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Non-collision periodic orbits with zero total angular momentum for the Newtonian Three-Body Problem

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Abstract

We study the existence of a periodic orbit without collisions, with zero total angular momentum, in the planar Three-Body Problem by means of Lagrangian reduction, variational methods and local analysis of the flow.

In a preceeding paper [**?**] we described a class of compact set of trajectories where to find minima for the Action-functional of the Three-Body Problem. In this paper we find critical points perturbatively in the masses, these critical points are in slightly bigger compact sets. We study a system composed of three bodies: two of them with small *different* masses compared with the mass of the third body. For the unperturbed problem we construct explicit solutions, which are regular critical points for the unperturbed Action-functional. These critical points can be *continued* for sufficiently small values of the masses. We prove the existence of periodic orbits without collisions. Moreover for the full Three-Body Problem on the manifold $J = 0$ we verify the existence of critical points at "infinity" composed of a Kepler-problem and an "escaping" body.

1 Introduction

In this paper we deal with the problem of the existence of periodic solutions in the Newtonian Three-Body Problem. In particular we consider the problem under the constrain that the total angular momentum J be zero.

For the Newtonian N-body problem in three dimensions, K.Sundman in a very old paper [**?**] proved that $J = 0$ is a necessary condition for total collision solutions. This fact implies that collision orbits must live in the submanifold of the phase space identified by the condition $J = 0$. Recently in [**?**] it is proved that in the N-body problem, under proper assumptions, collision orbits are not minima for the Action functional; (the assumptions are known to hold for $N \leq 4$) so that non-collision periodic solutions exist.

These facts led us to study the problem of the existence of periodic solutions and collision orbits by means of an Action functional reduced on the manifold of zero total angular momentum. Also in this case collision solutions are not minima and regular critical points, but now there is a lack of compactness and we cannot conclude immediately about the existence of regular critical points. In a preceeding paper [**?**] we studied the geometry of the reduced configuration manifold. Moreover by a weak Poincaré inequality, we showed that the lack of compactness is concentrated in a suitable neighborhood of collinear orbits. In this paper we prove the existence of non-collision periodic orbits on $J = 0$ in the case in which the system is composed of one large mass and two small *different* masses: the unperturbed system is essentially a small "Solar system" with two non-interacting small *different* "planets". For this system we exhibit a solution $\zeta_0(t)$ which is a C^1 critical point for the unperturbed Action-functional, and it lies in one of the compact sets that we found using a result proved in [**?**]. By means of *continuation* argument of critical points for the perturbed functional (see ([**?**]), we prove the existence of a solution for sufficiently small values of the masses. In the last section we study the behaviour of the Action functional on unbounded trajectories. Some of these trajectories describe the escape of one of the bodies.

2 Lagrangian reduction

We study a system composed of three point-like bodies whose masses are m_1, m_2, m_3, interacting with a newtonian potential. We will study the manifold of trajectories with zero total angular momentum, the system lies in a plane since every mechanical system composed of three material points interacting thorough a potential depending on the relative distances, with zero total angular momentum, has only planar motions. There is no loss of generality assuming that the system has the configuration space

$$Q \doteq \{\vec{q}_i \in \mathbf{R}^2; i = 1, 2, 3\} \simeq \mathbf{R}^6$$

In Q there are configurations on which the dynamics of our system is not defined. Those are the *Collision* configurations:

Double collisions

$$\{\vec{q}_1 = \vec{q}_2\} \quad \{\vec{q}_3 = \vec{q}_2\} \quad \{\vec{q}_1 = \vec{q}_3\}$$

Triple (total) collision

$$\{\vec{q}_1 = \vec{q}_2 = \vec{q}_3\}$$

The dynamics of the system is described by the Lagrangian:

$$L : TQ \rightarrow \mathbf{R}$$

$$L = \sum_{i=1}^{3} \frac{m_i}{2} \|\dot{\vec{q}}_i\|^2 + \sum_{i,j,i \neq j}^{3} \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|} \tag{1}$$

where we denote with $< .,. >$ $\|.\|$ respectively the scalar product and the norm in $\mathbf{R}^2$.

