Soliton-Like Asymptotics for a Classical Particle Interacting with a Scalar Wave Field

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Abstract We consider a scalar wave field translation invariantly coupled to a single particle. This Hamiltonian system admits soliton-like solutions, where the particle and comoving field travel with constant velocity. We prove that a solution of finite energy converges, in suitable local energy seminorms, to some soliton-like solutions in the long time limit $t \to \pm\infty$.

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1 Introduction and Main Results

Let us consider a mechanical particle coupled to a scalar wave field in a translation invariant manner. The equations of motion read, for $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$,

$$\begin{aligned} \dot{\phi}(x,t) &= \pi(x,t) \ , & \dot{\pi}(x,t) &= \Delta\phi(x,t) - \rho(x-q(t)) \ , \\ \dot{q}(t) &= p(t)/(1+p^2(t))^{1/2} \ , & \dot{p}(t) &= \textstyle\int d^3x \phi(x,t) \nabla\rho(x-q(t)) \ . \end{aligned} \tag{1}$$

Here $\phi(x,t)$ is the real scalar field, $q(t)$ the position of the particle, and ρ a form factor of compact support providing a cut-off in the interaction at small distances. All derivatives in (1) are understood in the sense of distributions. If we introduce the momentum p as canonically conjugate to q and the field $\pi(x)$ as canonically conjugate to $\phi(x)$, then (1) is a Hamiltonian system with Hamiltonian functional

$$h(\phi,q,\pi,p) = (1+p^2)^{1/2} + \frac{1}{2}\int d^3x \, (|\pi(x)|^2 + |\nabla\phi(x)|^2) + \int d^3x \, \phi(x)\rho(x-q) \ . \tag{2}$$

Note that the kinetic energy of the particle is relativistic and therefore $|\dot{q}| < 1$. Since the interaction is translation invariant, one expects soliton-like solutions to (1) of the form

$$\phi(x,t) = \phi_v(x - vt - q) \ , \quad q(t) = vt + q \tag{3}$$

with $v \in V = \{v \in \mathbb{R}^3 : \ |v| < 1\}$. Indeed they are easily determined. For $v \in V$ there is a unique function ϕ_v which makes (3) a solution to (1). It is given by

$$\phi_v(x) = -\int d^3y (4\pi|(y-x)_{\|} + \lambda(y-x)_{\perp}|)^{-1}\rho(y) \ , \quad \lambda = \sqrt{1-v^2} \tag{4}$$

as derived in Appendix 1. We have set $z = z_{\|} + z_{\perp}$, $z_{\|} \, \| v$ and $z_{\perp} \perp v$ for $z \in \mathbb{R}^3$.

We are interested in the long time asymptotics of a solution to (1), which turns out to be governed by the following basic mechanism: Whenever $\ddot{q}(t) \neq 0$, energy is transferred from the particle to infinity via the wave field. Since the energy is bounded from below, this must mean that $\ddot{q}(t) \to 0$ as $t \to \infty$. Indeed, such a result is proved in [5] using the boundedness of a certain energy dissipation functional. One would expect then that also the velocity has a limit, $\dot{q}(t) \to v$ for some $v \in V$, and consequently

$$\phi(q(t)+x,t) \to \phi_v(x) \quad \text{as} \quad t \to \infty \ . \tag{5}$$

Our main progress is to establish these limits.

Before entering into a more precise and technical discussion, it may be useful to give a general idea of our strategy. One first notices, that because of translation invariance the total momentum

$$P(\phi,q,\pi,p) = p - \int d^3x \, \pi(x) \ \nabla\phi(x) \tag{6}$$

is conserved. Through the canonical transformation $(\Phi(x), Q, \Pi(x), P) = (\phi(q+x), q, \pi(q+x), P(\phi,q,\pi,p))$ one obtains the new Hamiltonian

$$\begin{aligned} &H_P(\Phi,\Pi) = h(\phi,q,\pi,p) \\ &= \int \ d^3x \ \Big(\frac{1}{2}|\Pi(x)|^2 + \frac{1}{2}|\nabla\Phi(x)|^2 + \Phi(x)\rho(x)\Big) + \Big(1 + \Big(P + \int \ d^3x \ \Pi(x) \ \nabla\Phi(x)\Big)^2\Big)^{1/2} \ . \end{aligned}$$

Since Q is cyclic, we may regard P as a parameter and consider the reduced system (Φ, Π) only. Let us define

$$\pi_v(x) = -v \cdot \nabla\phi_v(x) \ , \quad P(v) = p_v + \int d^3x \ v \cdot \nabla\phi_v(x) \ \nabla\phi_v(x) \ , \quad p_v = v/(1-v^2)^{1/2} \ . \quad (7)$$

We will prove that (ϕ_v, π_v) is the unique critical point and global minimum of $H_{P(v)}$. Thus initial data close to (ϕ_v, π_v) must remain close forever by conservation of energy, which translates into the orbital stability of soliton-like solutions. For a general class of nonlinear wave equations with symmetries such orbital stability of soliton-like solutions is proved in the well known work [4]. Our argument here follows in essence Bambusi and Galgani [1] who discuss the coupled Lorentz-Maxwell equations.

