Quantum fluctuations in quantum lattice-systems with continuous symmetry[1]

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Abstract

We discuss conditions for the absence of spontaneous breakdown of continuous symmetries in quantum lattice systems at $T = 0$. Our analysis is based on Pitaevskii and Stringari's idea that the uncertainty relation can be employed to show quantum fluctuations. For the one-dimensional systems, it is shown that the ground state is invariant under the continuous transformation if a certain uniform susceptibility is finite. For the two- and three-dimensional systems, it is shown that truncated correlation functions cannot decay any more rapidly than $|r|^{-d+1}$ whenever the continuous symmetry is spontaneously broken. Both of these phenomena occur owing to quantum fluctuations. Our theorems cover a wide class of quantum lattice-systems having not-too-long-range interactions.

KEY WORDS: Quantum fluctuations; ground states; symmetry breaking; uncertainty relation; clustering.

1 Introduction

It is well known that continuous symmetries cannot be spontaneously broken in one- and two-dimensional systems at nonzero temperatures if the interactions are short range. Since Mermin and Wagner,[(1)] and Hohenberg[(2)] showed rigorous proofs, several papers have appeared, proving the invariance of the state under the continuous transformation.[(3 - 8)] These arguments, however, work only at finite temperatures.

Absence of symmetry breaking in the ground state of the one-dimensional quantum systems has been a long-standing question. This problem was discussed by using an extension of the Bogoliubov inequality[(9)] and using the uncertainty relation.[(10, 11)] Takada[(9)] argued the relation between the absence of long-range order and the dispersion form of the excitation spectrum, and thereby showed that, if the lowest excitation frequency has a gapless k-linear form, the ground state cannot show symmetry breaking. Pitaevskii and Stringari[(10)] proposed a zero-temperature analogue of the Bogoliubov inequality, using the uncertainty relation of the quantum mechanics. They presented a method for showing the absence of breakdown of continuous symmetry in the ground state. After that, Shastry[(11)] pointed out that one can complete the proof for the one-dimensional Heisenberg antiferromagnet combining their method and the infrared bound given by Dyson, Lieb and Simon.[(12)] The method proposed by Pitaevskii and Stringari[(10)] can be successfully applied only when we have a rigorous upper bound of the susceptibility at

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the whole momentum space. It is however difficult (to date) to obtain upper bounds of the momentum-dependent susceptibility in general quantum systems.

Another well-known theorem for short-range systems with continuous symmetry is the Nambu-Goldstone theorem, which states that there exist gapless elementary excitations whenever any continuous symmetry is spontaneously broken. This theorem was also proved for the lattice systems.[13] Furthermore, Martin[7] showed that some truncated correlation functions at finite temperatures have power-decay behavior slower than or equal to $|r|^{-1}$ in three-dimensional systems if continuous symmetry is spontaneously broken.

In the present paper, we extend the method by Pitaevskii and Stringari,[10] using the technique developed by Martin,[7] and thereby show conditions for the ground state of quantum systems being invariant under the continuous transformation. We obtain the following results on the continuous-symmetry breaking in the ground states.

1. In the one-dimensional system, if a certain uniform susceptibility is finite, the ground state has continuous symmetry, i.e.,

$$\omega(\sigma_\theta(A)) = \omega(A) \tag{1}$$

   for any local observable A. Here $\omega(\cdots)$ denotes the ground state and σ_θ denotes the continuous transformation under which the interactions of the Hamiltonian are invariant. (See theorem 1.)

2. In more-than-one-dimensional ($d > 1$) systems, if any continuous symmetry is spontaneously broken in the ground state as $\frac{d}{d\theta}\omega(\sigma_\theta(A))|_{\theta=0} \neq 0$ with a local observable A and if a certain uniform susceptibility is finite, the truncated two-point correlation function of A shows a power-decay slower than or equal to $O(1/r^{d-1})$. Here we denote the dimensionality of the system by d. (See theorem 2.)

Both of these phenomena occur as a consequence of quantum fluctuations. In our discussion, we define the ground state applying an infinitesimally small field. We derive these results, using rigorous inequalities and assuming the clustering property of this ground state. (Note that this assumption is quite reasonable, though it cannot be verified within the presently available techniques in mathematical physics.) These theorems are applicable to a wide class of quantum lattice systems having not-too-long-range interactions and continuous symmetries. Quantum spin systems, lattice fermion-systems and hard-core bose systems are included, for example.

2 Theorems and physical consequences

2.1 Preliminaries

We first give some notations. We denote the d-dimensional lattice by $\mathcal{L}$, which is taken as $\mathbf{Z}^d$. For each lattice point $x \in \mathcal{L}$, there are the algebra $\mathcal{A}_x$ of operators and the finite-dimensional Hilbert space $\mathcal{H}_x$. For any bounded subset $\Lambda \subset \mathcal{L}$, local operators which are defined on Λ generate the local algebra $\mathcal{A}_\Lambda$ of observables and the Hilbert space is given by $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$.

