbb{R} \backslash \bigcup_{j \in J_0 \cup \{\infty\}} \rho_j, \tag{3.5}$$

where $J \subseteq \mathbb{N}$, $J_0 = J \cup \{0\}$,

$$\begin{aligned} &\rho_0 = (-\infty, E_0), \quad \rho_\infty = (E_\infty, \infty), \\ &E_0 \le E_{2j-1} < E_{2j} \le E_\infty, \quad \rho_j = (E_{2j-1}, E_{2j}), \quad j \in J, \\ &-\infty < E_0 < E_\infty < \infty, \quad \rho_j \cap \rho_k = \emptyset \text{ for } j \neq k. \end{aligned} \tag{3.6}$$

(ii). The set of all accumulation points of the set $\{E_j\}_{j \in J_0 \cup \{\infty\}}$ *is assumed to be countable and denoted by*

$$\mathcal{A} = \{\tilde{E}_j\}_{j \in \tilde{J}}, \quad \tilde{J} \subseteq \mathbb{N}. \tag{3.7}$$

(iii). For all $n \in \mathbb{Z}$,

$$\xi(\lambda, n) = \frac{1}{2} \text{ for a.e. } \lambda \in \sigma_{ess}(H). \tag{3.8}$$

We emphasize that the notation employed in (3.6) implies that $E_{2\ell} = E_{2k+1}$ for some $k \in J_0 \cup \{\infty\}$ whenever $E_{2\ell} \in \sigma_d(H)$ ($\sigma_d(.)$ abbreviating the discrete spectrum).

Remark 3.2. *(i). Since by hypotheses* (3.5) *and* (3.6), $\sigma(H) \subset [E_0, E_\infty]$ *is bounded, the corresponding sequences* $\{a(n) > 0\}_{n \in \mathbb{Z}}$ *and* $\{b(n)\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ *associated with the difference expression* (2.2) *are necessarily bounded.*
(ii). Hypothesis (H.3.1)(i) implies that $\sigma(H)$ *is a countable union of closed intervals (which may degenerate to points) of the type,*

$$\sigma(H) = \left(\bigcup_{j \in J_0} \Sigma_j \right) \cup \left(\bigcup_{j \in \tilde{J}} \tilde{\Sigma}_j \right), \tag{3.9}$$

where

$$\Sigma_j = [E_{2j}, E_{2j}^{(r)}], \quad j \in J_0, \quad \tilde{\Sigma}_j = [\tilde{E}_j, \tilde{E}_j^{(r)}], \quad j \in \tilde{J}, \tag{3.10}$$

with

$$x^{(r)} = \inf\{E_n \in [E_0, E_\infty] | x < E_n\} \text{ for } x \in [E_0, E_\infty]. \tag{3.11}$$

Reflectionless conditions such as (3.8) have been used by a variety of authors for particular cases such as almost periodic potentials (see, e.g., [2], [3], [11], [14], [25], [34], [35]) and scattering theoretic situations (cf., e.g., [13], [16]). The following result further illustrates (H.3.1)(iii).

Lemma 3.3. *Suppose H is a bounded Jacobi operator and $\Omega \subset \sigma(H)$. Then the following conditions are equivalent.*

(i). For all $n \in \mathbb{Z}$, $\xi(\lambda, n) = \frac{1}{2}$ for a.e. $\lambda \in \Omega$.

(ii). For some $n_0 \in \mathbb{Z}$, $n_1 \in \mathbb{Z}\backslash\{n_0, n_0+1\}$,

$$\xi(\lambda, n_0) = \xi(\lambda, n_0+1) = \xi(\lambda, n_1) = \frac{1}{2} \textit{ for a.e } \lambda \in \Omega.$$

(iii). For some $n_0 \in \mathbb{Z}$,

$$M_+(\lambda + i0, n_0) = -\overline{M_-(\lambda + i0, n_0)} \textit{ for a.e. } \lambda \in \Omega.$$

Proof. Clearly (i) implies (ii). In order to prove that (ii) implies (iii) we first recall that $M_\pm(z, n_0)$, being Herglotz functions, have nontangential limits $z \to \lambda$ for a.e. $\lambda \in \mathbb{R}$. Next, consider a particular representation of $u_\pm(z, n, n_0)$ in (2.7) normalized by $u_\pm(z, n_0, n_0) = 1$, $z \in \mathbb{C}\backslash\sigma(H_{\pm,n_0})$, given by

$$u_\pm(z, n, n_0) = c(z, n, n_0) \mp a(n_0)^{-1} M_\pm(z, n_0) s(z, n, n_0). \tag{3.12}$$

