A PROOF OF THE LORENTZ–DIRAC EQUATION FOR CHARGED POINT PARTICLES

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Abstract. *A rigorous deduction of the nonrelativistic Lorentz–Dirac equation from the Maxwell–Lorentz system in the dipole approximation is given. Previously the validity of the equation had been checked, using the explicit solution of the complete system in terms of normal modes, only for very special initial data for the complete system; here instead the proof, valid for all initial data, is given by making rigorous the standard heuristic deduction, and this is obtained by overcoming a serious problem there involved. Finally a formula is given, which enables one to calculate, from the initial field, the initial acceleration to be used as an initial datum for the Lorentz–Dirac equation.*

1. Introduction

The dynamics of a classical point–like charged particle (with renormalized mass m charge e and position vector $\mathbf{q}$) is usually studied, in the non–relativistic approximation, taking as a mathematical model the so called Lorentz–Dirac (LD) equation

$$m\tau\dddot{\mathbf{q}} = m\ddot{\mathbf{q}} - \mathbf{F}(\mathbf{q}, t) \ ,$$

where $\mathbf{F}(\mathbf{q}, t)$ is a mechanical force and $\tau = 2e^2/3mc^3$ is a parameter having the dimension of a time. However, the fundamental model that should be studied is the Maxwell–Lorentz system (ML), namely Newton's law for the particle with Lorentz force due to the electromagnetic field, and Maxwell equations for the field, with a current due to the particle's motion. Now, the LD equation for the particle was deduced from the Maxwell–Lorentz system, in the non relativistic approximation [1], already at the beginning of the century, but such a deduction can only be considered as a heuristic one which leaves open some delicate and serious problems. In fact, even Dirac, when he introduced his relativistic version of the equation [2], could not deduce it in a clear way, and was obliged to state explicitly that he was actually postulating it. So, it might be that the LD equation is not a rigorous consequence of classical electrodynamics, and indeed there is a quite diffuse skepticism about its validity. In the present paper we prove (for a particular class of mechanical forces) equivalence of the linearized Maxwell–Lorentz system (the so called dipole approximation) with the Lorentz–Dirac equation.

The proof is obtained by overcoming a serious difficulty contained in the classical heuristic deductions of the LD equation [3,4] (see also [5]). Such a difficulty was already pointed out, at a heuristic level, by Yaghjian [6], who concluded that one should either avoid to study point particles or modify the model. On the contrary, in [7] (see also [8]) it was rigorously proved that the particle's motion corresponding to some particular initial data for the field (essentially, vanishing "free field") satisfies the LD equation. This was obtained by direct check using the explicit solution of the complete system in terms of normal modes. Here instead, we go back to the classical heuristic deduction, solve the difficulty pointed out by Yaghjian and obtain a simple and direct proof of the LD equation, which turns out to be valid for essentially all (smooth) initial data for the field (see theorem 3.1 below for a more precise statement). Moreover, we deduce a formula enabling one to calculate the initial acceleration to be inserted in the LD equation in order to obtain a particle's motion in agreement with the ML system in the dipole approximation. A similar formula was already contained in a paper by Kijowski [9], although with a completely different interpretation (see sect. 4 below).

In what follows we shall consider only the Maxwell–Lorentz system *in the dipole approximation* (see eq. (2.1) below), and for brevity the qualification "in the dipole approximation" will be understood.

To come to a precise statement we first recall that the dynamics of a point particle in the ML system is not well defined (indeed one has to calculate the electric field at the particle's position, where it is typically infinite); the usual way to overcome this problem (see e.g. [1, 10, 11, 12, 13]) consists in regularizing the model by taking a rigid extended particle (or equivalently a cutoff on the Fourier modes) and to study the limit of the interesting quantities as the particle's "radius" R tends to zero.

In order to study the behaviour of the particle, following [4] and many other authors, we first deduce a closed equation for its motion in the regularized system, and then study its limit as $R \to 0$. It is well known, heuristically, that such an equation converges to the LD equation *provided one neglects some terms* which are very small after a very short time. First we show that these terms, although very concentrated in time, produce very large effects. Indeed, we will show that the solution of the equation in which the "transient terms" are taken into account does not have a limit as $R \to 0$. This is a formalization of the fact, understood by Yaghjian, that such transient terms are very important. Then, following [7] we overcome the above difficulty by imposing a certain relation between the initial particle's velocity and the initial datum for the field. Precisely, motivated by the fact that a particle in uniform motion is followed by a field [10,14,15], which in the point limit is singular at the particle's position, we consider the class of initial fields differing from such an "adapted field" only by regular terms. It turns out that the contribution of the adapted field to the equation of motion exactly cancels the dangerous "transient terms".

