Equivalence of dynamical ensembles and Navier Stokes equations*

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Abstract: a reversible version of the Navier Stokes equation is studied. A conjecture emerges stating the equivalence between the reversible equation and the usual Navier Stokes equation. The latter appears as a statement of ensembles equivalence in the limit of infinite Reynolds number, which plays the role of the thermodynamic limit.

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In reference [GC1] a scheme for applying the chaoticity principle, also introduced there, to fluid motions was discussed, and in [G5] some other simple consequences were derived.

We begin by remarking that the usual derivation of the NS equations assumes on an empirical basis that there is a viscous force between fluid layers flowing at different speeds, and transforming kinetic energy into heat: it is desirable that such an assumption be justified microscopically, [Sp]. On the other hand it is difficult to see how a microscopically reversible equation of motion, non trivially driven by a non conservative force, could possibly lead to macroscopically irreversible equations of the NS type, with energy converted into heat in a stationary fashion. Unless a dissipation mechanism is *a priori* assumed at a microscopic level.

One could be tempted to say that irreversible macroscopic motion arises because of a mechanism like the one discussed by Grad and Lanford in their derivation of the Boltzmann equation. Note however that, although the Boltzmann equation is certainly irreversible, it seems that it cannot be used to derive the NS equations in a situation in which there are non conservative external forces acting on the system, at least not without extra assumptions besides the ones one is willing to assume in deriving the Boltzmann equation itself in a system driven to equilibrium.[1]

The physical problem that is stressed here, among many, is that the work

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[1] Note also that such a derivation will necessarily involve proving an existence theorem for the Boltzmann equation of a confined system for a very long (in fact I believe infinite) time and this is out of reach of the presently known techniques, [La].

performed by the forces on the system will not be dissipated and the system *will heat up* indefinitely, without reaching a stationary state, hence without reaching a state described by a stationary probability distribution for the NS equation.

Starting (to fix the ideas) from the reversible microscopic dynamics, the kind of "extra assumptions" consist essentially in supposing suitably large ratios between the microscopic time and length scales and the macroscopic ones: in order to make effectively infinite the time scale over which heat is created. If the scale ratios are very large and their relative values also suitably adjusted then the motions may be "well approximated" by a dissipative equation and reach a stationary state (very slowly drifting away from itself). *Strictly speaking the equations will still be reversible* (involving perhaps also other fields like the temperature field) and will be governed by approximately defined transport coefficients equal in the average (over time and space) to the observed ones.

The estimate of the size of the various scale ratios for this picture to be of practical interest, and the corresponding evaluation of the errors, is a difficult problem that, as far as I know, has not only no clear solution but the methods for its solution do not seem to exist yet, [Sp].

The above remarks also show that the macroscopic dissipative equations may perhaps be equivalent to reversible equations: the aim of this paper is to argue about the possibility of notions like "non equilibrium ensembles" and of interpreting the equivalence statement as a statement about ensembles equivalence.

For the above reasons it will be useful to imagine an idealized dissipation mechanism. Here we shall consider a dissipation due to the interaction of the system with a "thermostat" which will be modeled by a force acting on the system: the function of such forces will be that of *imposing* a constant energy dissipation rate. The force will be determined by Gauss' least constraint principle, [LA], [W].[2]

It is easy to derive the equations of a fluid in such a situation. We suppose that:

(1) the fluid is in a periodic box Ω with side L,

(2) that it is incompressible with density ρ,

(3) that the energy dissipation is:

$$\varepsilon=\eta\frac{1}{2}\int\sum_{ij}(\partial_i u_j+\partial_j u_i)^2\,d\,\underline{x}\,=\eta\int\underline{\omega}^{\,2}\,d\,\underline{x} \tag{1.1}$$

where $\underline{\omega}=\underline{\partial}\wedge\underline{u}$ is the vorticity field and η is a parameter equal to the *phenomenological* (*i.e.* experimentally determined) dynamical viscosity: so that $\nu=\eta\rho^{-1}$ is the viscosity.