Here $c(z, n, n_0)$, $s(z, n, n_0)$ are solutions of $\tau\psi(z) = z\psi(z)$, $z \in \mathbb{C}$ defined by

$$s(z, n_0, n_0) = c(z, n_0+1, n_0) = 0, \quad s(z, n_0+1, n_0) = c(z, n_0, n_0) = 1, \tag{3.13}$$

in particular, $c(z, n, n_0)$, $s(z, n, n_0)$ are polynomials with respect to z and real-valued for $z \in \mathbb{R}$. The requirement $\xi(\lambda, m) = \frac{1}{2}$, that is , $g(\lambda + i0, m) = \overline{-g(\lambda + i0, m)}$ for a.e. $\lambda \in \Omega$ then yields upon choosing $m = n_0$,

$$\mathrm{Re}[M_+(\lambda + i0, n_0)] = -\mathrm{Re}[M_-(\lambda + i0, n_0)] \quad \text{for a.e. } \lambda \in \Omega \tag{3.14}$$

and

$$\begin{aligned}&[M_+(\lambda + i0, n_0) M_-(\lambda + i0, n_0) - \overline{M_+(\lambda + i0, n_0) M_-(\lambda + i0, n_0)}] s(\lambda, m, n_0)^2 \\ &\quad - \{\mathrm{Im}[M_+(\lambda + i0, n_0)] - \mathrm{Im}[M_-(\lambda + i0, n_0)]\} 2ia(n_0) c(\lambda, m, n_0) s(\lambda, m, n_0) \\ &\quad = 0 \quad \text{for a.e. } \lambda \in \Omega.\end{aligned} \tag{3.15}$$

Taking $m = n_0 + 1$ in (3.15) yields

$$\begin{aligned} &M_+(\lambda + i0, n_0)M_-(\lambda + i0, n_0) - \overline{M_+(\lambda + i0, n_0)M_-(\lambda + i0, n_0)} \\ &\quad = 0 = \mathrm{Re}[M_+(\lambda + i0, n_0)]\{\mathrm{Im}[M_+(\lambda + i0, n_0)] - \mathrm{Im}[M_-(\lambda + i0, n_0)]\} \\ &\qquad\qquad \text{for a.e. } \lambda \in \Omega \end{aligned} \tag{3.16}$$

since $s(\lambda, n_0 + 1, n_0) = 1$. Taking $m = n_1$ in (3.15) finally proves

$$\mathrm{Im}[M_+(\lambda + i0, n_0)] = \mathrm{Im}[M_-(\lambda + i0, n_0)] \quad \text{for a.e. } \lambda \in \Omega \tag{3.17}$$

and hence (iii) since $c(\lambda, n_1, n_0)s(\lambda, n_1, n_0) \neq 0$ for a.e. $\lambda \in \mathbb{R}$. (iii) implies (i) by combining (2.3), (3.12), and the real-valuedness of $c(\lambda, n, n_0)$ and $s(\lambda, n, n_0)$ for $\lambda \in \mathbb{R}$. □

Next we turn to Dirichlet eigenvalues associated with τ corresponding to a Dirichlet boundary condition at $n \in \mathbb{Z}$. Associated with each spectral gap ρ_j we set

$$\mu_j(n) = \sup\{\{E_{2j-1}\} \cup \{\lambda \in \rho_j | g(\lambda, n) < 0\}\} \in \overline{\rho_j}, \quad j \in J. \tag{3.18}$$

The strict monotonicity of $g(\lambda, n)$ with respect to $\lambda \in \rho_j$, that is,

$$\frac{d}{d\lambda} g(\lambda, n) = \sum_{m \in \mathbb{Z}} G(\lambda, n, m)^2 > 0, \quad \lambda \in \rho_j, \tag{3.19}$$

then yields

$$\begin{matrix} g(\lambda, n) < 0, & \lambda \in (E_{2j-1}, \mu_j(n)), \\ g(\lambda, n) > 0, & \lambda \in (\mu_j(n), E_{2j}), \end{matrix} \quad j \in J. \tag{3.20}$$

A more detailed analysis of the exponential Herglotz representation (3.1) of $g(z, n)$ then yields

Lemma 3.4. *Assume (H.3.1)(i). Then*

$$\begin{aligned} g(z, n) &= |g(i, n)| \exp\left\{ \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] \xi(\lambda, n) d\lambda \right\} \\ &= \frac{-1}{z - E_\infty} \exp\left\{ \int_{E_0}^{E_\infty} \frac{\xi(\lambda, n) d\lambda}{\lambda - z} \right\} \\ &= \frac{-1}{(z - E_0)^{1/2}(z - E_\infty)^{1/2}} \prod_{j \in J} \left[\frac{z - \mu_j(n)}{(z - E_{2j-1})^{1/2}(z - E_{2j})^{1/2}} \right], \end{aligned} \tag{3.21}$$

where the square root branch used is defined by

$$z^{1/2} = |z^{1/2}| \exp[i \arg(z)/2], \quad -\pi < \arg(z) \leq \pi. \tag{3.22}$$