For simplicity, we present arguments for quantum systems with two-body interactions. We can easily extend the following arguments to models with more-than-two-body interactions. Let $\mathcal{L}$ be the translationally invariant lattice and H_Λ be the Hamiltonian in the finite-volume lattice $\Lambda \subset \mathcal{L}$, which is given by

$$H_\Lambda = \sum_{x,y\in\Lambda} \phi(x,y). \tag{2}$$

Here $\phi(x, y)$ denotes the translationally invariant interaction defined on $\mathcal{H}_x \otimes \mathcal{H}_y$ with the norm $\|\phi(x, y)\| = \psi(x - y)$. We restrict our discussions to the models that have not-too-long-range interactions satisfying

$$\sum_{x \in \mathcal{L}} |x|^2 \psi(x) < \infty \tag{3}$$

and that have, at least, the $U(1)$-continuous symmetry, i.e.,

$$[\phi(x, y), J_\Omega] = 0 \tag{4}$$

for any x, $y \in \Omega$ and local subset $\Omega \subset \mathcal{L}$. Here J_Ω denotes the generator of the (global) symmetry-transformations of operators in $\mathcal{A}_\Omega$. The continuous symmetry-transformation is given by

$$\sigma_\theta(A) = \exp(i\theta J_\Omega) A \exp(-i\theta J_\Omega) \tag{5}$$

for any $A \in \mathcal{A}_\Omega$.

To define the ground state, we select a proper order parameter and then apply the corresponding symmetry-breaking field. Let us define the ground state in the form

$$\omega(\cdots) = \lim_{B \downarrow 0} \lim_{\Lambda \uparrow \mathcal{L}} \lim_{\beta \uparrow \infty} \frac{\mathrm{Tr} \cdots \exp\{-\beta(H_\Lambda - B O_\Lambda)\}}{\mathrm{Tr} \exp\{-\beta(H_\Lambda - B O_\Lambda)\}}, \tag{6}$$

where O_Λ denotes the order-parameter operator and B is the real-valued symmetry breaking field. It is known that the limits are well-defined by choosing suitable sequences of Λ and B. (See Appendix A of ref. 14, for example.)

We restrict our discussions to the case that the order-parameter operator has a sublattice-translational invariance. Hence the ground state defined by (6) has the following sublattice-translational invariance

$$\omega(A) = \omega(\tau_x(A)) \tag{7}$$

for any $x \in \mathcal{L}_\mathrm{s}$ and $A \in \mathcal{A}_\Omega$ on a local subset Ω. Here τ_x denotes the space translation by x and $\mathcal{L}_\mathrm{s}$ denotes a set of sites in a sublattice. If we consider antiferromagnets on a bipartite lattice, for example, the order parameter is set as the staggered magnetization and $\mathcal{L}_\mathrm{s}$ is one of two sublattices. In ordinary ferromagnets, the ground state has the full lattice-translational invariance and hence $\mathcal{L}_\mathrm{s} = \mathcal{L}$.

In the following discussions, we assume the clustering property of the state,

$$|\omega(\tau_x(A)B) - \omega(\tau_x(A))\omega(B)| \le O\left(\frac{1}{|x|^\delta}\right) \tag{8}$$

with $\delta > 0$ for sufficiently large $|x|$ and any A, $B \in \mathcal{A}_\Omega$ on a local subset Ω. This property means that observations at two points separating far away from one another do not affect each other. Note that this is a quite natural assumption. It is believed that, by selecting a proper order parameter, the state $\omega(\cdots)$ becomes an pure state, i.e., it has the clustering property.

Remark: It is widely believed that any physically natural equilibrium state has the clustering property. In studies on finite systems, we sometimes encounter states which do not have the cluster property. For example, consider the ground state of the three-dimensional Heisenberg antiferromagnet. It is shown that the ground state of finite-volume systems is invariant under the global spin rotation[15, 16] and it has a long-range order in the infinite-volume limit.[17] Taking the infinite-volume limit of the ground state of finite systems, one can define a ground state that does not have the clustering property. However, as discussed in ref. 14, this symmetric ground state is unphysical and only a mathematical object. It is believed that in the thermodynamic limit this state is decomposed into pure states and one of the pure states appears as a natural state in the real system.[18, 19]

2.2 Main Theorems

In this section, we show our theorems. Physical consequences of the theorems will be discussed in sections 2.3 and 2.4, and proofs are given in section 2.5.

The statement that the state $\omega(\cdots)$ has the continuous symmetry is equivalent to

$$\frac{d}{d\theta}\omega(\sigma_\theta(A))\Big|_{\theta=0} = 0 \tag{9}$$

for any $A \in \mathcal{A}_\Lambda$ on any subset $\Lambda \subset \mathcal{L}$. We consider the transformation σ_θ in which J_Ω is given by $J_\Omega = \sum_{x\in\Omega} \tau_x(J_0)$ with a bounded self-adjoint operator $J_0 \in \mathcal{A}_0$. In this case, we have

$$\frac{d}{d\theta}\omega(\sigma_\theta(A))\Big|_{\theta=0} = i\omega([J_\Lambda, A]) \tag{10}$$

for any $A \in \mathcal{A}_\Lambda$ and any subset $\Lambda \subset \mathcal{L}$.

Without loss of generality, we consider the operator A on the subset $\Lambda = \{x \in \mathcal{L} : |x| \le r\}$, where r is a finite constant. To discuss the quantity $\omega([J_\Lambda, A])$, we use the sublattice-translational invariance (7) and hence we have

$$\omega([J_\Lambda, A]) = \frac{1}{|\Omega_{\rm S}|}\sum_{x\in\Omega_{\rm S}} \omega([J_\Omega, \tau_x(A)]) \tag{11}$$

for any $A \in \mathcal{A}_\Lambda$, where $\Omega_{\rm S} = \{x \in \mathcal{L}_{\rm S} : |x| \le R\}$ and $\Omega = \{x \in \mathcal{L} : |x_i| \le R_0 \text{ for } i = 1, \ldots, d\}$ with $R_0 = R + r$. [Though equation (11) holds by setting Ω as $\{x \in \mathcal{L} : |x| \le R + r\}$, we have taken Ω as the hyper-cubic lattice for convenience in later discussions.] Bounding the absolute value of the right-hand side of (11) with the uncertainty relation and the Kennedy-Lieb-Shastry inequality[(17)], and estimating the R dependence of the upper bound, we obtain the following lemma.