In such a way the essential difficulty is eliminated and the problem of taking the point limit is just reduced to a purely technical one, which can be dealt with by standard methods; we will in fact use Laplace transform. We thus obtain that the limit motion exists and satisfies the LD equation

$$m\ddot{\mathbf{q}} = m\tau\dddot{\mathbf{q}} + \mathbf{F}_L(t, \mathbf{A}_0, \dot{\mathbf{A}}_0) + \mathbf{F}(\mathbf{q}, t) \ , \tag{1.1}$$

where $\mathbf{F}_L(t, \mathbf{A}_0, \dot{\mathbf{A}}_0)$ is a function of time depending parametrically on the initial data for the field (it is in fact the linearization of the Lorentz force due to the free evolution of the initial fields), while $\mathbf{F}(\mathbf{q}, t)$ is a mechanical force that is here assumed to be the sum of a linear part and a part depending only on time, namely $\mathbf{F}(\mathbf{q}, t) = -\alpha\mathbf{q}+\mathbf{f}(t)$, with a constant α, and a given $\mathbf{f}$.

Our study also leads to solve the "problem of the initial acceleration". Let us recall it briefly. The solution of the Cauchy problem for eq. (1.1) requires knowledge of the initial particle's acceleration in addition to the mechanical state, namely position and velocity. Thus, since in the regularized Maxwell–Lorentz system the motion is completely determined by the initial mechanical state and the initial fields, one should determine the initial particle's acceleration from the initial data for the complete system, and use such a "correct acceleration" as an initial datum in the LD equation. We will prove that the correct initial particle's acceleration is given by

$$\ddot{\mathbf{q}}_0 = \frac{3}{2}\frac{c}{e}\lim_{r\to 0} r\frac{1}{4\pi r^2}\int_{S_r} \dot{\mathbf{A}}_0(\mathbf{x})d\sigma \ , \tag{1.2}$$

where S_r is the sphere of radius r centered at the origin (i.e. the point where the interaction of the particle with the field is localized in the dipole approximation) and $d\sigma$ its surface element. As a corollary, we obtain that each solution of the Cauchy problem for (1.1) is the point limit of the particle's motion corresponding to a definite solution of the ML system in the dipole approximation.

As anticipated above the technical tool used here in studying the point limit is the Laplace transform. The limit $R \to 0$ is obtained by exchanging the inverse Laplace transform and the point–limit. This requires an estimate which is non trivial.

The paper is organized as follows. In sect. 2 we revisit the classical deduction of the LD equation in order to point out the difficulties related to it. In sect. 3 we solve such difficulties, study the point–limit of the particle's motion and deduce the condition on the initial acceleration. Sect. 4 contains a brief discussion. The appendix contains the estimates needed in order to exchange the point limit and the inverse Laplace transform.

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2. On the classical deduction of the LD equation

Consider the regularized ML system in the dipole approximation. In the Coulomb gauge it has the form[12,16]

$$\frac{1}{c^2}\ddot{\mathbf{A}} - \triangle \mathbf{A} = \frac{4\pi e}{c}\mathbf{j}_{tr} \ , \quad m_0\ddot{\mathbf{q}} = -\int_{\mathbb{R}^3} \frac{e}{c}\rho_R(x)\dot{\mathbf{A}}(x)d^3x + \mathbf{F}(\mathbf{q},t) \ . \tag{2.1}$$

where $\mathbf{F}(\mathbf{q},t)$ represents a mechanical force, m_0 is the particle's bare mass, $\mathbf{A}$ the vector potential (which is subjected to the condition $div\mathbf{A} \equiv 0$), $\mathbf{j}_{tr}$ is the transversal part (i.e. the divergence free part) of the current $\mathbf{j}(\mathbf{x}) := e\dot{\mathbf{q}}\rho_R(\mathbf{x})$. Finally ρ_R is the charge distribution of the particle. We assume (as in [7]) that ρ_R is of the form

$$\rho_R(x) := \frac{1}{R^3}\mathcal{D}\left(\frac{x}{R}\right) \ , \tag{2.2}$$

where $\mathcal{D}$ is a fixed positive, normalized (in L^1), spherically symmetric C^∞ function tending to zero at infinity faster than any power, R is a positive parameter playing the role of particle's radius, and $x = |\mathbf{x}|$. We will also need the technical assumption that the Fourier transform of $\mathcal{D}$ is monotonically decreasing.

If $\mathbf{F}(\mathbf{q},t)$ is regular, then for $R > 0$ and in a suitable function space, system (2.1) is well posed (see [15]); in particular one has that $\mathbf{q}(.)$ is C^2.

We perform now the Fourier transform of the form factor (choosing the normalization convention which makes it an L^2 isometry) and of the fields

$$\mathbf{A}(\mathbf{x}) = \sum_{j=1,2} \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \hat{A}_j(\mathbf{k})\mathbf{e}_j(\mathbf{k})e^{i\mathbf{k}\mathbf{x}}d^3x \ ,$$

where, as usual $\mathbf{e}_j(\mathbf{k})$, $j = 1,2$ are unit polarization vectors, constituting with $\mathbf{k}$ an orthonormal basis of $\mathbb{R}^3$. Then, using the parameter variation formula, we can write the solution of the first of equations (2.1) in terms of the initial data for the field and of the particle's motion, namely

$$\hat{A}_j(\mathbf{k},t) = \hat{A}_{0,j}(\mathbf{k})\cos(\omega_k t) + \frac{\dot{\hat{A}}_{0,j}(\mathbf{k})}{\omega_k}\sin(\omega_k t) + \int_0^t \frac{4\pi ec}{\omega_k}\left[\dot{\mathbf{q}}(s)\cdot\mathbf{e}_j(\mathbf{k})\right]\hat{\rho}_R(k)\sin[\omega_k(t-s)]ds \ , \tag{2.3}$$