In particular, denoting by $\chi_\Omega(.)$ the characteristic function of the set $\Omega \subset \mathbb{R}$, one can represent $\xi(\lambda, n)$ by

$$\begin{aligned}
\xi(\lambda, n) = &\frac{1}{2}\left[\chi_{(E_0,\infty)}(\lambda) + \chi_{(E_\infty,\infty)}(\lambda)\right] \\
&+ \frac{1}{2}\sum_{j\in J}\left[\chi_{(E_{2j-1},\infty)}(\lambda) + \chi_{(E_{2j},\infty)}(\lambda) - 2\chi_{(\mu_j(n),\infty)}(\lambda)\right] \\
&= \frac{1}{2}\chi_{(E_0,E_\infty)}(\lambda) + \frac{1}{2}\sum_{j\in J}\left[\chi_{(E_{2j-1},\mu_j(n))}(\lambda) - \chi_{(\mu_j(n),E_{2j})}(\lambda)\right] \\
&+ \chi_{(E_\infty,\infty)}(\lambda) \quad \textit{for a.e. } \lambda \in \mathbb{R}.
\end{aligned} \tag{3.23}$$

For later purpose we observe that the Laurent expression of $g(z, n)$ near $\frac{1}{z} = 0$,

$$g(z, n) = -\frac{1}{z} - \frac{b(n)}{z^2} + 0(z^{-3}), \tag{3.24}$$

combined with (3.21) implies the trace formula (cf. [17])

$$b(n) = \frac{1}{2}(E_0 + E_\infty) + \frac{1}{2}\sum_{j\in J}[E_{2j-1} + E_{2j} - 2\mu_j(n)]. \tag{3.25}$$

Next, we denote for all $n \in \mathbb{Z}$,

$$\gamma_{\pm,j}(n) = -\lim_{\epsilon\downarrow 0} i\epsilon M_\pm(\mu_j(n) + i\epsilon, n) \geq 0, \quad j \in J, \tag{3.26}$$

$$\gamma_j(n) = \lim_{\epsilon\downarrow 0} i\epsilon g(\mu_j(n) + i\epsilon, n)^{-1} = \gamma_{+,j}(n) + \gamma_{-,j}(n) \geq 0, \quad j \in J, \tag{3.27}$$

$$\tilde{\gamma}_{\pm,j}(n) = -\lim_{\epsilon\downarrow 0} i\epsilon M_\pm(\tilde{E}_j + i\epsilon, n) \geq 0, \quad j \in \tilde{J}, \tag{3.28}$$

$$\tilde{\gamma}_j(n) = \lim_{\epsilon\downarrow 0} i\epsilon g(\tilde{E}_j + i\epsilon, n)^{-1} = \tilde{\gamma}_{+,j}(n) + \tilde{\gamma}_{-,j}(n) \geq 0, \quad j \in \tilde{J}, \tag{3.29}$$

where we used the fact that by the Herglotz property of $M_\pm(z, n)$, $-g(z, n)^{-1}$, the limits in (3.26)–(3.29) exist and take on nonnegative values (cf. (3.38) and (3.39)). Associated with the limits (3.26)–(3.29) are the following ones ($n \in \mathbb{Z}$),

$$\sigma_j(n) = \begin{cases} \lim_{\epsilon\downarrow 0} h(\mu_j(n)+i\epsilon, n) = \frac{\gamma_{+,j}(n)-\gamma_{-,j}(n)}{\gamma_j(n)} \in [-1,1] & \text{if } \gamma_j(n) > 0 \\ 2 & \text{if } \gamma_j(n) = 0 \end{cases}, \tag{3.30}$$

$$\tilde{\sigma}_j(n) = \begin{cases} \lim_{\epsilon\downarrow 0} h(\tilde{E}_j+i\epsilon, n) = \frac{\tilde{\gamma}_{+,j}(n)-\tilde{\gamma}_{-,j}(n)}{\tilde{\gamma}_j(n)} \in [-1,1] & \text{if } \tilde{\gamma}_j(n) > 0 \\ 2 & \text{if } \tilde{\gamma}_j(n) = 0 \end{cases}. \tag{3.31}$$

The actual value of $\sigma_j(n)$ (resp. $\tilde{\sigma}_j(n)$) if $\gamma_j(n) = 0$ (resp. $\tilde{\gamma}_j(n) = 0$) in (3.30) (resp. (3.31)) is immaterial. For notational convenience later on, we chose a value outside the interval $[-1,1]$ in this case.

We note that

$$\mu_j(n) \notin \mathcal{A} \text{ implies } \gamma_j(n) > 0. \tag{3.32}$$

Finally, we summarize the direct spectral problem in the following

Theorem 3.5. *Assume (H.3.1) and let $n \in \mathbb{Z}$. Then*

(i).

$$\begin{aligned} \sigma_p(H_{\pm,n}) = &\left\{ \mu_j(n) \in \overline{\rho_j} | \sigma_j(n) \in {}^{(-1,1]}_{[-1,1)} \right\}_{j\in J} \\ &\cup \left\{ \tilde{E}_j \in \mathcal{A} | \tilde{\sigma}_j(n) \in {}^{(-1,1]}_{[-1,1)} \right\}_{j\in \tilde{J}}. \end{aligned} \tag{3.33}$$

(ii). If $\mu_j(n) \in \sigma_p(H)$ and $\overset{(\sim)}{\gamma}_{+,j}(n) > 0$ (resp. $\overset{(\sim)}{\gamma}_{-,j}(n) > 0$) then $\overset{(\sim)}{\gamma}_{-,j}(n) > 0$ (resp. $\tilde{\gamma}_{+,j}(n) > 0$), that is, if $\mu_j(n) \in \sigma_p(H)$ then $\mu_j(n) \in \sigma_p(H_{+,n})$ if and only if $\mu_j(n) \in \sigma_p(H_{-,n})$.