Orbital stability by itself is not enough. It only ensures that initial data close to a soliton remain so and does not yield the convergence of $\dot{q}(t)$, even less the convergence (5). Thus we need an additional, not quite obvious argument which combines the limit $\ddot{q}(t) \to 0$ as $t \to \infty$ with the orbital stability in order to establish the long time asymptotics. As one essential input we will use the strong Huygens principle for the wave equation.

We recall some definitions and assumptions from [5]. For the form factor ρ we assume that

$$\rho \in C_0^\infty(\mathbb{R}^3) \ , \qquad \rho(x) = 0 \ \text{ for } \ |x| \geq R_\rho \ , \qquad \rho(x) = \rho_r(|x|) \ . \qquad (C)$$

We require that all "modes" of the wave field couple to the particle, which is formalized by the Wiener condition

$$\hat{\rho}(k) = \int d^3x \, e^{ikx} \rho(x) \neq 0 \qquad \text{for all} \quad k \in \mathbb{R}^3 \ . \qquad (W)$$

In [5] generic examples of form factors satisfying both (C) and (W) are constructed.

Next we introduce the phase space for (1). A point in phase space is referred to as state. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with norm $|\!|\cdot|\!|$ and scalar product $(\cdot,\cdot)$, and $D^{1,2}$ be the completion of the real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\phi(x)\| = |\!|\nabla\phi(x)|\!|$. Equivalently, using Sobolev's embedding theorem, $D^{1,2} = \{\phi(x) \in L^6(\mathbb{R}^3) : \ |\nabla\phi(x)| \in L^2(\mathbb{R}^3)\}$ (see [6]) . Let $|\!|\phi|\!|_R$ denote the norm in $L^2(B_R)$ for $R > 0$, where $B_R = \{x \in \mathbb{R}^3 : |x| \leq R\}$. Then the seminorms $\|\phi\|_R = |\!|\nabla\phi|\!|_R + |\!|\phi|\!|_R$ are continuous on $D^{1,2}$.

We denote by $\mathcal{E}$ the Hilbert space $D^{1,2} \oplus \mathbb{R}^3 \oplus L^2 \oplus \mathbb{R}^3$ with finite norm

$$\|Y\|_{\mathcal{E}} = \|\phi\| + |q| + |\!|\pi|\!| + |p| \quad \text{for} \quad Y = (\phi, q, \pi, p) \, .$$

For smooth $\phi(x)$ vanishing at infinity we have

$$\begin{aligned} -\frac{1}{8\pi}\int\!\!\int d^3x \ d^3y \ \frac{\rho(x)\rho(y)}{|x-y|} = \frac{1}{2}(\rho, \Delta^{-1}\rho) \ &\leq \ \frac{1}{2}|\!|\nabla\phi|\!|^2 + (\phi(x), \rho(x-q)) \\ &\leq \ |\!|\nabla\phi|\!|^2 - \frac{1}{2}(\rho, \Delta^{-1}\rho) \ . \end{aligned}$$

Therefore $\mathcal{E}$ is the space of finite energy states and in particular for the soliton-like solutions

$$y_{v,q}(t) = (\phi_v(x - vt - q), \ vt + q, \ \pi_v(x - vt - q), \ p_v) \qquad (8)$$

we have $\|y_{v,q}(t)\|_{\mathcal{E}} < \infty$. Note that $\|\phi_v\| < \infty$, but $|\!|\phi_v|\!| = \infty$.

Proposition 1.1 *For every $y^0 = (\phi^0\ ,\ q^0\ ,\ \pi^0\ ,\ p^0) \in \mathcal{E}$ the Hamiltonian system (1) has a unique solution $y(t) = (\phi(t),\ q(t),\ \pi(t),\ p(t)) \in C(\mathbb{R}, \mathcal{E})$ with $y(0) = y^0$. Energy and total momentum are conserved.*

We refer to Section 2 where also the precise notion of solution is explained.

On physical grounds one is tempted to conjecture that every solution $y(t)$ of finite energy will converge to some soliton-like solution as $t \to \infty$. We do not achieve such a global result in two respects: The $t = 0$ fields are required to decay at infinity so to have a finite energy. But in addition some smoothness is imposed. More severely, we do not control the asymptotics for the position in the form $q(t) \sim vt + q$. We only prove that $\dot{q}(t)$ has a limit.

Theorem 1.2 *Let (C) and (W) hold. The initial state $y^0 = (\phi^0\ , q^0\ , \pi^0\ , p^0) \in \mathcal{E}$ is required to have the following decay at infinity: For some $R_0 > 0$ the functions $\phi^0(x)$,$\pi^0(x)$ are C^2,C^1-differentiable outside the ball B_{R_0}, respectively, and*

$$|\phi^0(x)| + |x|(|\nabla\phi^0(x)| + |\pi^0(x)|) + |x|^2(|\nabla\nabla\phi^0(x)| + |\nabla\pi^0(x)|) = \mathcal{O}(|x|^{-\sigma}) \quad \text{as} \quad |x| \to \infty \quad (9)$$

with some $\sigma > 1/2$. Let $y(t) \in C(\mathbb{R}, \mathcal{E})$ be the solution to (1) with $y(0) = y^0$. Then there exists a velocity $v \in V$ such that for every $R > 0$

$$\lim_{t\to\infty} \Big(\|\phi(q(t) + \cdot\ ,\ t) - \phi_v(\cdot)\|_R + |\pi(q(t) + \cdot\ ,\ t) - \pi_v(\cdot)|_R + |\dot{q}(t) - v| \Big) = 0\ . \qquad (10)$$

Remarks (i) Since the Hamiltonian system (1) is invariant under time-reversal, our results also hold for $t \to -\infty$.
(ii) The assumption (C) can be weakened to finite differentiability and to some decay of $\rho(x)$ at infinity.