Lemma. Let the interaction satisfy (3) and (4), and assume that the ground state (6) satisfies (7) and (8). Consider $\mathcal{A}_\Lambda$ on the subset $\Lambda = \{x \in \mathcal{L} : |x| \le r\}$, where r is a finite constant, and let $\Omega_{\rm S} = \{x \in \mathcal{L}_{\rm S} : |x| \le R\}$ and $\Omega = \{x \in \mathcal{L} : |x_i| \le R_0 \text{ for } i = 1, \ldots, d\}$ with $R_0 = R + r$. Furthermore, define the uniform susceptibility of J by

$$\chi_J = \lim_{\Omega\uparrow\mathcal{L}} \frac{2}{|\Omega|}\int_0^\infty d\lambda \{\omega(J_\Omega J_\Omega(i\lambda)) - \omega^2(J_\Omega)\} \ge 0 \tag{12}$$

assuming existence of the limit,[3] where $J_\Omega(t)$ is the time-evolved operator of J_Ω. Then, the right-hand side of (11) is bounded as

$$\left|\frac{1}{|\Omega_{\rm s}|}\sum_{x\in\Omega_{\rm s}} \omega([J_\Omega, \tau_x(A)])\right|^2 \le \begin{cases} O(R^{d-1-\delta}) \cdot \sqrt{\chi_J} & (0 < \delta < d) \\ O(R^{-1}\ln R) \cdot \sqrt{\chi_J} & (\delta = d) \\ O(R^{-1}) \cdot \sqrt{\chi_J} & (\delta > d) \end{cases} \tag{13}$$

for sufficiently large R and any $A \in \mathcal{A}_\Lambda$.

[3]Mathematically speaking, existence of the limit in (12) may be nontrivial. It should be remarked that this definition of the uniform susceptibility is equivalent to the standard one, which has been used in many papers in physics. See Appendix.

We will give a proof in section 2.5. As shown in the proof, this lemma comes from the uncertainty relation of the quantum mechanics. Hence the inequality (13) can show purely quantum effects. In the following, we discuss physical consequences of the bound in each dimension. It should be remarked that these results are applicable to various models on arbitrary lattices that have the translation invariance. Selecting bonds of the non-vanishing interactions $\phi(x, y)$, we can define various lattices on $\mathbf{Z}^d$. The results depend only on the dimensionality d of the lattice.

First, we discuss one-dimensional systems, in which $\mathcal{L} = \mathbf{Z}$. By taking the $R \to \infty$ limit of (13), the above lemma shows conditions for the absence of continuous-symmetry breaking in one-dimensional systems.

Theorem 1. Let $\mathcal{L}$ be a one-dimensional lattice and the interaction $\phi(x, y)$ satisfy (3) and (4). Assume the ground state (6) satisfies the properties (7) and (8). If the infinite volume limit in the definition (12) of the uniform susceptibility exists and if this susceptibility χ_J is not diverging, the ground state (6) is invariant under the continuous transformation σ_θ, i.e.,

$$\frac{d}{d\theta}\omega(\sigma_\theta(A))\bigg|_{\theta=0} = 0 \tag{14}$$

for any $A \in \mathcal{A}_\Lambda$ on any finite subset Λ.

Physical meanings of this theorem are discussed in section 2.3. An advantageous point of this theorem is that the results depend only on the "uniform" susceptibility, not on other momentum-dependent susceptibilities. The condition that the uniform susceptibility is finite (or vanishing) is physically important. (See examples in the next section.) We cannot improve the condition without further detailed properties of the model, since the uniform susceptibility is finite or diverging, depending on each model.

Next, we discuss two- and three-dimensional systems. For these systems, we consider the case that the continuous symmetry is spontaneously broken. Slight modifications of the lemma give the following bound for a truncated two-point correlation function.

Theorem 2. Let $\mathcal{L}$ be a more-than-one-dimensional ($d > 1$) lattice, and $\phi(x, y)$ satisfy (3) and (4). Assume that the ground state (6) satisfies the conditions (7) and (8). If continuous symmetry is spontaneously broken in the ground state (6), i.e., $\omega([J_\Lambda, A]) \neq 0$ with an operator $A \in \mathcal{A}_\Lambda$ on an arbitrary subset $\Lambda \subset \mathcal{L}$, and if the infinite volume limit in (12) exists and $\chi_J < \infty$, the truncated two-point correlation function of A shows the slow clustering as

$$|\omega(A^*\tau_x(A)) - \omega(A^*)\omega(\tau_x(A))| \geq O\left(\frac{1}{|x|^{d-1}}\right) \tag{15}$$

for sufficiently large $|x|$ with $x \in \mathcal{L}_\mathrm{S}$.

We discuss the meaning of this theorem in section 2.4 and give a proof in section 2.5. Under some conditions, this theorem states that the truncated correlation function of A cannot show any exponential decay in the ordered ground state. This result hence corresponds to an extension of the Nambu-Goldstone theorem. This theorem shows the conditions for existence of quantum fluctuations and shows strong correlation between the fluctuations. (Remember that in the classical model there is no fluctuation in the ground state and hence the truncated two-point correlation function vanishes.) The condition for the uniform susceptibility appears in the theorem again and it is important in this case, as well. (See examples in section 2.4.)