(iii). The following cases may occur:

$\mu_j(n) \in \rho_j$ and $\sigma_j(n) \in \{-1,1\}$,
$\mu_j(n) \in \sigma_p(H)$ and $\sigma_j(n) \in [-1,1]$ implying that $\mu_j(n) \in \sigma_p(H_{\pm,n})$,
$\mu_j(n) = \tilde{E}_k$ and $\sigma_j(n) = \tilde{\sigma}_k(n)$.

Moreover, we have

$$\overset{(\sim)}{\sigma}_j(n) \in [-1,1] \qquad \text{if} \quad \lim_{\epsilon\downarrow 0} i\epsilon g(\overset{(\sim)}{E}_j + i\epsilon, n)^{-1} > 0,$$

$$\overset{(\sim)}{\sigma}_j(n) = 2 \qquad \text{if} \quad \lim_{\epsilon\downarrow 0} i\epsilon g(\overset{(\sim)}{E}_j + i\epsilon, n)^{-1} = 0.$$

(iv).

$$
\begin{aligned}
\sigma_{ac}(H) &= \sigma_{ac}(H_{\pm,n}) \\
&= \overline{\{\lambda \in [E_0, E_\infty] | \xi(\lambda, n_0) = 1/2\}}^{ess} \quad \textit{for some } n_0 \in \mathbb{Z} \\
&= \overline{\bigcup_{\substack{j \in J \\ E_{2j} \neq E_{2j}^{(r)}}} \Sigma_j},
\end{aligned} \tag{3.34}
$$

$$
\sigma_{sc}(H) = \sigma_{sc}(H_{\pm,n}) = \emptyset, \tag{3.35}
$$

$\sigma_{ac}(H)$ being of uniform spectral multiplicity two whereas $\sigma_p(H)$, $\sigma_p(H_\pm, n)$, and $\sigma_{ac}(H_{\pm,n})$ are all simple. In addition, if $d\nu_{\pm,n}$ denote the measures associated with the Herglotz representations of $M_\pm(z,n)$, that is,

$$
M_+(z,n) = \int_{\mathbb{R}} \frac{d\nu_{+,n}(\lambda)}{\lambda - z}, \quad M_-(z,n) = z - b(n) + \int_{\mathbb{R}} \frac{d\nu_{-,n}(\lambda)}{\lambda - z}, \tag{3.36}
$$

then

$$
d\nu_{+,n,ac} = d\nu_{-,n,ac}, \qquad d\nu_{\pm,n,sc} = \emptyset. \tag{3.37}
$$

$d\nu_{\pm,n}$ are both supported on infinite sets. (Here $\sigma_{ac}(.)$, $\sigma_{sc}(.)$ abbreviate absolutely and singularly continuous spectra, respectively, $d\nu = d\nu_{ac} + d\nu_{sc} + d\nu_{pp}$ denotes the usual Lebesgue decomposition of measures, and $\overline{A}^{ess}$ denotes the essential closure of $A \subset \mathbb{R}$ with respect to Lebesgue measure, i.e., $\overline{A}^{ess} = \{\lambda \in \mathbb{R} | m(A \cap (\lambda - \epsilon, \lambda + \epsilon)) > 0$ for all $\epsilon > 0\}$, $m(.)$ denoting the Lebesgue measure.)

Proof. If $F(z)$ denotes a Herglotz function with representation

$$
F(z) = cz + d + \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right] d\omega(\lambda), \quad c \geq 0, \quad d \in \mathbb{R}, \tag{3.38}
$$

then

$$
\omega(\{\lambda_0\}) = -\lim_{\epsilon \downarrow 0} i\epsilon F(\lambda_0 + i\epsilon) \quad \text{for all } \lambda_0 \in \mathbb{R} \tag{3.39}
$$

yields (i), taking into account (3.26)–(3.29).