where $\hat{A}_{0,j}$ and $\dot{\hat{A}}_{0,j}$ are the Fourier transforms of the initial fields $\mathbf{A}_0$, $\dot{\mathbf{A}}_0$, while $\hat{\rho}_R(k) = \hat{\mathcal{D}}(kR)$ is the Fourier transform of ρ_R, and $\omega_k := ck$. Substituting (2.3) in the Newton equation for the particle (i.e. in the second of (2.1)) and performing the angular integrations one obtains a closed integro-differential equation for the particle's motion, namely

$$m_0\ddot{\mathbf{q}}(t) = \int_0^t \dot{\mathbf{q}}(s)\ddot{M}_R(t-s)ds + \mathbf{F}(\mathbf{q},t) + \mathbf{F}_L^R(t,\mathbf{A}_0,\dot{\mathbf{A}}_0) \ , \tag{2.4}$$

where

$$
\begin{aligned}
M_R(t) &:= \frac{32}{3}\pi^2\frac{e^2}{c^2}\int_0^\infty |\hat\rho_R(k)|^2\cos(ckt)dk \\
&= \frac{1}{R}\frac{32}{3}\pi^2\frac{e^2}{c^2}\int_0^\infty \left|\hat{\mathcal{D}}(s)\right|^2\cos\left(cs\frac{t}{R}\right)ds \ , \\
\mathbf{F}_L^R(t,A_0,\dot A_0) &:= \sum_j\int_{\mathbb{R}^3} e\hat\rho_R^*(k)k\hat A_{0,j}(\mathbf{k})\sin(ckt)\mathbf{e}_j(\mathbf{k})d^3k \\
&\quad -\sum_j\int_{\mathbb{R}^3} e\hat\rho_R^*(k)\frac{1}{c}\dot{\hat A}_{0,j}(\mathbf{k})\cos(ckt)\mathbf{e}_j(\mathbf{k})d^3k \ .
\end{aligned}
\tag{2.5}
$$

Remark that $\mathbf{F}_L$ is the linearization of the Lorentz force due to the free evolution of the initial data for the fields. ¿From (2.4) it is easy to prove more regularity of $\mathbf{q}(.)$. In particular if $\mathbf{F}(\mathbf{q},t)$ is C^1, and the initial data data for the field are such that $\mathbf{F}_L^R$ is C^1 as a function of time, then one has that $\mathbf{q}(.)$ is C^3. So it is possible to perform two integrations by parts obtaining

$$
(m_0+M_R(0))\ddot{\mathbf{q}}(t) = \dot{\mathbf{q}}_0\dot M_R(t)+\ddot{\mathbf{q}}_0M_R(t)+\int_0^t\dddot{\mathbf{q}}(s)M_R(t-s)ds+\mathbf{F}(\mathbf{q},t)+\mathbf{F}_L^R(t,A_0,\dot A_0) \ . \tag{2.6}
$$

At this step *the standard deduction of the LD equation neglects the two terms $\dot{\mathbf{q}}_0\dot M_R$ and $\ddot{\mathbf{q}}_0M_R$ as pertaining to a very short transient* (one has $M_R(t)=N(t/R)/R$, for a regular N which goes to zero faster than any power as its argument tends to infinity, see (2.5)). Then, taking m_0 as a function of R in such a way that $m=m_0+M_R(0)$ is fixed, namely

$$
m_0(R) := m - M_R(0) \ , \tag{2.7}
$$

where m is identified with the physical mass, one is reduced to the equation

$$
m\ddot{\mathbf{q}} = \int_0^\infty \dddot{\mathbf{q}}(s)M_R(t-s)ds+\mathbf{F}(\mathbf{q},t)+\mathbf{F}_L^R(t,A_0,\dot A_0) \quad . \tag{2.8}
$$

It is easy, at a heuristic level, to take the limit of (2.8) and to show that it gives the LD equation.

We claim that, notwithstanding the fact that they are very concentrated in time, the terms $\dot{\mathbf{q}}_0\dot M_R$ and $\ddot{\mathbf{q}}_0M_R$ produce very important effects, so that they cannot be neglected. In particular we are going to prove that, due to their existence, *the solution of the Cauchy problem for (2.6) (and therefore for (2.1)) does not have a limit as* $R\to 0$.