In [5] we consider the system (1) with an additional external potential $V(q)$ which confines the particle. In this case the system has stationary solutions of the form $(\phi_{(q^*)}(x), q^*, 0, 0)$ where $\nabla V(q^*) = 0$ and $\phi_{(q^*)}(x) = \phi_0(x - q^*)$. If the set of critical points for $V(q)$ is discrete, then the solution $y(t)$ converges locally to some stationary state in the sense that $\|\phi(x,t) - \phi_{(q^*)}(x)\|_R + |\pi(x,t)|_R$ vanishes and $\dot{q}(t) \to 0$ as $t \to \infty$. In Theorem 1.2 we prove the same kind of convergence provided we substitute $\dot{q}(t) \to v$ and consider the fields close to the particle.

Soliton-like asymptotics of type (5) are proved in [7] for some translation invariant 1D completely integrable equations and in [3] for some class of 1D first and second order nonlinear translation invariant wave equations. Soliton-like asymptotics are also proved for small perturbations of soliton-like solutions to 2D and 3D nonlinear Schrödinger equations with a potential term with power decay at infinity [8] and to 1D nonlinear translation invariant Schrödinger equations [2].

2 Conservation laws and relaxation of the acceleration

We consider the Cauchy problem for the Hamiltonian system (1), which we write as

$$\dot{y}(t) = F(y(t))\ , \quad y(0) = y^0\ , \qquad (11)$$

where $y(t) = (\phi(t), q(t), \pi(t), p(t))$, $y^0 = (\phi^0 , q^0 , \pi^0 , p^0) \in \mathcal{E}$ and for $y = (\phi(x), q, \pi(x), p) \in \mathcal{E}$ we denote

$$F(y) = \Big(\pi(x), \;\; p/(1+p^2)^{1/2}, \;\; \Delta\phi(x) - \rho(x-q), \;\; \int d^3x \phi(x) \nabla\rho(x-q)\Big) \; . \tag{12}$$

All derivatives in (11) and (12) are understood in the sense of distributions. To define what we mean by a solution $y(t) \in C(\mathbb{R}, \mathcal{E})$, we introduce first a suitable space of test functions and of distributions.

Definition 2.1 *$\mathcal{D}$ denotes the space $D \oplus \mathbb{R}^3 \oplus D \oplus \mathbb{R}^3$, where $D = C_0^\infty(\mathbb{R}^3)$ is the space of real test functions. $\mathcal{D}^*$ denotes the dual space $D^* \oplus \mathbb{R}^3 \oplus D^* \oplus \mathbb{R}^3$, where D^* is the space of real distributions on $\mathbb{R}^3$. The pairing between $\mathcal{D}^*$ and $\mathcal{D}$ is written as $< \cdot, \cdot >$.*

Note that $F(y) \in \mathcal{D}^*$ for $y \in \mathcal{E}$.

Definition 2.2 *$y(t) \in C(\mathbb{R}, \mathcal{E})$ is said to be a solution to Eq. (11), equivalently to the system (1), iff for all $t \in \mathbb{R}$ and for every $w \in \mathcal{D}$*

$$< y(t), w > - < y(0), w > = \int_0^t ds < F(y(s)), w > . \tag{13}$$

The following lemma states existence and some properties of the solution to the Cauchy problem (11).

Lemma 2.3 *Let (C) hold. Then*
(i) For every $y^0 \in \mathcal{E}$ the Cauchy problem (11) has a unique solution $y(t) \in C(\mathbb{R}, \mathcal{E})$.
(ii) For every $t \in \mathbb{R}$ the map $w_t : y_0 \mapsto y(t)$ is continuous on $\mathcal{E}$.
(iii) The energy and total momentum are conserved, i.e. for every $t \in \mathbb{R}$

$$h(y(t)) = h(y^0) \quad \textit{and} \quad P(y(t)) = P(y^0) \; . \tag{14}$$

Proof We refer to [5, Lemma 2.1] where all items are proved except for total momentum conservation. For smooth initial data ϕ^0, π^0 of compact support momentum conservation follows by partial integration. This conservation extends to all of $\mathcal{E}$ by (ii) and because smooth initial data of compact support are dense in $D^{1,2} \oplus L^2$. $\square$

We restate the relaxation of the acceleration [5].