2.3 One-dimensional systems

First we discuss one-dimensional systems, whose lattice is set as $\mathbf{Z}$. Among the assumptions of Theorem 1, the finiteness of χ_J is physically important. It determines whether the ground state shows symmetry breaking or not. To clarify the meaning of Theorem 1, we display three examples.

Example 1. *Spin SU(2) symmetry.* Let us first consider the one-dimensional spin-S Heisenberg antiferromagnet on the lattice $\mathcal{L}(=\mathbf{Z})$. The Hamiltonian in $\Lambda \subset \mathcal{L}$ is given by

$$H_\Lambda = \sum_{\langle i,j\rangle \in \Lambda} (S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z), \tag{16}$$

where S_i^α ($\alpha = x,\, y,\, z$) denote the spin operators on the site i satisfying $[S_j^\alpha, S_k^\beta] = i\delta_{jk}\epsilon_{\alpha\beta\gamma}S_j^\gamma$ with $\mathbf{S}^2 = S(S+1)$. The summation runs over all the nearest-neighbor sites. As a generator of the $U(1)$ rotation, we take

$$J_\Lambda = \sum_{i\in\Lambda} S_i^z. \tag{17}$$

This model clearly satisfies the conditions (3) and (4). Setting the order-parameter operator of the antiferromagnetism as

$$O_\Lambda = \sum_{i\in\Lambda} (-1)^i S_i^x, \tag{18}$$

we define the ground state $\omega(\cdots)$ by (6). By definition, the ground state satisfies the sublattice-translation invariance (7). In this model, the quantity χ_J is the uniform magnetic susceptibility of the ground state $\omega(\cdots)$. It has been proved in refs. 12 and 17 that χ_J is bounded from above by a finite constant for the Heisenberg antiferromagnets on hyper-cubic lattices.[4] Finally we assume that $\omega(\cdots)$ satisfies the clustering property. Under this assumption, Theorem 1 hence states that the ground state $\omega(\cdots)$ has the spin-rotational symmetry.

For the system whose uniform susceptibility is not diverging, Theorem 1 states that quantum fluctuations suppress spin ordering, even if some momentum-dependent susceptibility is diverging. The correlation of $k=0$ is however special. Theorem 1 does not exclude the possibility of ferromagnetism, since in the ferromagnets the uniform transverse susceptibility, which is nothing but χ_J, diverges. As is well known, the one-dimensional Heisenberg ferromagnet has the fully ordered ground state. Thus the spin long-range correlation with the zero momentum can survive quantum fluctuations. Furthermore Theorem 1 says that ferrimagnetism can occur as well. Some models indeed show the ferrimagnetic order even in the one-dimensional system.[(20)] In the ferrimagnetism, antiferromagnetic long-range order coexists with ferromagnetic order. From the theorem we learn that this antiferromagnetic order can appear owing to the existence of ferromagnetic order.

Example 2. *Spin O(2) symmetry.* Next we consider the one-dimensional spin-S XY ferromagnet, whose Hamiltonian is

$$H = -\sum_{\langle i,j\rangle \in \Lambda} (S_i^x S_j^x + S_i^y S_j^y). \tag{19}$$

[4] Though, in refs. 12 and 17, they discussed only antiferromagnets without any magnetic field, their arguments can be easily extended to the Hamiltonian with the staggered magnetic field, $H_\Lambda - BO_\Lambda$, and hence we can show that their bound on the susceptibility holds for this system as well.

The summation runs over all nearest-neighbor sites. This model is invariant under the $O(2)$ rotation, whose generator is $J_\Lambda = \sum_{i\in\Lambda} S_i^z$. This model is expected to have strong correlation at $k = 0$. We hence set the order parameter as $O_\Lambda = \sum_{i\in\Lambda} S_i^x$, thereby defining the ground state by (6). The Hamiltonian and the ground state clearly satisfy the conditions (3), (4) and (7). Since this model has only the $O(2)$ symmetry and may have weak S^z-correlation, situations are different from the ferromagnets in the above example. We expect χ_J is not diverging in the XY model and hence, from Theorem 1, the ground state has the $O(2)$ rotational invariance.

Example 3. *U(1)-gauge symmetry of fermions.* Let us consider the breakdown of the $U(1)$-gauge symmetry of fermions. (The Hilbert space of fermion systems is not a simple tensor product of the local Hilbert spaces and hence some modifications to the notations are needed. Furthermore, each observable in the algebra $\mathcal{A}_\Omega$ should contain multiplets of an even number of fermion operators, so that $[A, B] = 0$ for any $A \in \mathcal{A}_{\Lambda_1}$ and $B \in \mathcal{A}_{\Lambda_2}$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$. Thereby our theorems still work for the fermion systems, as well.)

As an example of correlated lattice-fermions, we consider the one-dimensional Hubbard model, whose Hamiltonian is given by

$$H_\Lambda = -t \sum_{\langle i,j\rangle\in\Lambda} \sum_{\sigma=\uparrow,\downarrow} (c_{i\sigma}^* c_{j\sigma} + c_{j\sigma}^* c_{i\sigma}) + U \sum_{i\in\Lambda} n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i\in\Lambda} (n_{i\uparrow} + n_{i\downarrow}). \tag{20}$$

The summation of the hopping term runs over all the neatest-neighbor sites. We denote the creation operator of the fermion at site i with spin σ by $c_{i\sigma}^*$ and the number operator of the fermion by $n_{i\sigma}$. The generator of the gauge transformation is given by $J_\Lambda = \sum_{i\in\Lambda}(n_{i\uparrow} + n_{i\downarrow})$ and hence χ_J is the uniform charge susceptibility, or the compressibility. This model satisfies the conditions (3) and (4). Under the assumption of the clustering property, Theorem 1 states for this model that, if the compressibility is finite, there is no breakdown of the $U(1)$-gauge symmetry.