For simplicity we consider the case of vanishing initial field, and vanishing mechanical force, and take the Laplace transform of the original eq. (2.6). Denoting by $\bar{\mathbf{q}}=\bar{\mathbf{q}}(p)$ the Laplace transform of $\mathbf{q}$, and by $\bar M_R(p)$ the Laplace transform of M_R, we obtain

$$
m(\bar{\mathbf{q}}p^2-\mathbf{q}_0p-\dot{\mathbf{q}}_0) = \bar M_R(p)(\bar{\mathbf{q}}p^3-\mathbf{q}_0p^2-\dot{\mathbf{q}}_0p-\ddot{\mathbf{q}}_0)+\dot{\mathbf{q}}_0[p\bar M_R(p)-M_R(0)]+\bar M_R(p)\ddot{\mathbf{q}}_0 \ . \tag{2.9}
$$

A first remark is that the last term at r.h.s., namely $\bar M_R\ddot{\mathbf{q}}_0$, exactly cancels the other term proportional to $\ddot{\mathbf{q}}_0$, namely the one coming from the Laplace transform of $\dddot{\mathbf{q}}$. As a consequence, the solution of the Cauchy problem for eq. (2.6) requires knowledge of $\mathbf{q}_0$, and $\dot{\mathbf{q}}_0$ only. So, the effect of the "transient term" $M\ddot{\mathbf{q}}_0$ is the following: *the solution of the Cauchy problem for equation (2.6) does not require knowing the initial particle's acceleration* $\ddot{\mathbf{q}}_0$.

It is easy to see (at least heuristically) that $\bar{M}(p) \to m\tau$ as $R \to 0$ (see (5.3)), while $M_R(0) \to +\infty$, so that all terms in (2.9) have a well defined limit as $R \to 0$, except the term proportional to $M_R(0)$, which causes the limit of $\bar{\mathbf{q}}$ not to exist. This term comes from the Laplace transform of $\dot{\mathbf{q}}_0\dot{M}(t)$; so, we conclude that *the term* $\dot{M}\dot{\mathbf{q}}_0$ *causes the point limit of the particle's motion not to exist*, at least when $\dot{\mathbf{q}}_0 \neq 0$. Such a result was already proved, with a different method, in [7]. We conclude that a correct deduction of the LD equation cannot be obtained by neglecting the "transient terms" of equation (2.6). We recall that Yaghjian in [6] made some heuristic considerations in order to show that the transient terms we considered here are important. This led him to conclude that equation (2.6) should be modified in order to deal with small particles, and he suggested a possible modification. Here, we take a different point of view: we will retain the model and show that, provided initial data are chosen in a good way, the point limit dynamics exists.

3. Point-limit of the particle's motion and Lorentz–Dirac equation

We recall that in the ref. [7] it was proved that the point limit of the particle's motion in the ML exists also when the initial particle's velocity is different from zero, *provided* one considers initial data for the fields "adapted" to the initial particle's velocity. Precisely, in [7], for each $\mathbf{v} \in \mathbb{R}^3$ a field $\mathbf{X}^R_{\mathbf{v}}(\mathbf{x}) = \mathbf{X}^R(\mathbf{v}, \mathbf{x})$ was considered, whose components are given in Fourier transform by

$$\hat{X}^R_{\mathbf{v},j}(\mathbf{k}) = \hat{X}^R_j(\mathbf{v}, \mathbf{k}) := \frac{4\pi e}{c}\hat{\rho}_R(k)\frac{\mathbf{v}\cdot\mathbf{e}_j(\mathbf{k})}{k^2} \ , \quad j = 1, 2 \ , \tag{3.1}$$

and then it was proved that the particle's motion corresponding to initial data of the form $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}^R_{\dot{\mathbf{q}}_0} + \mathbf{A}'_0, \dot{\mathbf{A}}_0)$, where $(\mathbf{A}'_0, \dot{\mathbf{A}}_0)$ are regular fields, has a limit as the radius R tends to zero.

Motivated by the above considerations we study initial data of the form

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}^R_{\dot{\mathbf{q}}_0} + \mathbf{A}'_0, \mathbf{X}^R_{\mathbf{a}} + \dot{\mathbf{A}}'_0) \ , \tag{3.2}$$

where $\mathbf{a} \in \mathbb{R}^3$ is a parameter, and $(\mathbf{A}'_0, \dot{\mathbf{A}}'_0)$, are regular (say C^∞_c, where c stands for "compactly supported") fields. By a simple calculation one has

$$\mathbf{F}^R_L(t, \mathbf{X}^R_{\dot{\mathbf{q}}_0}, \mathbf{X}^R_{\mathbf{a}}) = -\dot{\mathbf{q}}_0\dot{M}_R(t) - \mathbf{a}M_R(t) \ , \tag{3.3}$$

so that, corresponding to the above kind of initial data, eq. (2.6) takes the form

$$m\ddot{\mathbf{q}} = (\ddot{\mathbf{q}}_0 - \mathbf{a})M_R(t) + \int_0^\infty \ddot{\mathbf{q}}(t-s)M_R(s)ds + \mathbf{F}^R_L(t, \mathbf{A}'_0, \dot{\mathbf{A}}'_0) + \mathbf{F}(\mathbf{q}, t) \ . \tag{3.4}$$

In order to study rigorously the limit $R \to 0$ of its solution we will use the Laplace transform, for this reason we will consider only the linear case in which the mechanical force reduces to

$$\mathbf{F}(\mathbf{q}, t) = -\alpha \mathbf{q} + \mathbf{f}(t) \ , \tag{3.5}$$

with a given $\mathbf{f}$.