Imposing that the fluid is not viscous but subject to the constraint that (1.1) is a constant of motion, implies that the Euler equations for the velocity and pressure fields $\underline{u}\,,p$ are modified into:

[2] This type of thermostatting force is essentially the well known *Nosé–Hoover* thermostat and it has been studied in particular in the fundamental paper [ECM2] which led to the formulation of the chaotic hypothesis as a reinterpretation of the earlier Ruelle's principle, [R2], [G2], [GC2].

$$\dot{\underline{u}} + \underset{\sim}{u} \cdot \underset{\sim}{\partial}\, \underline{u} = -\frac{1}{\rho}\partial p + \underline{g} + \alpha \Delta \underline{u}, \qquad \underline{\partial} \cdot \underline{u} = 0 \tag{1.2}$$

where Δ is the Laplace operator and α is *not* the viscosity but rather it is the multiplier necessary to impose the constraint that (1.1) is an integral of motion. A simple computation shows that:

$$\alpha(\underline{u}) = \frac{\int \left(\underline{\partial} \wedge \underline{g} \cdot \underline{\omega} + \underline{\omega} \cdot (\underset{\sim}{\omega} \cdot \underset{\sim}{\partial}\, \underline{u}) \right) d\underline{x}}{\int \underline{\omega}^2\, d\underline{x}} \tag{1.3}$$

This has odd symmetry in $\underline{u}$, so that (1.2) is *reversible*: if V_t is the flow describing the equation solution (so that $t \to V_t \underline{u} = \underline{u}(t)$ is the solution with initial data $\underline{u}$) then the transformation $i : \underline{u} \to -\underline{u}$ *anticommutes* with the time evolution V_t:

$$i\, V_t = V_{-t}\, i \tag{1.4}$$

The *reversible* equation (1.2), (1.3) will be called the Gaussian Navier Stokes equation, or GNS.

A similar equation, with a constraint on the energy contained in each "momentum shell" to be constant was considered in [ZJ], which is the first paper in which the idea of a reversible Navier Stokes equation is advanced and studied. The energy content of each "momentum shell" was fixed to be the value predicted by Kolmogorov theory, [LL].

Existence of global solutions to the equations (1.2),(1.3) is non trivial even if the initial datum $\underline{u}_0$ is C^∞. In view of the conjecture that follows this might be a reassuring tribute to the principle of conservation of difficulties.

In fact let the velocity field $\underline{u} = \sum_{\underline{k} \neq \underline{0}} \underline{\gamma}_{\underline{k}} e^{i \underline{k} \cdot \underline{x}}$ be represented in Fourier series with $\underline{\gamma}_{\underline{k}} = \overline{\underline{\gamma}}_{-\underline{k}}$ and $\underline{k} \cdot \underline{\gamma}_{\underline{k}} = 0$ (incompressibility condition); here $\underline{k}$ has components that are integer multiples of the "lowest momentum" $k_0 = \frac{2\pi}{L}$. Then consider the equation:

$$\begin{aligned}
&\dot{\underline{\gamma}}_{\underline{k}} = -\alpha \underline{k}^2 \underline{\gamma}_{\underline{k}} - i \sum_{\underline{k}_1 + \underline{k}_2 = \underline{k}} (\underline{\gamma}_{\underline{k}_1} \cdot \underline{k}_2)\, \Pi_{\underline{k}}\, \underline{\gamma}_{\underline{k}_2} + \underline{g}_{\underline{k}} \\
&\alpha = \alpha_i + \alpha_e, \qquad \alpha_e = \frac{\sum_{\underline{k} \neq \underline{0}} \underline{k}^2 \underline{g}_{\underline{k}} \cdot \overline{\underline{\gamma}}_{\underline{k}}}{\sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}_{\underline{k}}|^2} \\
&\alpha_i = \frac{-i \sum_{\underline{k}_1 + \underline{k}_2 + \underline{k}_3 = \underline{0}} \underline{k}_3^2 (\underline{\gamma}_{\underline{k}_1} \cdot \underline{k}_2)(\underline{\gamma}_{\underline{k}_2} \cdot \underline{\gamma}_{\underline{k}_3})}{\sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}_{\underline{k}}|^2}
\end{aligned} \tag{1.5}$$

where the $\underline{k}$'s take only the values $0 < |\underline{k}| < K$ for some *momentum cut–off* $K > 0$ and $\Pi_{\underline{k}}$ is the orthogonal projection on the plane orthogonal to $\underline{k}$. This is an equation that defines a "truncation on the momentum sphere K of the

GNS equations". The actual GNS equations (1.2), (1.3) have the same form with $K=+\infty$.