Here we mention about the model proposed by Essler et al.[21] In their model the ground state has superconductivity even in the one-dimensional system. It should be noted that the compressibility is diverging in the ground state of their model, and hence Theorem 1 is not applicable to their model.

2.4 Two- and three-dimensional systems

In this section we discuss two- and three-dimensional systems, whose lattice is taken as $\mathbf{Z}^2$ or $\mathbf{Z}^3$. To clarify the meaning of Theorem 2, let us consider two examples.

Example 4. We again discuss the spin-symmetry breaking of the Heisenberg antiferromagnet (16). Here we take the lattice $\mathcal{L}$ as $\mathbf{Z}^2$ or $\mathbf{Z}^3$. We set the order-parameter operator as $O_\Lambda = \sum_{r\in\Lambda} S_r^x \exp(i\mathbf{q}\cdot\mathbf{r})$ with $\mathbf{q} = (\pi, \ldots, \pi)$ and the generator of rotation as $J_\Lambda = \sum_{r\in\Lambda} S_r^z$. This model hence satisfies the conditions (3) and (4), and the ground state $\omega(\cdots)$ defined by (6) satisfies (7). The occurrence of symmetry breaking in $\omega(\cdots)$ is proved for the two-dimensional $S \geq 1$ models[22, 23] and for the three-dimensional arbitrary-S models.[17, 23] Existence of long-range order is also proved for anisotropic Heisenberg antiferromagnets.[24 - 26] In these models, the ground state hence shows

$$\omega([J_\Lambda, S_r^y]) = -i\omega(S_r^x) \neq 0. \tag{21}$$

Furthermore, the finiteness of χ_J is proved in refs. 12 and 17. (See also the footnote 3 on page 6.) Using these results and assuming the clustering property of $\omega(\cdots)$, we find from

Theorem 2 that the transverse-spin correlation shows the slow clustering as

$$|\omega(S_0^y S_r^y)| \geq O\left(\frac{1}{|r|^{d-1}}\right) \tag{22}$$

for $0,\ r \in \mathcal{L}_{\mathrm{S}}$ and for sufficiently large $|r|$. Note that $\omega(S_r^y) = 0$ by definition of the ground state. Hence (22) shows that there are quantum fluctuations in the ground state and they are strongly correlated. Shastry[11] showed that the transverse-structure factor diverges as $\omega(S_k^y S_{-k}^y) \sim 1/|\mathbf{k} - \mathbf{q}|$ at $\mathbf{k} \simeq \mathbf{q}$ in the ground state with Néel order. This indicates that the transverse-correlation function decays as $\omega(S_0^y S_r^y) \sim (-1)^r/|r|^{d-1}$. Thus this example shows that our bound (15) is optimal.

It may be worth mentioning about another Nambu-Goldstone-type theorem for the excitation spectrum of the Heisenberg antiferromagnets.[27 - 29] It states that the Néel-ordered ground state has a gapless excitation spectrum and the lowest frequency of excitations is bounded from above by a gapless k-linear form around $\mathbf{k} \simeq \mathbf{0}$ and $\mathbf{q}$. These two Nambu-Goldstone-type theorems may closely relate to each other.

Furthermore we discuss the ferromagnetic Heisenberg model, in which χ_J is diverging, to clarify the significance of the condition on χ_J. The ground state of the ferromagnet can be written as a direct product of local spins and it does not fluctuate. Hence the truncated two-point correlation function is always vanishing. Thus the Heisenberg ferromagnet is a special model, which does not contain quantum fluctuations in the ground state. Our theorem successfully excludes this special case.

Example 5. Finally we consider lattice fermion-systems, e.g., the Hubbard model (20) on $\mathbf{Z}^2$ or $\mathbf{Z}^3$, and consider the spontaneous breakdown of the $U(1)$-gauge symmetry. As the order parameter, we take, for example,

$$O_\Lambda = \sum_{i \in \Lambda} O_i^+ = \sum_{i \in \Lambda} (c_{i\uparrow}^* c_{i\downarrow}^* + c_{i\downarrow} c_{i\uparrow}). \tag{23}$$

One can take other types of order parameters, as well. The generator of gauge transformation is given by $J_\Lambda = \sum_{i \in \Lambda}(n_{i\uparrow} + n_{i\downarrow})$ and hence χ_J denotes the charge susceptibility. Assume that the ground state defined by (6) shows superconductivity and satisfies

$$\omega([J_\Lambda, O_j^-]) = 2i\omega(O_j^+) \neq 0, \tag{24}$$

where $O_j^- = ic_{j\uparrow}^* c_{j\downarrow}^* - ic_{j\downarrow} c_{j\uparrow}$. For this system, Theorem 2 states that, if the compressibility is finite, we have

$$|\omega(O_0^- O_r^-)| \geq O\left(\frac{1}{|r|^{d-1}}\right) \tag{25}$$

for sufficiently large $|r|$.

It should be remarked that the Coulomb interaction does not satisfy the condition (3) and hence Theorem 2 is not applicable to the models that contain the Coulomb interactions. Decay of correlation functions in these systems may closely relate to the Anderson-Higgs phenomena and it is out of scope of this paper.