In order to prove (ii) one can argue as follows. $\overset{(\sim)}{\gamma}_{+,j}(n) > 0$ implies $\mu_j(n) \in \sigma_p(H_{+,n})$ by (2.16), (3.26), and (3.27) and $\overset{(\sim)}{\gamma}_j(n) > 0$ yields

$$\lim_{\epsilon \downarrow 0} \epsilon g(\mu_j(n) + i\epsilon, n) = 0$$

by (3.27) and (3.29). Since $\mu_j(n) \in \sigma_p(H)$, (2.13), (2.15), (3.26), and (3.27) yield

$$0 < \overset{(\sim)}{\gamma}_j(n) = -\lim_{\epsilon \downarrow 0} i\epsilon g(\mu_j(n) + i\epsilon, n+1) = \frac{\overset{(\sim)}{\gamma}_{+,j}\, \overset{(\sim)}{\gamma}_{-,j}}{\overset{(\sim)}{\gamma}_{+,j} + \overset{(\sim)}{\gamma}_{-,j}}. \tag{3.40}$$

Hence $\overset{(\sim)}{\gamma}_{-,j} > 0$ and thus $\mu_j(n) \in \sigma_p(H_{-,n})$. Alternatively, one can invoke the eigenfunction $u_+(\mu_j(n), m)$ which then satisfies $u_+(\mu_j(n), n) = 0$, $u_+(\mu_j(n), .) \in \ell^2(\mathbb{Z})$ since $\mu_j(n) \in \sigma_p(H) \cap \sigma_p(H_{+,n})$. The limit point property of H at $\pm\infty$ then yields $u_-(\mu_j(n), .) = C u_+(\mu_j(n), .)$ for some constant C and again one concludes that $\mu_j(n) \in \sigma_p(H_{-,n})$.

(iii) is clear from (3.30) and (3.31).

Next, define

$$\Sigma_{\pm,n,sc} = \{\lambda \in [E_0, E_\infty] | \lim_{\epsilon \downarrow 0} \mathrm{Im}[m_\pm(\lambda + i\epsilon, n)] \text{ exists and equals } +\infty\}, \tag{3.41}$$

$$\begin{aligned} \Sigma_{sc} = \{\lambda \in [E_0, E_\infty] | &\lim_{\epsilon \downarrow 0} \mathrm{Im}[g(\lambda + i\epsilon, n_0) + g(\lambda + i\epsilon, n_0 + 1)] \\ &\text{exists and equals } +\infty\} \text{ for some } n_0 \in \mathbb{Z}. \end{aligned} \tag{3.42}$$

Then $\Sigma_{\pm,n,sc}$ and Σ_{sc} are minimal supports (cf., e.g., [4], [21], [22], [32], [33]) of $d\nu_{\pm,n,sc}$ and $d\nu_{sc}^{tr}$, where $d\nu^{tr} = d\nu_{1,1} + d\nu_{2,2}$ abbreviates the trace measure of the 2×2 matrix-valued spectral measure $d\nu_{p,q}$, $1 \le p, q \le 2$ of H (derived from $g(z, n_0)$, $g(z, n_0 + 1)$, $h(z, n_0)$, cf. (2.12)–(2.14)). By (2.12) one has

$$-g(z, n)^{-1} = M_+(z, n) + M_-(z, n) \tag{3.43}$$

and by the reflectionless property (3.8),

$$-\lim_{\epsilon \downarrow 0} g(\lambda + i\epsilon, n)^{-1} = 2i \lim_{\epsilon \downarrow 0} \mathrm{Im}[M_\pm(\lambda + i\epsilon, n)] \text{ for a.e. } \lambda \in \sigma_{ess}(H). \tag{3.44}$$

Consider

$$\sigma(H)^0 = (\bigcup_{j\in J_0} \Sigma_j^0) \cup (\bigcup_{j\in\tilde{J}} \tilde{\Sigma}_j^0), \quad \Sigma_j^0 = (E_{2j}, E_{2j}^{(r)}), \quad \tilde{\Sigma}_j^0 = (\tilde{E}_j, \tilde{E}_j^{(r)}), \quad (3.45)$$

where A^0 denotes the open interior of $A \subseteq \mathbb{R}$. Then the representation (3.21) shows that $\Sigma_{sc} \cap \sigma(H)^0 = \emptyset$. But $\sigma(H)\backslash\sigma(H)^0$ is countable by Hypothesis (H.3.1) and hence $\sigma_{sc}(H) = \emptyset$. (3.44) then also yields $\Sigma_{\pm,sc} \cap \sigma(H)^0 = \emptyset$ and $\sigma_{sc}(H_{\pm,n}) = \emptyset$, $d\nu_{\pm,n,sc} = 0$ since $\sigma(H_{\pm,n})\backslash\sigma(H)^0$ is countable as well. Next, we recall that $(n_0 \in \mathbb{Z})$

$$\Sigma_{\pm,ac} = \{\lambda \in [E_0, E_\infty] | 0 < \lim_{\epsilon\downarrow 0} \mathrm{Im}[M_\pm(\lambda + i\epsilon, n_0)] < \infty \text{ exists}\}, \quad (3.46)$$

$$\Sigma_{ac} = \{\lambda \in [E_0, E_\infty] | 0 < \lim_{\epsilon\downarrow 0} \mathrm{Im}[g(\lambda + i\epsilon, n_0)] < \infty \text{ exists}\} \quad (3.47)$$

are minimal supports of $d\nu_{\pm,n,ac}$ and $d\nu_{ac}^{tr}$, respectively. By (3.44) one infers $d\nu_{+,n,ac} = d\nu_{-,n,ac}$ and hence (3.37). (3.34) then follows from (3.8), (3.44), (3.46), (3.47), and Theorem 5.2 of [17] which states

$$\sigma_{ac}(H) = \overline{\{\lambda \in [E_0, E_\infty] | 0 < \xi(\lambda, n_0) < 1\}}^{ess}. \quad (3.48)$$