Theorem 3.1. *Let $\mathbf{a} \in \mathrm{I\!R}^3$ be fixed; consider the solution of the Cauchy problem for system (2.1) with force given by (3.5), with $\mathbf{f} \in C^1([0,+\infty]) \cap L^1([0,+\infty])$ and $\alpha \in \mathrm{I\!R}$. Take initial data of the form*

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}^R_{\dot{\mathbf{q}}_0} + \mathbf{A}'_0, \mathbf{X}^R_{\mathbf{a}} + \dot{\mathbf{A}}'_0) \ , \tag{3.6}$$

with $(\mathbf{A}'_0, \dot{\mathbf{A}}'_0) \in C^\infty_c \times C^\infty_c$, and $\mathbf{X}^R$ defined by (3.1). For each positive R let $\mathbf{q}^R(t)$ be the corresponding particle's motion. Then, as R tends to zero, $\mathbf{q}^R(.)$ converges in the sense of distributions to the solution of the Cauchy problem

$$\begin{aligned} m\ddot{\mathbf{q}} &= m\tau\dddot{\mathbf{q}} + \mathbf{F}(\mathbf{q}, t) + \mathbf{F}^{R=0}_L(t, \mathbf{A}'_0, \dot{\mathbf{A}}'_0) \\ \mathbf{q}_0 &= \mathbf{q}_0 \ , \quad \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}_0 \ , \quad \ddot{\mathbf{q}}_0 = \mathbf{a} \ , \end{aligned} \tag{3.7}$$

with $\mathbf{F}^{R=0}_L(t, \mathbf{A}'_0, \dot{\mathbf{A}}'_0)$ defined by (2.5).

The idea of the proof is to solve (3.4) by Laplace transform, take the limit of the Laplace transform of the solution as $R \to 0$ and compare it with the Laplace transform of the solution of (3.7). This can be made rigorous through some technical estimates which are deferred to the appendix.

The above theorem has a double content. Indeed, on the one hand it ensures that the point limit of the particle's motion satisfies the LD equation with a Lorentz force due to the free evolution of a part of the initial fields, which might be called "free fields". On the other hand it gives a relation between the initial acceleration and the "non free" part of the initial field. We will show below that it is also possible to deduce the initial particle's acceleration from the initial field.

Remark 3.2. *In ref. [7] it was proved, with a different method, that, at least when $\mathbf{F}(\mathbf{q}, t) = -\alpha \mathbf{q}$ with $\alpha > 0$, the convergence of $\mathbf{q}^R$ is actually in $H^2([-T,T], \mathrm{I\!R}^3)$, so that the particle's position and velocity converge uniformly as the radius tends to zero.*

Remark 3.3. *Theorem 3.1 also holds if the fields $(\mathbf{A}'_0, \dot{\mathbf{A}}'_0)$, are much more irregular than assumed in the statement. For example it is enough to assume that such fields give rise to a Lorentz force $\mathbf{F}^0_L(., \mathbf{A}'_0, \dot{\mathbf{A}}'_0)$ which is $C^1([0,+\infty]) \cap L^1([0,+\infty])$.*

¿From theorem 3.1 we also have the following important

Corollary 3.4. *Let $\mathbf{q}(t)$ be a solution of the Cauchy problem (3.7) for the LD equation; then $\mathbf{q}(.)$ is the limit of the particle's motion corresponding to the solution of the ML system (2.1) with initial data $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}^R_{\dot{\mathbf{q}}_0} + \mathbf{A}'_0, \mathbf{X}^R_{\ddot{\mathbf{q}}_0} + \dot{\mathbf{A}}'_0)$.*

Concerning the initial acceleration to be inserted in the LD equation we have the following

Proposition 3.5. *For a field of the form* $\mathbf{A} = \mathbf{X}^0_{\mathbf{a}} + \mathbf{A}'$, *with* $\mathbf{a} \in \mathbb{R}^3$ *and* $\mathbf{A}' \in C^\infty_c$, *one has*

$$\mathbf{a} = \frac{3}{2}\frac{c}{e}\lim_{r\to 0} r\frac{1}{4\pi r^2}\int_{S_r}\mathbf{A}(\mathbf{x})d\sigma \ , \tag{3.8}$$

where, as above S_r *is the sphere of radius* r *centered at the origin, and* $d\sigma$ *its surface element.*

Proof. It is just a calculation. First notice that $\mathbf{A}'$ does not contribute to the limit in (3.8). So denote by $\mathbf{Z}(r)$ the average of $\mathbf{X}^0_{\mathbf{a}}$ over a sphere of radius r. We calculate it. Recall that the Fourier transform commutes with the spherical average [17], so that by calculating the angular integration in Fourier transform (see (3.1)) one has

$$\hat{\mathbf{Z}}(k) = 4\pi\frac{e}{c}\frac{2}{3}\frac{\mathbf{a}}{k^2}\frac{1}{(2\pi)^{3/2}} \ ,$$

which gives $\mathbf{Z}(r) = 2e\mathbf{a}/(3cr)$. From this (3.8) immediately follows. □

Remark 3.6. *Equation (3.8) enables one to explicitly calculate the initial acceleration in terms of the initial field; the corresponding formula is given*

$$\ddot{\mathbf{q}}_0 = \frac{3}{2}\frac{c}{e}\lim_{r\to 0} r\frac{1}{4\pi r^2}\int_{S_r}\dot{\mathbf{A}}_0(\mathbf{x})d\sigma \ . \tag{1.2}$$

So, the initial acceleration is proportional to the coefficient of r^{-1} in the power expansion of the spherical average of the electric field.