Proposition 2.4 *Let all assumptions of Theorem 1.2 be fulfilled. Then*

$$\lim_{t\to\infty} \ddot{q}(t) = 0. \tag{15}$$

Proof The system (1) is identical to the system (2) of [5] with $V(q) \equiv 0$. The zero potential satisfies the condition (P_w) from the Remark at the end of [5, Section 4]. Thus the convergence follows from [5, Lemma 4.1] . $\square$

3 Canonical transformation and reduced system

Since the total momentum is conserved, it is natural to use P as a new coordinate. To maintain the symplectic structure we have to canonically complete the coordinate transformation.

Definition 3.1 *Let $T:\mathcal{E}\to\mathcal{E}$ be defined by*

$$T : y = (\phi, q, \pi, p) \mapsto Y = (\Phi(x),\ Q,\ \Pi(x),\ P) = (\phi(q{+}x),\ q,\ \pi(q{+}x),\ P(\phi, q, \pi, p)) \quad , \tag{16}$$

where $P(\phi, q, \pi, p)$ is the total momentum (6).

Remarks i) T is continuous on $\mathcal{E}$ and Fréchet differentiable at points $y = (\phi, q, \pi, p)$ with sufficiently smooth $\phi(x), \pi(x)$, but it is not everywhere differentiable.

ii) In the T-coordinates the solitons (8) are stationary except for the coordinate Q,

$$Ty_v(t) = (\phi_v(x),\ vt + q,\ \pi_v(x),\ P(v)) \tag{17}$$

with $P(v)$ the total momentum of the soliton as defined in (7).

Let $H(Y) = h(T^{-1}Y)$ for $Y = (\Phi, Q, \Pi, P) \in \mathcal{E}$. Then

$$\begin{aligned} H(\Phi, Q, \Pi, P) &= H_P(\Phi, \Pi) = h(\Phi(x-Q),\ Q,\ \Pi(x-Q),\ P + \int d^3x\ \Pi(x)\ \nabla\Phi(x)) \\ &= \int d^3x\ \Big(\frac{1}{2}|\Pi(x)|^2 + \frac{1}{2}|\nabla\Phi(x)|^2 + \Phi(x)\rho(x)\Big) + \Big(1 + \Big(P + \int d^3x\ \Pi(x)\ \nabla\Phi(x)\Big)^2\Big)^{1/2} \quad . \end{aligned}$$

The functionals $H(Y)$ and $h(y)$ are Fréchet-differentiable on the phase space $\mathcal{E}$.

Proposition 3.2 *Let $y(t) \in C(\mathrm{I\!R}, \mathcal{E})$ be a solution to the system (1). Then $Y(t) = Ty(t) \in C(\mathrm{I\!R}, \mathcal{E})$ is a solution to the Hamiltonian system*

$$\begin{aligned} \dot{\Phi} &= \frac{\delta H}{\delta \Pi} \quad , \quad \dot{\Pi} = -\frac{\delta H}{\delta \Phi} \quad , \\ \dot{Q} &= \frac{\partial H}{\partial P} \quad , \quad \dot{P} = -\frac{\partial H}{\partial Q} \quad , \end{aligned} \tag{18}$$

understood in the sense of distributions, compare with (13).

Proof The equations for $\dot{\Phi}$, $\dot{\Pi}$ and $\dot{Q}$ can be checked by direct computation, while the one for $\dot{P}$ follows from Lemma 2.3. □

Q is a cyclic coordinate. Hence the system (18) is equivalent to a reduced Hamiltonian system for Φ and Π only, which can be written as

$$\dot{\Phi} = \frac{\delta H_P}{\delta \Pi} \ , \qquad \dot{\Pi} = -\frac{\delta H_P}{\delta \Phi} \quad . \tag{19}$$

For every $P \in \mathrm{I\!R}^3$ the functional H_P is Fréchet differentiable on the Hilbert space $\mathcal{F} = D^{1,2} \oplus L^2$.

Proposition 3.3 *For every $v \in V$ the functional $H_{P(v)}$ has the lower bound*

$$H_{P(v)}(\Phi,\Pi) - H_{P(v)}(\phi_v,\pi_v) \geq \frac{1-|v|}{2}\Big(\|\Phi-\phi_v\|^2 + |\Pi-\pi_v|^2\Big) \qquad (20)$$

on the space $\mathcal{F}$. Besides (ϕ_v,π_v), $H_{P(v)}$ has no other critical point in $\mathcal{F}$.

Proof Denoting $\Phi - \phi_v = \phi$ and $\Pi - \pi_v = \pi$ we have

$$H_{P(v)}(\phi_v+\phi,\pi_v+\pi) - H_{P(v)}(\phi_v,\pi_v) = \int d^3x(\pi_v(x)\pi(x) + \nabla\phi_v(x)\cdot\nabla\phi(x) + \rho(x)\phi(x))$$
$$+\frac{1}{2}\int\ d^3x\ (|\pi(x)|^2 + |\nabla\phi(x)|^2) + (1+(p_v+m)^2)^{1/2} - (1+p_v^2)^{1/2}\quad , \qquad (21)$$

where $p_v = P(v) + \int\ d^3x\ \pi_v(x)\ \nabla\phi_v(x)$ and

$$m = \int\ d^3x\ (\pi(x)\ \nabla\phi_v(x) + \pi_v(x)\ \nabla\phi(x) + \pi(x)\ \nabla\phi(x))\ .$$