Finally, spectral multiplicity two on $\sigma_{ac}(H)$ is a consequence of (3.8) and (3.44); $\sigma_p(H)$ is simple since H is in the limit point case at $\pm\infty$, and half-line spectra $\sigma(H_{\pm,n})$ are well-known to be simple. $d\nu_{\pm,n}$ are both supported on infinite sets since $H_{\pm,n}$ are defined on the discrete half-lines $\mathbb{Z} \cap (0, \pm\infty)$. $\square$

That $H, H_{\pm,n}$ have purely absolutely continuous spectra on $\sigma(H)^0$ (cf. (3.45)) also follows from Theorem 3.1 in [27].

4 The Inverse Spectral Problem

In this section we describe our principal new result on the isospectral set of self-adjoint Jacobi operators satisfying Hypothesis (H.3.1).

We start by introducing the following hypothesis.

Hypothesis 4.1. *(i). Let*

$$\Sigma = \mathbb{R}\backslash \bigcup_{j\in J_0\cup\{\infty\}} \rho_j, \quad (4.1)$$

where $J \subseteq \mathbb{N}$, $J_0 = J \cup \{0\}$,

$$\begin{aligned} &\rho_0 = (-\infty, E_0), \quad \rho_\infty = (E_\infty, \infty), \\ &\rho_j = (E_{2j-1}, E_{2j}), \quad E_0 \leq E_{2j-1} < E_{2j} \leq E_\infty, j \in J, \\ &-\infty < E_0 < E_\infty < \infty, \quad \rho_j \cap \rho_k = \emptyset \text{ for } j \neq k. \end{aligned} \tag{4.2}$$

By Σ_d *we denote the set of isolated (discrete) points of* Σ.

(ii). The set $\mathcal{A}$ *of all accumulation points of the set* $\{E_j\}_{j \in J_0 \cup \{\infty\}}$ *is assumed to be countable and denoted by*

$$\mathcal{A} = \{\tilde{E}_j\}_{j \in \tilde{J}}, \quad \tilde{J} \subseteq \mathbb{N}. \tag{4.3}$$

(iii). We introduce the set $\{\mu_j \in \overline{\rho_j}\}_{j \in J}$ *and define* $g(z)$ *as in* (3.21). *In addition, we introduce*

$$\{(\mu_j, \sigma_j) \in \overline{\rho_j} \times [-1, 1]\}_{j \in J}, \tag{4.4}$$

where

$$\sigma_j \in \{-1, 1\} \quad \text{if} \quad \mu_j \in \rho_j, \tag{4.5}$$

$$\sigma_j \in (-1, 1) \quad \text{if} \quad \mu_j \in \partial\rho_j \cap \Sigma_d. \text{ In this case} \tag{4.6}$$

$$\mu_j = \mu_k \text{ and } \sigma_j = \sigma_k \text{ for some } j \neq k \in J.$$

Next, we consider

$$\{(\tilde{E}_j, \tilde{\sigma}_j) \in \mathcal{A} \times \{[-1, 1] \cup \{2\}\}\}_{j \in \tilde{J}}, \tag{4.7}$$

where

$$\begin{aligned} &\tilde{\sigma}_j \in [-1, 1] \text{ if } \lim_{\epsilon \downarrow 0} i\epsilon g(\tilde{E}_j + i\epsilon)^{-1} > 0, \\ &\tilde{\sigma}_j = 2 \text{ if } \lim_{\epsilon \downarrow 0} i\epsilon g(\tilde{E}_j + i\epsilon)^{-1} = 0. \end{aligned} \tag{4.8}$$

Finally,

$$\text{if } \mu_j = \tilde{E}_k \text{ for some } j \in J, k \in \tilde{J}, \text{ then } \sigma_j = \tilde{\sigma}_k. \tag{4.9}$$

(iv). If $\Sigma = \overline{\Sigma_d}$, *the index sets*

$$J_\pm = \{j \in J | \sigma_j \in^{(-1,1]}_{[-1,1)}\} \text{ are infinite.} \tag{4.10}$$

Remark 4.2. *Conditions (i) and (ii) just reintroduce the necessary notation from Hypothesis (H.3.1).* (4.4)–(4.8) *in condition (iii) takes care of items (ii) and (iii) in Theorem 3.5. In particular, the fact that two Dirichlet eigenvalues must simultaneously hit a point in* $\sigma_d(H)$ *is taken into account in* (4.6). *If only a single Dirichlet eigenvalue* μ_j *would hit a point* $E^* \in \sigma_d(H)$, *then, since* E^* *necessarily occurs twice in the product* (3.21), *the term* $[z-\mu_j]/[(z-E^*)^{1/2}(z-E^*)^{1/2}]$ *simply drops out and one would have "lost"* E^*. *In other words, such a deformation of* $\mu_j(n)$ *would be nonisospectral. (A detailed account of such (non)isospectral deformations will appear in [19].) Condition* (4.9) *is a consistency requirement and condition (iv) reflects the fact that we are working with infinite matrix operators on the discrete half-lines* $\mathbb{Z} \cap (0, \pm\infty)$.