Remark 3.7. *The initial acceleration to be inserted in the LD equation is not free: in order to obtain a particle's motion which is coherent with the Maxwell–Lorentz system one must use the initial acceleration given by (1.2).*

Remark 3.8. *For the class of initial fields allowed by theorem 3.1 one has, in the point limit, a formula similar to (1.2) relating the initial particle's velocity and the initial value of the vector potential* $\mathbf{A}_0$, *namely*

$$\dot{\mathbf{q}}_0 = \frac{3}{2}\frac{c}{e}\lim_{r\to 0} r\frac{1}{4\pi r^2}\int_{S_r}\mathbf{A}_0(\mathbf{x})d\sigma \ . \tag{3.9}$$

Concerning the behaviour of the singularity of the fields for $t > 0$ we have

Remark 3.9. *The solutions of the limit field equation*

$$\frac{1}{c^2}\ddot{\mathbf{A}} - \triangle\mathbf{A} = \frac{4\pi}{c}\left(\dot{\mathbf{q}}\delta\right)_{\mathrm{tr}} \tag{3.10}$$

(where the index "tr" denotes, as above, the divergence free part) satisfy

$$\ddot{\mathbf{q}}(t) = \frac{3}{2}\frac{c}{e}\lim_{r\to 0} r\frac{1}{4\pi r^2}\int_{S_r}\dot{\mathbf{A}}(\mathbf{x},t)d\sigma \ , \quad \dot{\mathbf{q}}(t) = \frac{3}{2}\frac{c}{e}\lim_{r\to 0} r\frac{1}{4\pi r^2}\int_{S_r}\mathbf{A}(\mathbf{x},t)d\sigma \tag{3.11}$$

(to prove (3.11) use retarded potentials).

Remark 3.10. *We did not prove that, as $R \to 0$, the field converges in a suitable sense to a solution of (3.10). In the case $\alpha > 0$, $\mathbf{f}(t) \equiv 0$, this could be obtained easily by exploiting the strong convergence result for the particle's motion given in [7] (see remark 3.2).*

A general study of the behaviour of the field in the point limit is beyond the aim of this note and is deferred to future work.

4. Discussion

We have thus proved that the Lorentz–Dirac equation is essentially equivalent to the Maxwell–Lorentz system in the dipole approximation. In this connection we remark that, since (due to mass renormalization), the original Newton's equation for the particle, which involves the bare mass, is not defined in the point–limit, one can say that the Lorentz–Dirac equation is the point–like version of Newton equation with Lorentz force when the self interaction is taken into account. However, such an interpretation is possible only if a prescription is given for the initial particle's acceleration, and this is afforded by (1.2), which allows the practical use of the LD equation. We also remark that the generic runaway behaviour, which is well known to be present in the Lorentz–Dirac equation, turns out as a consequence to be present (and generic) also in the Maxwell–Lorentz system in dipole approximation. Moreover, some interesting phenomena recently found in the LD equation [18] turn out to pertain also to the Maxwell–Lorentz system (in the dipole approximation).

As recalled in the introduction, in [7] it was already proved, in the case $\mathbf{F}(\mathbf{q}, t) = -\alpha \mathbf{q}$ with $\alpha > 0$, that the limit particle's motion corresponding to adapted initial data (namely of the form $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}^0_{\dot{\mathbf{q}}_0}, 0))$ satisfy the LD equation. So the present paper (which uses a different and simpler technique) gives an extension of such a result. Moreover, we give here a recipe for the calculation of the correct initial acceleration.

On the other hand we think that the analysis of ref. [7] (which is very close to that of van Kampen [12]) gives a geometrical insight in the dynamics, which could be very useful for further developments of the theory of point particles, e.g. for the study of non–runaway motions, and in investigating the possible existence of a renormalized Hamiltonian. Moreover, the technique of paper [7] gives a much stronger convergence for the particle's motion (see remark 3.2 above).

Concerning formula (1.2) for the initial acceleration, we already recalled that it is essentially contained in a paper by Kijowski. Indeed Kijowski realized that the field calculated via the retarded potentials has the asymptotic behaviour (3.11) on the particle's position, and he interpreted eqs. (3.11) as boundary conditions that have to be imposed on the fields, and subsequently he upgraded the first of (3.11) to the status of dynamical equation for the particle, introducing a new boundary condition at the particle's position. Our result is that the first of (3.11) has to be explicitly imposed at $t = 0$, if the point–limit dynamics has to exist. On the contrary, the astonishing fact happens that the particle's acceleration immediately adapts itself to the initial electric field, so that the second of (3.11) is automatically satisfied at $t = 0$. No boundary conditions are needed for positive times (except those at infinity).

5. Appendix: Proof of theorem 3.1.

For simplicity we will deal explicitly only with the case $(\mathbf{A}'_0, \dot{\mathbf{A}}'_0) = 0$ and $\mathbf{F}(\mathbf{q}, t) \equiv 0$. The proof in the general case requires only minor changes; in particular, when $\alpha \neq 0$ the present technique gives a convergence slightly weaker than that ensured by lemma 5.2 below.