Soliton-like solutions (3) satisfy

$$\pi_v(x) = -v\cdot\nabla\phi_v(x)\ , \qquad -\Delta\pi_v(x) + \rho(x) = v\cdot\nabla\phi_v(x). \qquad (22)$$

Setting $v = (1+p_v^2)^{-1/2}p_v$ and inserting in the first integral of (21) we obtain

$$\begin{aligned}&H_{P(v)}(\phi_v+\phi,\pi_v+\pi) - H_{P(v)}(\phi_v,\pi_v)\\ &= \frac{1}{2}\int\ d^3x\ (|\pi(x)|^2 + |\nabla\phi(x)|^2) + (1+p_v^2)^{-1/2}\int\ d^3x\ \pi(x)\ p_v\cdot\nabla\phi(x)\\ &-(1+p_v^2)^{-1/2}p_v\cdot m + (1+(p_v+m)^2)^{1/2} - (1+p_v^2)^{1/2}\ .\end{aligned}$$

Since the expression in the third line is nonnegative, the lower bound (20) follows by using $|(1+p_v^2)^{-1/2}p_v| = |v|$.

If $(\Phi,\Pi) \in \mathcal{F}$ is a critical point for $H_{P(v)}$, then it satisfies

$$0 = \Pi(x) + (1+\tilde{p}^2)^{-1/2}\tilde{p}\cdot\nabla\Phi(x)\ , \quad 0 = -\Delta\Phi(x) + \rho(x) - (1+\tilde{p}^2)^{-1/2}\tilde{p}\cdot\nabla\Pi(x)\ ,$$

where $\tilde{p} = P(v) + \int\ d^3x\ \Pi(x)\ \nabla\Phi(x)$. This system coincides with the system (22) for soliton-like solutions provided we set the velocity $\tilde{v} = (1+\tilde{p}^2)^{-1/2}\tilde{p}$. Hence $\Phi = \phi_{\tilde{v}}$, $\Pi = \pi_{\tilde{v}}$ and $P(\tilde{v}) = P(v)$. Since $P(v) = \kappa(|v|)v$ with $\kappa(|v|) \geq 0$ and $|P(v)| = \kappa(|v|)|v|$ is a monotone increasing function of $|v| \in [0,1[$, as proved in Appendix 1, we conclude that $v = \tilde{v}$. □

Remark Proposition 3.2 is not really needed for the proof of Theorem 1.2. However it shows directly that (ϕ_v,π_v) is a critical point, using (19) and (17), and suggests an investigation of the stability through a lower bound as in (20). In Appendix 2 we sketch the derivation of Proposition 3.2 for sufficiently smooth solutions based only on the invariance of the symplectic structure. We expect a similar proposition to hold for other translation invariant systems similar to (1).

4 Orbital stability of solitons

We follow [1] and deduce orbital stability from the conservation of H_P together with its lower bound (20).

Proposition 4.1 *Let us fix some $v \in V$ and $q \in \mathbb{R}^3$. Let $y(t) = (\phi(t), q(t), \pi(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ be a solution to the system (1) with initial state $y(0) = y^0 = (\phi^0, q^0, \pi^0, p^0) \in \mathcal{E}$ and denote*

$$\delta = \|\phi^0(x) - \phi_v(x-q)\| + |\pi^0(x) - \pi_v(x-q)| + |p^0 - p_v| \ . \tag{23}$$

Then for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\|\phi(q(t)+x,t) - \phi_v(x)\| + |\pi(q(t)+x,t) - \pi_v(x)| + |p(t) - p_v| \leq \varepsilon \quad for \quad all \quad t \in \mathbb{R} \tag{24}$$

provided $\delta \leq \delta(\varepsilon)$.

Proof We denote by P^0 the total momentum of the considered solution $y(t)$. There exists a soliton-like solution (8) corresponding to some $\tilde{v} \in V$ and having the same total momentum $P(\tilde{v}) = P^0$. Then (23) implies $|P^0 - P(v)| = |P(\tilde{v}) - P(v)| = \mathcal{O}(\delta)$, hence also $|\tilde{v} - v| = \mathcal{O}(\delta)$ and

$$\|\phi^0(x) - \phi_{\tilde{v}}(x-q)\| + |\pi^0(x) - \pi_{\tilde{v}}(x-q)| + |p^0 - p_{\tilde{v}}| = \mathcal{O}(\delta) \ .$$

Therefore denoting $(\Phi^0, Q^0, \Pi^0, P^0) = Ty^0$ we have

$$H_{P(\tilde{v})}(\Phi^0 \ , \ \Pi^0) - H_{P(\tilde{v})}(\phi_{\tilde{v}} \ , \ p_{\tilde{v}}) = \mathcal{O}(\delta) \ . \tag{25}$$

Total momentum and energy conservation (14) imply for $(\Phi(t), Q(t), \Pi(t), P^0) = Ty(t)$

$$H_{P(\tilde{v})}(\Phi(t), \ \Pi(t)) = H(Ty(t)) = H_{P(\tilde{v})}(\Phi^0 \ , \ \Pi^0) \quad \text{for} \quad t \in \mathbb{R} \ .$$