Given Hypothesis (H.4.1) we define the set of Dirichlet and accumulation data

$$\mathcal{D}_\Sigma = \{\{(\mu_j, \sigma_j) \in \overline{\rho_j} \times [-1,1]\}_{j\in J}, \{\tilde\sigma_j\}_{j\in\tilde J} | \text{assuming (H.4.1)}\}. \tag{4.11}$$

The isospectral set of self-adjoint reflectionless Jacobi operators H satisfying (H.3.1) with $\sigma(H) = \Sigma$ is denoted by

$$I(\Sigma) = \{\text{Jacobi operators } H \text{ in } \ell^2(\mathbb{Z}) | \sigma(H) = \Sigma\}. \tag{4.12}$$

Theorem 4.3. *Suppose* Σ *satisfies (H.4.1). Then the map*

$$\begin{cases} I(\Sigma) \to \mathcal{D}_\Sigma \\ H \to \{\{(\mu_j^\circ, \sigma_j^\circ)\}_{j\in J}, \{\tilde\sigma_j^\circ\}_{j\in\tilde J}\}, \end{cases} \tag{4.13}$$

constructed in Theorem 3.5 is a bijection, where

$$\Sigma(H) = \Sigma, \tag{4.14}$$

$$\sigma_p(H_{\pm,n_0}) = \{\mu_j^\circ \in \overline{\rho_j} | \sigma_j^\circ \in {}^{(-1,1]}_{[-1,1)}\}_{j\in J} \cup \{\tilde E_j \in \mathcal{A} | \tilde\sigma_j^\circ \in {}^{(-1,1]}_{[-1,1)}\}_{j\in\tilde J} \quad \textit{for some } n_0 \in \mathbb{Z}. \tag{4.15}$$

Proof. We first show that the map (4.13) is surjective. Fix a point

$$\{\{(\mu_j^\circ, \sigma_j^\circ)\}_{j\in J}, \{\tilde\sigma_j^\circ\}_{j\in\tilde J}\} \in \mathcal{D}_\Sigma. \tag{4.16}$$

We shall construct a unique Jacobi operator $H \in I(\Sigma)$ satisfying (4.14) and (4.15). Given (4.16), define $g(z,n_0)$ as in (3.21). Let ν_{n_0} be the measure in the Herglotz representation of $-g(z,n_0)^{-1}$, that is

$$-g(z,n_0)^{-1} = z - b(n_0) + \int_{\mathbb{R}} \frac{d\nu_{n_0}(\lambda)}{\lambda - z}, \tag{4.17}$$

with

$$b(n_0) = \frac{1}{2}(E_0 + E_\infty) + \frac{1}{2}\sum_{j\in J}[E_{2j-1} + E_{2j} - 2\mu_j^\circ]. \tag{4.18}$$

Next, we split up $\nu_{n_0} = \nu_{+,n_0} + \nu_{-,n_0}$ as follows. Since the pure point part of ν_{n_0} is supported on $\{\mu_j^\circ \in \overline{\rho_j}\}$ we define

$$\nu_{\pm,n_0}(\{\mu_j^\circ\}) = \frac{1}{2}(1 \pm \sigma_j^\circ)\nu_{n_0}(\{\mu_j^\circ\}) \tag{4.19}$$

and similarly,

$$\nu_{\pm,n_0}(\{\tilde{E}_j\}) = \frac{1}{2}(1 \pm \tilde{\sigma}_j^\circ)\nu_{n_0}(\{\tilde{E}_j\}). \tag{4.20}$$

(The split up of the pure point part in (4.19) resembles the one in Theorem 3.6 of [18] in the case of Schrödinger operators with purely discrete spectra.) The absolutely continuous part of ν_{n_0} is then split up according to Lemma 3.3, respectively (3.44), by

$$\nu_{\pm,n_0,ac} = \frac{1}{2}\nu_{n_0,ac}. \tag{4.21}$$

We note that

$$\nu_{n_0,sc} = \nu_{\pm,n_0,sc} = 0 \tag{4.22}$$

by the argument following (3.45). Next, define

$$a(n_0) = \left[\int_{\mathbb{R}} d\nu_{+,n_0}(\lambda)\right]^{1/2}, \quad a(n_0-1) = \left[\int_{\mathbb{R}} d\nu_{-,n_0}(\lambda)\right]^{1/2} \tag{4.23}$$

and consider the probability measures

$$\omega_{+,n_0} = a(n_0)^{-2}\nu_{+,n_0}, \quad \omega_{-,n_0} = a(n_0-1)^{-2}\nu_{-,n_0} \tag{4.24}$$