We split the proof into two lemmas.

Lemma 5.1. *There exists a positive r_*, and a positive w_* such that, for $0 < r \leq r_*$ and $w > w_*$, the solution of (3.4) with $\mathbf{A}'_0 = \dot{\mathbf{A}}'_0 = 0$ and $(\mathbf{q}, t) \equiv 0$ is given by*

$$\mathbf{q}^R(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \bar{\mathbf{q}}^R(w + is) e^{(w+is)t} ds \ , \tag{5.1}$$

where

$$\bar{\mathbf{q}}^R(p) = \frac{\mathbf{q}_0}{p} + \frac{\dot{\mathbf{q}}_0}{p^2} + \frac{\mathbf{a}\bar{M}_R(p)}{(m - p\bar{M}_R(p))p^2} \ . \tag{5.2}$$

Proof. By the standard theory of Laplace transform the candidate to be the Laplace transform of the solution of (3.4) is (5.2). We will prove that (5.2) is analytic in the region $\mathrm{Re}(p) \geq w_*$, and moreover tends to zero like $|p|^{-1}$ as $|p| \to \infty$, so that it is the Laplace transform of the function $\mathbf{q}^R$ given by (5.1).

The only nontrivial term to be studied is the last one in (5.2). So, we consider only the case $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = 0$. First notice that $\bar{M}_R$ is bounded and analytic for $\mathrm{Re}(p) \geq \epsilon > 0$. We give now some useful formulas which follow directly form the definition of $\bar{M}_R$, namely

$$m - p\bar{M}_R(p) = m - \frac{32}{3}\pi^2 \frac{e^2}{c^2} p^2 \int_0^\infty \frac{|\hat{\rho}_R(k)|^2}{p^2 + (ck)^2} dk = m_0 + \frac{32}{3}\pi^2 e^2 \int_0^\infty \frac{k^2 |\hat{\rho}_R(k)|^2}{p^2 + (ck)^2} dk \ , \tag{5.3}$$

$$\bar{M}_R(p) = \frac{16}{3i}\pi^2 \frac{e^2}{c^2} \int_{-\infty}^{\infty} \frac{|\hat{\rho}_R(k)|^2}{ck - ip} dk \ , \tag{5.4}$$

where the function $\hat{\rho}_R(k)$ has been extended to the whole real axis by symmetry.

Fix $R > 0$. From the last of (5.3) one has that $m - p\bar{M}_R(p) \to m_0(R)$ as $|p| \to \infty$, and therefore the last term of (5.2) behaves as $|p|^{-2}$ when $|p| \to \infty$. We prove now that the function (5.3) does not vanish for $\mathrm{Re}(p) \geq w_*$ with a suitable w_* independent of R.

Denoting $p^2 = \nu + i\omega$, one has that the imaginary part of $m - p\bar{M}_R$ is proportional through inessential coefficients to (use (5.3))

$$\omega \int_0^\infty \frac{|\hat{\rho}_R(k)|^2 (ck)^2}{(\nu + (ck)^2)^2 + \omega^2} dk \ , \tag{5.5}$$

which vanishes only for $\omega = 0$. So $m - p\bar{M}_R$ can vanish only on the real axis (i.e. when $\omega = 0$). We are led to look for *real* solutions of the equation $m - p\bar{M}_R(p) = 0$. Taking the derivative with respect to p one checks that the l.h.s. is monotonically decreasing (for positive p), and therefore such an equation has at most one positive solution. Moreover, the l.h.s. is a regular function of R and p, and therefore, by the implicit function theorem, the solution depends continuously on R. So, for small R the only solution of our equation is close to the solution of the limit equation $m - \tau m p = 0$. Therefore there exists w_* such that (5.2) is analytic in the region $\mathrm{Re}(p) \geq w_*$ (it is the sum of functions which are the inverse of nonvanishing analytic functions). □

Lemma 5.2. *As $R \to 0$ the function defined by (5.1), (5.2) converges, in the dual to $H^1([-T,T])$, to*

$$\frac{1}{2\pi} \int_{\mathrm{Re}(p)=w\geq w_*} \left(\frac{\mathbf{q}_0}{p} + \frac{\dot{\mathbf{q}}_0}{p} + \frac{m\tau\mathbf{a}}{(m - \tau m p)p^2} \right) e^{pt} d\mathrm{Im}(p) \ . \tag{5.6}$$

Proof. Denote $p = w + is$. We will prove that $\mathbf{q}^R(t)e^{-wt}$ (which is obviously extended to the whole $\mathbb{R}$ by zero) converges in the dual to $H^1(\mathbb{R})$ as $R \to 0$ to the function (5.6). To this end we have to prove that for any $f \in H^1$ the integral of f multiplied by $\mathbf{q}^R(t)e^{-wt}$ converges as $R \to 0$. In practice we will prove that $\bar{\mathbf{q}}^R(w + is)$ is uniformly bounded, so that Lebesgue theorem gives the result.