2.5 Proof of Theorems

In this section, we shall show proofs of Lemma, Theorem 1 and Theorem 2.

Proof of Lemma. As in ref. 10, we use the following two inequalities; one is the uncertainty relation,[10]

$$|\omega([C,A])|^2 \le \omega(\{\Delta C^*, \Delta C\})\omega(\{\Delta A^*, \Delta A\}) \tag{26}$$

for any A, $C \in \mathcal{A}_\Omega$ with $\Delta C = C - \omega(C)$ and $\Delta A = A - \omega(A)$, and the other is Kennedy, Lieb and Shastry's inequality,[17]

$$\omega(\{\Delta C^*, \Delta C\})^2 \le D(C)\omega([[C^*, H_\Omega], C]) \tag{27}$$

for any $C \in \mathcal{A}_\Omega$. Here H_Ω denotes the Hamiltonian on Ω and $D(C)$ denotes the Duhamel two-point function of C,

$$D(C) = \lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}}\lim_{\beta\uparrow\infty}\int_0^\beta d\lambda\{\omega_{\Lambda,B}(C^*C(i\lambda)) - \omega_{\Lambda,B}(C^*)\omega_{\Lambda,B}(C)\}, \tag{28}$$

where

$$\omega_{\Lambda,B}(\cdots) = \frac{\mathrm{Tr}[\cdots\exp\{-\beta(H_\Lambda - BO_\Lambda)\}]}{\mathrm{Tr}[\exp\{-\beta(H_\Lambda - BO_\Lambda)\}]} \tag{29}$$

and

$$C(t) = \exp\{it(H_\Lambda - BO_\Lambda)\}C\exp\{-it(H_\Lambda - BO_\Lambda)\}. \tag{30}$$

Both inequalities (26) and (27) were first obtained for finite-volume systems. Taking the thermodynamic limit of the inequalities, one obtains (26) and (27). Combining (26) and (27), we have

$$|\omega([C,A])|^2 \le \left\{D(C)\omega([[C^*, H_\Omega], C])\right\}^{1/2}\omega(\{\Delta A^*, \Delta A\}) \tag{31}$$

for any A, $C \in \mathcal{A}_\Omega$, where $\Delta A = A - \omega(A)$. Setting A as $A_{\Omega_\mathrm{S}} = |\Omega_\mathrm{S}|^{-1}\sum_{x\in\Omega_\mathrm{S}}\tau_x(A)$ with $A \in \mathcal{A}_\Lambda$ and $C = J_\Omega$ in (31), we obtain an upper bound of (11).

To estimate properly the R dependence of the right-hand side of (31), we use the smooth action[5, 6] of J_Ω. We set the operator C as

$$C = J_f = \sum_{x\in\mathcal{L}} f(x)J_x, \tag{32}$$

where $f(x) = 1$ for $x \in \Omega$, and $f(x) \to 0$ as $|x| \to \infty$. Defining $x_{\max}$ by $x_{\max} = \max_i |x_i|$, we set the function $f(x)$ in the form

$$f(x) = \begin{cases} 1 & (x_{\max} < R_0) \\ 2 - x_{\max}/R_0 & (R_0 \le x_{\max} \le 2R_0) \\ 0 & (2R_0 < x_{\max}). \end{cases} \tag{33}$$

Hence the operator $C(= J_f)$ is defined on the subset $\Omega' = \{x \in \mathcal{L} : |x_i| \le 2R_0 \text{ for } i = 1, \ldots, d\}$. Thus, we have

$$\begin{aligned} |\omega([J_\Omega, A_{\Omega_\mathrm{S}}])|^2 &= |\omega([J_f, A_{\Omega_\mathrm{S}}])|^2 \\ &\le \left\{D(J_f)\omega([[J_f, H_{\Omega'}], J_f])\right\}^{1/2}\omega(\{\Delta A^*_{\Omega_\mathrm{S}}, \Delta A_{\Omega_\mathrm{S}}\}) \end{aligned} \tag{34}$$

for any $A \in \mathcal{A}_\Lambda$, where $\Delta A_{\Omega_\mathrm{S}} = |\Omega_\mathrm{S}|^{-1}\sum_{x\in\Omega_\mathrm{S}}\tau_x(A) - \omega(A)$. From now, we discuss the right-hand side of (34) estimating the R dependence in the large R limit.

Let us first discuss $D(J_f)$. The operator J_f can be decomposed as

$$J_f = \frac{1}{R_0}\sum_{n=0}^{R_0-1}\Big(\sum_{x\in\Omega(n)} J_x\Big) = \frac{1}{R_0}\sum_{n=0}^{R_0-1} J_{\Omega(n)}, \tag{35}$$

where $\Omega(n)$ denotes the hyper-cubic lattice defined by

$$\Omega(n) = \{x \in \mathcal{L} : |x_i| \le R_0 + n \text{ for } i = 1, \dots, d\}. \tag{36}$$