In order to simplify the notations we put:

$$\vec{q}_i = (q_i^1, q_i^2) \doteq q_i$$

The Angular momentum J of the system is conserved. It is a scalar since we consider only rotation in the plane of the motion. We describe the motion for fixed values of J (in particular $J = 0$) with a reduced Lagrangian obtained by the procedure described by Routh.

Definition 1.1

A Mechanical System is a triple $(M, < .,. >_M, V)$ where:

$(M, <.,.>_M)$ is a Riemmanian Manifold of dimension n, and $V : M \to \mathbf{R}$ a function (the potential) which is smooth excluding only a finite number of compact sets.

For this system we can construct a Lagrangian:

$$\begin{aligned} L : TM &\longrightarrow \mathbf{R} \\ (m, v_m) &\longrightarrow L(m, v_m) \end{aligned} \tag{2}$$

where:

$$L(m, v_m) \doteq \frac{1}{2} < v_m, v_m >_M + (V \circ \tau_M)(m, v_m)$$

$$\begin{aligned} \tau_M : TM &\longrightarrow M \\ (m, v_m) &\longrightarrow m \end{aligned} \tag{3}$$

Assume that on M acts freely a Lie group G:

$$\begin{aligned} \Phi : M \times G &\longrightarrow M \\ q, m &\longrightarrow \Phi_g(m) \end{aligned} \tag{4}$$

with lifted action on the tangent bundle TM:

$$\begin{aligned} \Phi^T : TM \times G &\longrightarrow TM \\ (m, v_m), g &\longrightarrow (\Phi_g(m), T\Phi_g(v_m)) \end{aligned} \tag{5}$$

We assume that G is a symmetry for the system, so that:

$$L \circ \Phi_g^T(m, v_m) = L(m, v_m)$$

Then Noether theorem implies the existence of integrals of the motion.

The Three Body Problem can be easily described in this context observing that we can put:

$$M \doteq Q, \quad < .,. >_Q \doteq \sum_{i=1}^{3} m_i < .,. >$$

and the whole symmetry group is given by the diagonal action on Q of:

$$G_0 \doteq T_{\mathbf{R}^2} \times O(2)$$

but for the Routh procedure we are interested in:

$$G \doteq T_{\mathbf{R}^2} \times SO(2)$$

where we denote with $T_{\mathbf{R}^2}$ the translations group in the plane. G is an Abelian Lie group.

Remark 2.1

*In the sequel, the presence of the planar reflections S_2 in the whole symmetry group G_0 of our system will play a crucial role in the study of the reduced configuration space (see ([**?**]). S_2 symmetry will survive the Lagrangian reduction.*

If G is an Abelian symmetry group, one can find a complete set of *cyclic coordinates*, there exists a system of coordinates on M such that the action of G on M can be written additively:

$$m \in M \qquad m = (x_1 ......x_r, \theta_{r+1} ......\theta_n) = (\vec{x}, \vec{\theta})$$

$$\Phi_g(m) = \Phi_g(\vec{x}, \vec{\theta}) = (\vec{x}, \vec{\theta} + \vec{\lambda})$$

with $\vec{\lambda} \in \mathbf{R}^a \times (S^1)^b \quad a + b = n - r - 1$ (in our case $n = 6, a = 2, b = 1$).

Remark 2.2

The additivity of the action Φ and the invariance condition $L \circ \Phi_g^T(m, v_m) = L(m, v_m)$ imply that

$$L(m, v_m) = L(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}})$$

In the case of *cyclic coordinates* the Euler-Lagraian equations have the form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \quad i = 1...r$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = 0 \quad i = r + 1...n$$

Therefore the Integrals of motion, we have the following coordinate representation:

$$J_i = \frac{\partial L}{\partial \dot{\theta}_i}$$

Due to the Abelian character of G these are independent integrals of the motion of the system. Now we can recall the theorem which describe the reduction "a la Routh":