Again we will consider only the case $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = 0$. Notice that $\bar{M}$ is globally bounded uniformly in R. Now, we are going to estimate (from below) uniformly in R the denominator of (5.2). We first consider the case of large s, and estimate $m - p\bar{M}_R$ by the modulus of its imaginary part. Then we consider the case of finite s.

So, consider the imaginary part of $m - p\bar{M}$, which is given by (5.5). Calculating the derivative of (5.5) with respect to R it is easy to see that (5.5) is a function of R which is monotonically decreasing for $\omega > 0$. Therefore, for $R \leq R_*$ one has

$$|\mathrm{Im}(p\bar{M}_R(p))| \geq |\mathrm{Im}(p\bar{M}_{R_*}(p))| \geq Cs^{-2} \ , \quad \forall |s| \geq S \ ,$$

with S and C *independent of* R. We study now $|s| \leq S$. By the standard theory of the Hilbert transform [17] (based on the idea of taking the boundary value of functions analytic on a half plane) using $|\hat{\mathcal{D}}|^2 \in H^1(\mathbb{R})$ one has, as $R \to 0$,

$$\frac{1}{2i} \int_{-\infty}^{\infty} \frac{|\hat{\mathcal{D}}(b)|^2}{cb - iRp} db \to \frac{1}{2i} P \int_{-\infty}^{\infty} \frac{|\hat{\mathcal{D}}(b)|^2}{cb} db + \frac{\pi}{2c} \left|\hat{\mathcal{D}}(0)\right|^2 = \frac{\pi}{2c} \left|\hat{\mathcal{D}}(0)\right|^2 = \frac{1}{16\pi^2 c} \ ,$$

where $P\int_{-\infty}^{\infty}$ denotes the principal value (recall that p is fixed and $\mathrm{Re}(p) > 0$). It follows that, for $|s| < S$ and R small enough one has

$$m - \bar{M}_R(p)p = m - m\tau p + o(R^0) \ .$$

So, for fixed $w = \mathrm{Re}(p)$, one has

$$\left| \frac{\mathbf{a}\bar{M}_R(p)}{p^2(m - p\bar{M}_R(p))} \right| \leq C \ .$$

□

References

[1] H.A. Lorentz: The theory of Electrons, Dover, (New York 1952); first edition 1909.

[2] P.A.M. Dirac: *Classical Theory of Radiating Electrons.* Proc. Roy. Soc. **A167**, 148-168 (1938).

[3] G. Morpurgo: Introduzione alla fisica delle particelle. Zanichelli (Bologna, 1987)

[4] D. Bohm, M. Weinstein: *The Self Oscillations of a Charged Particle.* Phys. Rev. **74**, 1789-1798 (1948).

[5] W.T. Grandy, Ali Aghazadeh: *Radiative Corrections for Extended Charged Particles in Classical Elecrodynamics.* Ann. Phys. **142**, 284-298 (1982).

[6] A.D. Yaghjian: Relativistic Dynamics of a Charged Sphere. Lect. Notes. Phys. m.11, Springer Verlag (Berlin Heidelberg, 1992).

[7] D. Bambusi, D. Noja: *Classical Electrodynamics of Point Particles and Mass Renormalization. Some preliminary results.* Lett. Math. Phys. In print.

[8] D. Bambusi, D. Noja: *Rigorous results on classical electrodynamics of point particles in the dipole approximation and proof of the Abraham–Lorentz–Dirac equation.* Quaderno $\mathrm{n}^0$24/94 Dipartimento di Matematica "F.Enriques", University of Milano (1994).

[9] J. Kijowski: *Electrodynamic of Moving Particles* G.R.G. **26**, 167–201 (1994)

[10] M. Abraham: *Prinzipien der Dynamik des Elektrons.* Ann. der Phys. **10**, 105-179 (1903).

[11] H.A. Kramers: *Nonrelativistic Quantum Electrodynamics and the Correspondence Principle*, Rapports et discussion du huitieme Conseil de Physique Solvay. Published by R. Stoops (Brussels 1950).

[12] N.G. Van Kampen: *Contribution to the quantum theory of light scattering* Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26** No. 15 (1951).

[13] R.P. Feynman, R.B. Leighton, M. Sands: The Feynman lectures in physics, Mainly Electromagnetis and Matter, Vol. II. Addison–Wesley (Redwood City, 1963)

[14] L. Galgani, C. Angaroni, L. Forti, A. Giorgilli, F. Guerra: *Classical Electrodynamics as a Nonlinear Dynamical System.* Phys. Lett. A, **139**, 221-230, (1989).

[15] D. Bambusi, L. Galgani: *Some Rigorous Results on the Pauli-Fierz Model of Classical Electrodynamics.* Ann. Inst. H. Poincaré, Physique théorique **58**, 155-171 (1993).

[16] S. Coleman, R. E. Norton: *Runaway Modes in Model Field Theories* Phys. Rev. **125**, 1422-1428 (1962).

[17] E.M. Stein, G. Weiss: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press (Princeton, New Jersey 1971)

[18] A. Carati, P. Delzanno, L. Galgani, J. Sassarini: *Nonuniqueness properties of solutions of the Lorentz Dirac equation* Nonlinearity. **8** 65–79 (1995).