Now we consider the finite-volume lattice $\Lambda(\supset \Omega(R_0))$ and introduce the function

$$D_{\Lambda,B}(A, C) = \lim_{\beta\uparrow\infty}\int_0^\beta d\lambda\{\omega_{\Lambda,B}(A^* C(i\lambda)) - \omega_{\Lambda,B}(A^*)\omega_{\Lambda,B}(C)\} \tag{37}$$

for $A, C \in \mathcal{A}_\Lambda$, where $C(t)$ is the time-evolved operator of C, given in (30). This function clearly satisfies $D_{\Lambda,B}(A, A) \ge 0$ and the linearity $D_{\Lambda,B}(A, aC_1 + bC_2) = aD_{\Lambda,B}(A, C_1) + bD_{\Lambda,B}(A, C_2)$ for any $a, b \in \mathbf{C}$ and $A, C_1, C_2 \in \mathcal{A}_\Lambda$. We hence regard $D_{\Lambda,B}(A, C)$ as the inner product. Inserting J_f into $D_{\Lambda,B}$, we obtain

$$\begin{aligned} D_{\Lambda,B}(J_f, J_f) &= \frac{1}{{R_0}^2}\sum_{n=0}^{R_0-1}\sum_{m=0}^{R_0-1} D_{\Lambda,B}(J_{\Omega(n)}, J_{\Omega(m)}) \\ &\le \frac{1}{{R_0}^2}\sum_{n=0}^{R_0-1}\sum_{m=0}^{R_0-1} |D_{\Lambda,B}(J_{\Omega(n)}, J_{\Omega(m)})| \\ &\le \frac{1}{{R_0}^2}\sum_{n=0}^{R_0-1}\sum_{m=0}^{R_0-1} \{D_{\Lambda,B}(J_{\Omega(n)}, J_{\Omega(n)}) D_{\Lambda,B}(J_{\Omega(m)}, J_{\Omega(m)})\}^{1/2}, \end{aligned} \tag{38}$$

where we have used the Schwarz inequality. Taking the thermodynamic limit of the system, we have $\lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}} D_{\Lambda,B}(A, A) = D(A)$ and hence from (38) we obtain

$$D(J_f) \le \frac{1}{{R_0}^2}\sum_{n=0}^{R_0-1}\sum_{m=0}^{R_0-1}\{D(J_{\Omega(n)})D(J_{\Omega(m)})\}^{1/2}. \tag{39}$$

The function $D(A)$ can be written as $D(A) = 2\int_0^\infty d\lambda\{\omega(AA(i\lambda)) - \omega^2(A)\}$ for an arbitrary self-adjoint operator A, where $A(i\lambda)$ denotes the time-evolved operator of A. Hence, in the large R limit, $D(J_{\Omega(n)})$ relates to the uniform susceptibility in the form

$$\chi_J = \lim_{R_0\uparrow\infty}\frac{1}{|\Omega(n)|} D(J_{\Omega(n)}). \tag{40}$$

(See also Appendix.) For sufficiently large R, using $|\Omega(n)| = (2R_0 + 2n + 1)^d$ and $R_0 = R + r$, we have

$$D(J_{\Omega(n)}) = \{(2R_0 + 2n + 1)^d + o(R^d)\}\chi_J \le 4^d(R + r)^d\chi_J \tag{41}$$

and hence, from (39), we obtain

$$D(J_f) \le 4^d(R + r)^d\chi_J. \tag{42}$$

Next, we discuss other parts in the right-hand side of (34). Since calculations of $\omega(\{\Delta A^*_{\Omega_S}, \Delta A_{\Omega_S}\})$ have been published in ref. 7, we adopt the results and do not repeat the calculations here. Thereby we have an upper bound

$$\omega(\{\Delta A^*_{\Omega_S}, \Delta A_{\Omega_S}\}) \le \begin{cases} O(R^{-\delta}) & (0<\delta<d) \\ O(R^{-d}\ln R) & (\delta = d) \\ O(R^{-d}) & (\delta > d), \end{cases} \tag{43}$$

where we have used the clustering property (8). Calculations of $\omega([[J_f, H], J_f])$ are also given in ref. 7. Though the definition of the smooth function $f(x)$ is different from ours, the derivations and results of ref. 7 still hold only by changing the spherical supports to the hyper-cubic ones. Thus we have

$$\omega([[J_f, H], J_f]) \le M\|J_0\|^2 R^{d-2} \sum_x |x|^2 \psi(x), \tag{44}$$

where M is a positive finite constant. If we use J_Ω instead of J_f in (44), $\omega([[J_\Omega, H], J_\Omega])$ can be bounded by the form R^{d-1}. Thus in (44) the double commutator is better estimated due to the smooth action. Inserting (42)–(44) into (34), we obtain (13). ∎

Proof of Theorem 1. Setting $d = 1$ in Lemma, taking the $R \to \infty$ limit, and using (10) and (11), one obtains (14) for any $\delta > 0$, if $\chi_J < \infty$. ∎

Proof of Theorem 2. Consider the case that all hypotheses of this theorem are satisfied and furthermore assume that the truncated two-point correlation function of A decays faster than $1/|x|^{d-1}$, i.e.,

$$|\omega(A^*\tau_x(A)) - \omega(A^*)\omega(\tau_x(A))| \le o\left(\frac{1}{|x|^{d-1}}\right). \tag{45}$$

Here $o(|x|^{-d+1})$ denotes a number that is lower order than $|x|^{-d+1}$. Using (45) instead of the clustering property (8), one can obtain

$$\omega(\{\Delta A^*_{\Omega_S}, \Delta A_{\Omega_S}\}) \le o(R^{-d+1}) \tag{46}$$

instead of (43). Thus, slightly modifying the proof of Lemma, we obtain

$$\left|\frac{1}{|\Omega_s|}\sum_{x\in\Omega_s} \omega([J_\Omega, \tau_x(A)])\right|^2 \le o(R^0), \tag{47}$$

where $o(R^0)$ denotes a number that vanishes in the $R \to \infty$ limit. (Remember that we are in the condition $\chi < \infty$.) In the $R \to \infty$ limit, (47) shows $\omega([J_\Lambda, A]) = 0$. This clearly contradicts with the condition $\omega([J_\Lambda, A]) \neq 0$ and hence, by contradiction, we arrive at (15). ∎