Theorem A

Let $(M, <.,.>_M, V, G)$ be a Mechanical System with Symmetry group G which is an Abelian Lie group which acts freely and transitively. Assume that

$$\det \left(\frac{\partial^2 L}{\partial \dot{\theta}_i \partial \dot{\theta}_j} \right) \neq 0 \tag{6}$$

Define:

$$R_\nu(\vec{x}, \dot{\vec{x}}) \doteq L(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}}) - \sum_i \nu_i \dot{\theta}_i \tag{7}$$

where $\dot{\theta}_i$ are computed inverting:

$$\nu_i = J_i(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}})$$

Then R_ν is a Lagrangian which describe the motion constrained to

$$\{(m, v_m) \in TM \quad / \quad J(m, v_m) = \nu\}$$

with configuration manifold

$$\tilde{M} \doteq M/G$$

(the reduced configuration space).

For interesting geometrical descriptions of this theorem one can see [**?**], [**?**].

Remark 2.3

If the Lagrangian L is a positive function then also R_0 is positive. In fact one has:

$$R_0(\vec{x}, \dot{\vec{x}}) \doteq L(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}}(\vec{x}, \dot{\vec{x}}))$$

where $\dot{\vec{\theta}}(\vec{x}, \dot{\vec{x}})$ is the unique solution of

$$J_i(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}}) = 0 \qquad i = r+1...n$$

Let us now apply the previous Theorem to the Three Body Problem.

$$M \doteq Q$$

where:

$$Q \doteq \{\vec{q}_i \in \mathbf{R}^2; i = 1, 2, 3\} \simeq \mathbf{R}^6$$

the metric:

$$< .,. >_Q \doteq \sum_{i=1}^{3} m_i < .,. >$$

where $< .,. >$ is the standard scalar product in $\mathbf{R^2}$; the potential:

$$V(q) \doteq \sum_{i,j,i\neq j}^{3} \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|}$$

and the symmetry group turns out to be:

$$G \doteq T_{\mathbf{R}^2} \times SO(2)$$

Now we find the *cyclic coordinates* for our system. This will be done in some steps.

First step:

$$\mathcal{T}_1 : Q \to Q$$

$$\begin{cases} x_1 = q_2 - q_1 \\ x_2 = q_3 - q_1 \\ x_3 = (m_1 q_1 + m_2 q_2 + m_3 q_3)/(m_1 + m_2 + m_3) \end{cases} \tag{8}$$

by means of $\mathcal{T}_1$ after some algebraic manipulations (1) becomes:

$$L = \frac{\mu}{2}\|\dot{x_3}\|^2 + \frac{1}{2}\sum_{i=1}^{3} \tilde{M}_{ij} < \dot{x_i}, \dot{x_j} > + \frac{m_1 m_2}{\|x_1\|} + \frac{m_1 m_3}{\|x_2\|} + \frac{m_2 m_3}{\|x_2 - x_1\|} \tag{9}$$

where $\|x\| =< x, x >$ and

$$\tilde{M} \doteq \begin{pmatrix} M_{11} & -M_{12} \\ -M_{12} & M_{22} \end{pmatrix} \tag{10}$$

$$M_{11} \doteq \frac{m_2(m_1+m_3)}{\mu} \qquad M_{22} \doteq \frac{m_3(m_1+m_2)}{\mu} \qquad M_{12} \doteq \frac{m_2 m_3}{\mu} \qquad \mu \doteq \sum_{i=1}^{3} m_i \tag{11}$$

Now in the second step, we consider the following transformation:

$$\begin{aligned} \mathcal{T}_2 : Q \quad &\longrightarrow \quad Q \\ (x_1, x_2, x_3) \quad &\longrightarrow \quad (\rho_1, \theta_1, \rho_2, \theta_2, x_3) \end{aligned} \tag{12}$$

where:

$$\begin{cases} x_i^1 = \rho_i \cos\theta_i \\ x_i^2 = \rho_i \sin\theta_i \end{cases} \tag{13}$$