Hence (25) and (20) with $\tilde{v}$ instead of v imply

$$\|\Phi(t) - \phi_{\tilde{v}}\| + |\Pi(t) - \pi_{\tilde{v}}| = \mathcal{O}(\delta) \tag{26}$$

uniformly in $t \in \mathbb{R}$. On the other hand, total momentum conservation implies

$$p(t) = P(\tilde{v}) + < \Pi(t), \nabla\Phi(t) > \quad \text{for} \quad t \in \mathbb{R} \ .$$

Therefore (26) leads to

$$|p(t) - p_{\tilde{v}}| = \mathcal{O}(\delta) \tag{27}$$

uniformly in $t \in \mathbb{R}$. Finally (26), (27) together imply (24) because $|\tilde{v} - v| = \mathcal{O}(\delta)$. $\square$

5 Soliton-like asymptotics

We combine orbital stability and relaxation of the acceleration to prove Theorem 1.2.

Proposition 5.1 *Let the assumptions of Theorem 1.2 be fulfilled. Then for every $\delta > 0$ there exist a $t_* = t_*(\delta)$ and a solution $y_*(t) = (\phi_*(x,t), q_*(t), \pi_*(x,t), p_*(t)) \in C([t_*, \infty), \mathcal{E})$ to the system (1) such that*

i) $y_*(t)$ coincides with $y(t)$ in some future cone,

$$q_*(t) = \quad q(t) \qquad for \quad t \geq t_* \ , \tag{28}$$
$$\phi_*(x,t) = \quad \phi(x,t) \qquad for \quad |x - q(t_*)| < t - t_* \ . \tag{29}$$

ii) $y_*(t_*)$ is close to $y_{v,q}(t_*)$ with some $v = v(\delta) \in V$ and $q = q(\delta) \in \mathbb{R}^3$,

$$\|y_*(t_*) - y_{v,q}(t_*)\|_{\mathcal{E}} \leq \delta \ . \tag{30}$$

This proposition leads to the

Proof of Theorem 1.2 Proposition 4.1 and (30) imply that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|\phi_*(q_*(t) + x, t) - \phi_v(x)\| + |\!|\pi_*(q_*(t) + x, t) - \pi_v(x)|\!| + |\dot{q}_*(t) - v| \leq \varepsilon \ \text{ for } \ t > t_* \ .$$

Therefore, using (28) and (29), for every $R > 0$

$$\begin{aligned} &\|\phi(q(t) + x, \ t) - \phi_v(x)\|_R \ + \ |\!|\pi(q(t) + x, \ t) - \pi_v(x)|\!|_R \ + \ |\dot{q}(t) - v| \\ = \ &\|\phi_*(q_*(t) + x, t) - \phi_v(x)\|_R + \ |\!|\pi_*(q_*(t) + x, t) - \pi_v(x)|\!|_R + \ |\dot{q}_*(t) - v| \ \ \leq \varepsilon, \ t > t_* + R. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude (10). □

Proof of Proposition 5.1 The Kirchhoff formula asserts that

$$\phi(x,t) = \phi_r(x,t) + \phi_0(x,t) \ \text{ for } \ |x - q^0| < t - R_\rho \ , \tag{31}$$

where

$$\phi_r(x,t) \ = \ -\int \frac{d^3y}{4\pi|x-y|}\rho(y - q(t - |x-y|)) \ , \tag{32}$$
$$\phi_0(x,t) \ = \ \frac{1}{4\pi t}\int_{S_t(x)} d^2y \ \pi^0(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S_t(x)} d^2y \ \phi^0(y)\right) . \tag{33}$$

By (15) for every $\varepsilon > 0$ there exists t_ε such that

$$|\ddot{q}(t)| \leq \varepsilon \quad \text{for} \quad t \geq t_\varepsilon \quad \text{and} \quad t_\varepsilon \to \infty \quad \text{as} \quad \varepsilon \to 0 \ . \tag{34}$$

Let us define

$$t_{0,\varepsilon} = t_\varepsilon + R_\rho \ \ , \qquad t_{1,\varepsilon} = t_{0,\varepsilon} + R_\rho \ \ , \qquad t_{2,\varepsilon} = t_{1,\varepsilon} + 2R_\rho/(1 - q_1) \ .$$

Then we modify $q(t)$ by

$$q_\varepsilon(t) = \left\{ \begin{array}{lll} q(t) & \text{for} & t \geq t_{0,\varepsilon} \ , \\ q_\varepsilon + v_\varepsilon(t - t_{0,\varepsilon}) & \text{for} & t \leq t_{0,\varepsilon} \ , \end{array} \right. \tag{35}$$

where $q_\varepsilon = q(t_{0,\varepsilon})$ and $v_\varepsilon = \dot{q}(t_{0,\varepsilon})$. Then $q_\varepsilon(t) \in C^1(\mathbb{R})$ and (34) implies that

$$|\ddot{q}_\varepsilon(t)| \leq \varepsilon \quad \text{for} \ \ \text{all} \ \ t \in \mathbb{R}. \tag{36}$$

Let us modify the initial values $\phi^0(x) \in D^{1,2}$, $\pi^0(x) \in L^2$ by cutting off a large ball with the center at the point q_ε .