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A Definitions of the uniform susceptibility

We give a comment on the definition of the uniform susceptibility (12). In the literature, the susceptibility is usually defined by

$$X_J \equiv \lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}} \frac{1}{|\Lambda|} D_{\Lambda,B}(J_\Lambda), \tag{48}$$

where

$$D_{\Lambda,B}(A) = \lim_{\beta\uparrow\infty}\int_0^\beta d\lambda \{\omega^\beta_{\Lambda,B}(A^* A(i\lambda)) - \omega^\beta_{\Lambda,B}(A^*)\omega^\beta_{\Lambda,B}(A)\} \tag{49}$$

with

$$\omega^\beta_{\Lambda,B}(\cdots) = \frac{\mathrm{Tr}[\cdots \exp\{-\beta(H_\Lambda - BO_\Lambda)\}]}{\mathrm{Tr}[\exp\{-\beta(H_\Lambda - BO_\Lambda)\}]}. \tag{50}$$

For an arbitrary self-adjoint operator $A \in \mathcal{A}_\Lambda$, $D_{\Lambda,B}(A)$ can be written as

$$D_{\Lambda,B}(A) = 2\int_0^\infty d\lambda \{\omega^{\beta=\infty}_{\Lambda,B}(AA(i\lambda)) - \omega^{\beta=\infty}_{\Lambda,B}(A)\omega^{\beta=\infty}_{\Lambda,B}(A)\}. \tag{51}$$

In (48), the limits are taken so that the state $\omega^\beta_{\Lambda,B}(\cdots)$ converges. Here we assume that the limits of the quantity in (48) exist and that X_J is well-defined. Our definition of the uniform susceptibility is however different from (48). In this paper, we have defined the uniform susceptibility as follows

$$\begin{aligned} \chi_J &\equiv \lim_{\Omega\uparrow\mathcal{L}}\lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}} \frac{1}{|\Omega|} D_{\Lambda,B}(J_\Omega) \\ &= \lim_{\Omega\uparrow\mathcal{L}} \frac{2}{|\Omega|}\int_0^\infty d\lambda \{\omega(J_\Omega J_\Omega(i\lambda)) - \omega(J_\Omega)\omega(J_\Omega)\} \end{aligned} \tag{52}$$

taking suitable subsequences of Λ and B, where Ω is set as the hyper-cubic subsets $\{x \in \mathcal{L} : |x_i| \le R_0 \text{ for } i = 1,\ldots,d\}$ and

$$\omega(\cdots) = \lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}}\lim_{\beta\uparrow\infty} \omega^\beta_{\Lambda,B}(\cdots). \tag{53}$$

In this Appendix, we shall show that these two definitions are equivalent and hence χ_J converges to X_J.

Consider a finite subset $\Lambda(\supset \Omega)$ and a function $g(x)$ defined by

$$g(x) = \begin{cases} 1 & (x \in \Omega) \\ 0 & (x \notin \Omega), \end{cases} \tag{54}$$

then we have $J_\Omega = \sum_{x\in\Lambda} g(x) J_x$ and

$$\begin{aligned} \frac{1}{|\Omega|} D_{\Lambda,B}(J_\Omega) &= \frac{1}{|\Omega|}\frac{1}{|\Lambda|} D_{\Lambda,B}(\sum_k g_{-k} J_k) \\ &= \frac{1}{|\Omega|}\frac{1}{|\Lambda|}\sum_k |g_k|^2 D_{\Lambda,B}(J_k), \end{aligned} \tag{55}$$

where $J_k = |\Lambda|^{-1/2}\sum_{x\in\Lambda} J_x \exp(ikx)$ and $g_k = \sum_{x\in\Lambda} g(x)\exp(ikx)$. In the thermodynamic limit, (55) can be written as

$$\lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}} \frac{1}{|\Omega|} D_{\Lambda,B}(J_\Omega) = \frac{1}{|\Omega|}\int_{|k_i|\le\pi} \frac{d^d k}{(2\pi)^d} |g_k|^2 X_J(k), \tag{56}$$

where

$$X_J(k) = \lim_{B\downarrow 0}\lim_{\Lambda\uparrow\mathcal{L}} D_{\Lambda,B}(J_k). \tag{57}$$

The function $|\Omega|^{-1}|g_k|^2$ has the following two properties;

$$\int_{|k_i|\le\pi} \frac{d^dk}{(2\pi)^d}\frac{1}{|\Omega|}|g_k|^2 = 1 \tag{58}$$

and

$$\begin{aligned}\lim_{\Omega\uparrow\mathcal{L}} \frac{1}{|\Omega|}|g_k|^2 &= \lim_{R_0\uparrow\infty} \frac{1}{(2R_0+1)^d}\left\{\prod_{i=1}^{d}\frac{\sin k_i(R_0+1/2)}{\sin k_i/2}\right\}^2 \\ &= 0 \end{aligned}\tag{59}$$

for any k satisfying $k \ne 0$ and $|k_i| \le \pi$. Hence it converges to the Dirac's delta function,

$$\lim_{\Omega\uparrow\mathcal{L}} \frac{1}{|\Omega|}|g_k|^2 = (2\pi)^d\delta(k) \tag{60}$$

for $|k_i| \le \pi$. Inserting (60) into (56) and using $X_J(k=0) = X_J$, we thus find that $\chi_J = X_J$.

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