(which are both supported on infinite sets). $\omega_{\pm,n_0}$ enable one to compute $H_{\pm,n_0}$ by the moment approach as outlined, for instance, in [1], Ch. 4 and [7], Ch. 7. One obtains,

$$\begin{aligned} a(n) &= \int_{\mathbb{R}} \lambda s(\lambda,n,n_0)s(\lambda,n+1,n_0)d\omega_{\pm,n_0}(\lambda), \\ b(n) &= \int_{\mathbb{R}} \lambda s(\lambda,n,n_0)^2 d\omega_{\pm,n_0}(\lambda), \quad n \lesseqqgtr n_0 \pm 1 \end{aligned} \tag{4.25}$$

which determines H and (4.15). Introducing

$$\begin{aligned} M_+(z,n_0) &= a(n_0)^2 \int_{\mathbb{R}} \frac{d\omega_{+,n_0}(\lambda)}{\lambda - z}, \\ M_-(z,n_0) &= z - b(n_0) + a(n_0-1)^2 \int_{\mathbb{R}} \frac{d\omega_{-,n_0}(\lambda)}{\lambda - z}, \end{aligned} \tag{4.26}$$

one verifies (4.14) using (2.12), (2.14), and (2.15). It remains to show that the map (4.13) is injective. Suppose $H_1 \in I(\Sigma)$ and $H_2 \in I(\Sigma)$ are both mapped to the same point in (4.16). Then one infers $\nu_{\pm,1,n_0} = \nu_{\pm,2,n_0}$ and $b_1(n_0) = b_2(n_0)$ (where, in obvious notation, $\nu_{\pm,j,n_0}$ and b_j refer to $H_j, j = 1,2$) and hence $H_1 = H_2$. $\square$

We conclude with a simple example illustrating an explicit construction to the effect that an accumulation point of eigenvalues of H may or may not be an eigenvalue of H.

Example 4.4. *Suppose H satisfies (H.3.1), H has pure point spectrum only, and $\mathcal{A} \neq \emptyset$. Let $\tilde{E}_{j_0} \in \mathcal{A}$ and define*

$$\tilde{\gamma}_{j_0} = \lim_{\epsilon \downarrow 0} i\epsilon g(\tilde{E}_{j_0} + i\epsilon, 0)^{-1} \tag{4.27}$$

and

$$g_\delta(z,0) = -[-g(z,0)^{-1} - (\delta - \tilde{\gamma}_{j_0})(z - \tilde{E}_{j_0})^{-1}]^{-1}, \quad \delta \geq 0. \tag{4.28}$$

Then $\tilde{\gamma} \geq 0$ and g_δ is a Herglotz function corresponding to a pure point measure in its representation of the type (3.38). *Computing the zeros $\mu_{\delta,j}$ of $g_\delta(z,0)$ and choosing $\sigma_{\delta,j}$, $\tilde{\sigma}_{\delta,j} \in [-1,1] \times \{2\}$ according to (H.4.1) yields a corresponding Jacobi operator H_δ by Theorem 4.3. Since*

$$\lim_{\epsilon \downarrow 0} i\epsilon g_\delta(\tilde{E}_{j_0} + i\epsilon, 0)^{-1} = \delta, \tag{4.29}$$

one obtains the following case distinctions.

(i). $\delta = 0,$ *then* $\tilde{E}_{j_0} \notin \sigma_p(H_{\delta,\pm})$.

(ii). $\delta > 0,\ \tilde{\sigma}_{\delta,j_0} \in \{\pm 1\}$, *then* $\tilde{E}_{j_0} \in \sigma_p(H_{\delta,\tilde{\sigma}_{\delta,j_0}}),\quad \tilde{E}_{j_0} \notin \sigma_p(H_\delta)$.

(iii). $\delta > 0,\ \tilde{\sigma}_{\delta,j_0} \in (-1,1)$, *then* $\tilde{E}_{j_0} \in \sigma_p(H_{\delta,\pm}) \cap \sigma_p(H_\delta)$.

Case (i) is clear (in this case $\tilde{E}_{j_0}$ *may or may not belong to* $\sigma_p(H)$*). Case (ii) follows from Theorem 3.5 (ii). In case (iii) one has* $\lim_{\epsilon\downarrow 0} \epsilon g_\delta(\tilde{E}_{j_0} + i\epsilon, 0) = 0$ *but* $-\lim_{\epsilon\downarrow 0} i\epsilon g_\delta(\tilde{E}_{j_0} + i\epsilon, 1) > 0$ *in analogy to* (3.40)*.* $\tilde{E}_{j_0} \in \sigma_p(H_\delta)$ *then follows from* (2.15)*.*

Acknowledgements.

F.G. would like to thank Barry Simon for numerous discussions and joint work on inverse spectral problems which helped to shore up the foundations for this paper. M.K. wishes to thank Walter Craig for discussions and the Departments of Mathematics at Brown University and the University of Missouri-Columbia for an invitation which made this work possible.

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