Lemma 5.2 *For every $\varepsilon > 0$ there exist $\phi^0_\varepsilon(x) \in D^{1,2}$, $\pi^0_\varepsilon(x) \in L^2$ such that*

$$\phi^0_\varepsilon(x) = \begin{cases} \phi^0(x), \\ 0 \ , \end{cases} \quad \pi^0_\varepsilon(x) = \begin{cases} \pi^0(x) & \text{for} \quad |x - q_\varepsilon| > t_\varepsilon \ , \\ 0 & \text{for} \quad |x - q_\varepsilon| < t_\varepsilon - 1, \end{cases} \tag{37}$$

and moreover,

$$\|\phi^0_\varepsilon\| + |\pi^0_\varepsilon| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{38}$$

Proof Since $\phi^0 \in D^{1,2}$ and $\pi^0 \in L^2$ we have

$$\int_{|x-q_\varepsilon|>t_\varepsilon-1} d^3x \ (|\nabla\phi^0(x)|^2 + |\pi^0(x)|^2) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

because of $t_\varepsilon \to \infty$ and $|q_\varepsilon| = \mathcal{O}(q_1 \cdot t_\varepsilon)$ where $0 < q_1 < 1$. □

Further, let us define the corresponding modification $\phi_\varepsilon(x,t)$ for the solution (31),

$$\phi_\varepsilon(x,t) = \phi_{r,\varepsilon}(x,t) + \phi_{0,\varepsilon}(x,t) \quad \text{for} \quad x \in \mathbb{R}^3 \quad \text{and} \quad t > 0, \tag{39}$$

where

$$\phi_{r,\varepsilon}(x,t) = -\int \frac{d^3y}{4\pi|x-y|}\rho(y - q_\varepsilon(t - |x-y|)) \ , \tag{40}$$

$$\phi_{0,\varepsilon}(x,t) = \frac{1}{4\pi t}\int_{S_t(x)} d^2y \ \pi^0_\varepsilon(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S_t(x)} d^2y \ \phi^0_\varepsilon(y)\right). \tag{41}$$

Then $\phi_\varepsilon(x,t)$ is a solution to the wave equation

$$\ddot{\phi}_\varepsilon(x,t) = \Delta\phi_\varepsilon(x,t) - \rho(x - q_\varepsilon(t)) \quad \text{for} \quad t > 0 \ . \tag{42}$$

By (40), (32), and (35) we have

$$\phi_{r,\varepsilon}(x,t) = \phi_r(x,t) \qquad \text{for} \quad |x - q_\varepsilon| < t - t_{1,\varepsilon} \ , \tag{43}$$

$$\phi_{r,\varepsilon}(x,t) = \phi_{v_\varepsilon}(x - v_\varepsilon t - q_\varepsilon) \quad \text{for} \quad |x - q_\varepsilon| > t - t_\varepsilon \ . \tag{44}$$

Now, let us fix $T > 0$. Then $|\dot{q}_\varepsilon(t) - v_\varepsilon| = \mathcal{O}(\varepsilon)$ uniformly in $t_\varepsilon \le t \le t_\varepsilon + T$ due to (36). Therefore (40) and (44) imply

$$\sup_{x\in\mathbb{R}^3, \ 0<t-t_\varepsilon<T}\Big(|\dot{\phi}_{r,\varepsilon} - \dot{\phi}_{v_\varepsilon,q_\varepsilon}| + |\nabla\phi_{r,\varepsilon} - \nabla\phi_{v_\varepsilon,q_\varepsilon}| + |\phi_{r,\varepsilon} - \phi_{v_\varepsilon,q_\varepsilon}|\Big) = \mathcal{O}(\varepsilon) \ , \tag{45}$$

where we denote $\phi_{v,q}(x,t) = \phi_v(x - vt - q)$. On the other hand (37) and (33), (41) imply

$$\phi_{0,\varepsilon}(x,t) = \phi_0(x,t) \quad \text{for} \quad |x - q_\varepsilon| < t - t_\varepsilon \ . \tag{46}$$

In addition, by conservation of energy for the solution $\phi_{0,\varepsilon}(x,t)$ of the free wave equation and by (38) we have

$$\sup_{t>0}\Big(\|\phi_{0,\varepsilon}(\cdot,t)\| + |\dot{\phi}_{0,\varepsilon}(\cdot,t)|\Big) \to 0 \quad \text{as} \quad \varepsilon \to 0 \ . \tag{47}$$

Finally, we define

$$y_*(t) = (\phi_\varepsilon(\cdot,t), q(t), \dot{\phi}_\varepsilon(\cdot,t), p(t)) \quad \text{for} \quad t > t_* = t_{2,\varepsilon} \ . \tag{48}$$

It is easy to check that the time t_* and the function $y_*(t)$ for $t \geq t_*$ satisfy all requirements of Proposition 5.1 with $v(\delta) = v_\varepsilon$ and $q(\delta) = q_\varepsilon$ provided one chooses $\varepsilon > 0$ sufficiently small. Firstly, $y_*(t) \in C([t_*, \infty), \mathcal{E})$ is a solution to the system (1) for $t > t_*$. Indeed, (43), (46) and (31), (39) imply for large enough $t_{1,\varepsilon}$

$$\phi_\varepsilon(x,t) = \phi(x,t) \quad \text{for} \quad |x - q_\varepsilon| < t - t_{1,\varepsilon} \ . \tag{49}$$

Therefore (35) implies that $y_*(t)$ together with $y(t)$ is a solution to the system (1) in the region $|x - q_\varepsilon| < t - t_{1,\varepsilon}$. On the other hand for $|x - q_\varepsilon| > t - t_{1,\varepsilon}$ and $t > t_{2,\varepsilon}$ we have $\rho(x - q(t)) = 0$ and $q_\varepsilon(t) = q(t)$, hence $y_*(t)$ is a solution to the system (1) also in this region due to (42). Secondly, (29) and (28) follow from (49) and (35), and (30) follows from (48) and (39) due to (45) and (47). □

6 Appendix 1. Soliton-like solutions

1. *For every $v \in V$ the function ϕ_v in the soliton-like solution (3), (8) is given by Eq. (4).*
Proof The system (22) for a soliton-like solution reads $(v \cdot \nabla)^2 \phi_v(x) = \Delta\phi_v(x) + \rho(x)$, which through Fourier transform becomes

$$\hat{\phi}_v(k) = \hat{\rho}(k)/(k^2 - (v \cdot k)^2) \ . \tag{50}$$

□

2. *For the total momentum $P(v)$ of the soliton-like solution (8) we have $P(v) = \kappa(|v|)v$ with $\kappa(|v|) \geq 0$ and $|P(v)| = \kappa(|v|)|v|$ is a monotone increasing function of $|v| \in [0, 1[$.*
Proof Parseval identity and (50) imply

$$\begin{aligned} P(v) &= p_v + \int d^3x \ v \cdot \nabla\phi_v(x) \ \nabla\phi_v(x) \\ &= \frac{v}{\sqrt{1-v^2}} + (2\pi)^{-3} \int d^3k \frac{(v \cdot k)\hat{\rho}(k)\overline{k\hat{\rho}(k)}}{(k^2 - (v \cdot k)^2)^2} \ . \end{aligned}$$

Hence $P(v) = \kappa(|v|)v$ with $\kappa(|v|) \geq 0$ and for $v \neq 0$

$$|P(v)| = \frac{|v|}{\sqrt{1-v^2}} + \frac{1}{(2\pi)^3|v|} \int d^3k \frac{|(v \cdot k)\hat{\rho}(k)|^2}{(k^2 - (v \cdot k)^2)^2} \ .$$

□

7 Appendix 2. Invariance of symplectic structure

The canonical equivalence of the Hamiltonian systems (1) and (18) can be seen from the Lagrangian viewpoint. We remain at the formal level. For a complete mathematical justification we would have to develop some theory of infinite dimensional Hamiltonian systems which is beyond the scope of this paper.

By definition we have $H(\Phi, Q, \Pi, P) = h(\phi, q, \pi, p)$ with the arguments related through the canonical transformation T. To each Hamiltonian we associate a Lagrangian through the

Legendre transformation

$$
\begin{aligned}
l(\phi,q,\dot{\phi},\dot{q}) = & <\pi,\dot{\phi}> + p\cdot\dot{q} - h(\phi,q,\pi,p) \ , & \dot{\phi} = \frac{\delta h}{\delta\pi} \ , \quad \dot{q} = \frac{\partial h}{\partial p} \ , \\
L(\Phi,Q,\dot{\Phi},\dot{Q}) = & <\Pi,\dot{\Phi}> + P\cdot\dot{Q} - H(\Phi,Q,\Pi,P) \ , & \dot{\Phi} = \frac{\delta H}{\delta\Pi} \ , \quad \dot{Q} = \frac{\partial H}{\partial P} \ .
\end{aligned}
$$

These Legendre transforms are well defined because the Hamiltonian functionals are convex in the momenta. We claim the identity $L(\Phi,Q,\dot{\Phi},\dot{Q}) = l(\phi,q,\dot{\phi},\dot{q})$. Clearly we have to check the invariance of the canonical 1-form,

$$
<\Pi,\dot{\Phi}> + P\cdot\dot{Q} = <\pi,\dot{\phi}> + p\cdot\dot{q} \ . \tag{51}
$$

For this purpose we substitute

$$
\begin{array}{llll}
\Pi(x) & = \pi(q+x) \ , & \dot{\Phi}(x) & = \dot{\phi}(q+x) + \dot{q}\cdot\nabla\phi(q+x) \ , \\
P & = p - \int d^3x \ \dot{\phi}\cdot\nabla\phi \ , & \dot{Q} & = \dot{q} \ .
\end{array}
$$

The left hand side of (51) becomes then

$$
<\pi(q+x),\dot{\phi}(q+x) + \dot{q}\cdot\nabla\phi(q+x)> + (p - <\pi(x),\nabla\phi(x)>)\cdot\dot{q} = <\pi,\dot{\phi}> + p\cdot\dot{q} \ .
$$

Since $l = L$, the corresponding action functionals are identical when transformed by T. The dynamical trajectories are stationary points of the respective action functionals. Therefore the two Hamiltonian systems are equivalent.

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