For simplicity we suppose that the mode $\underline{k}=\underline{0}$ is *absent*, *i.e.* $\underline{\gamma}_{\underline{0}}=\underline{0}$: this can be done if, as we suppose, the external force $\underline{g}$ does not have a zero mode component (*i.e.* if it has zero average).

Calling $V_t^K\,\underline{u}$ the solution to the equation (1.5) with initial datum $\underline{u}\in C^\infty$, and with components with mode $\underline{k}$ greater than K set equal to zero, it should follow that $V_t^K\,\underline{u}\xrightarrow[K\to\infty]{}V_t\,\underline{u}$ uniformly in $\underline{x}$ and in t for t in any bounded interval. This is however *not* so easy and in fact I do not know whether (1.5) admits a global solution that can be constructed in this way. [3]

For the purpose of comparison with the NS equation we shall therefore use the equations (1.5) and compare them with the corresponding truncation of the NS equation (with *constant* viscosity ν) subject to the non conservative field $\underline{g}$:

$$\dot{\underline{u}}_{\underline{k}}=-\nu\,\underline{k}^2\,\underline{\gamma}_{\underline{k}}-i\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}\underline{\gamma}_{\underline{k}_1}\cdot\underline{k}_2\,\Pi_{\underline{k}}\,\underline{\gamma}_{\underline{k}_2}+\underline{g}_{\underline{k}} \tag{1.6}$$

with the same K. We shall take K very large, fixing it once and for all and dropping it from the places where it would belong as a label (to simplify the notations).

In order that the resulting cut–off equations be physically acceptable, and supposing that $\underline{g}_{\underline{k}}\ne\underline{0}$ only for $|\underline{k}|\sim k_0$, one shall have to fix K much larger than the *Kolmogorov scale* $K=(\varepsilon_k\nu^{-1})^{1/4}$, where ε_k is the average dissipation of the solutions to (1.6) with $K=+\infty$, determined on the basis of heuristic dimensional considerations by $\varepsilon_k\sim\sqrt{|\underline{g}|^3L}$: see [LL].

We shall also call the truncated equations (1.5),(1.6) respectively GNS and NS equations omitting the "truncated": below we always refer to the truncated equations, unless otherwise stated.

The solutions of the equations (1.6) with initial datum $\underline{u}$ will be denoted $V_t^{ns}\,\underline{u}$. And we shall admit that there is a unique stationary distribution μ_{ns} describing the statistics of all initial data $\underline{u}$ that are randomly chosen with a "Liouville distribution", *i.e.* with a distribution $\mu_0(d\underline{\gamma})$ proportional to the volume measure $\prod_{|\underline{k}|<K}d\underline{\gamma}_{\underline{k}}$.

This means that given "any observable" F on the phase space $\mathcal{F}$ (of the velocity fields with momentum cut–off K) it is:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(V_t^{ns}\,\underline{\gamma})dt=\int_{\mathcal{F}}F(\underline{\gamma}')\mu_{ns}(d\underline{\gamma}') \tag{1.7}$$

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[3] Examining the theory of Leray, [L], for the NS equations one realizes that the "only" difficulty there was the absence of a uniform *a priori estimate* on the total vorticity $\int\underline{\omega}^2\,d\underline{x}$: in the equation (1.2),(1.3) the uniform boundedness of the total vorticity is simply automatic (as it is a constant of motion). But the theory of Leray also made essential use of the constancy of the viscosity coefficient, or more precisely of its positivity and boundedness away from 0, see [G1]. The latter property is false for the coefficient α which, by reversibility, takes values of either sign (although, see below, we expect that it is "more often" positive than negative on "typical" motions).

for all choices of $\underline{\gamma}$ except a set of zero Liouville measure. The distribution μ_{ns} will be called the SRB distribution for the Eq. (1.6) (see the "zero-th law" in [UF], [GC2]).

We make the corresponding assumptions for the solutions of the cut–off GNS equations (1.5) (with the same cut–off) and the relative flow V_t^{gns}; we call μ_{gns} the associated SRB distribution, [R2], [ER].

In particular we shall study the observable $\sigma(\underline{\gamma})$ that we call the *entropy production rate* and that is given by the divergence of the r.h.s. of Eq. (1.5).

If D_K is the number of modes $\underline{k}$ with $0 < |\underline{k}| < K$ then the number of (independent) components of $\{\underline{\gamma}_{\underline{k}}\}$ is $2D_K$ and, see (1.5), setting $\overline{D}_K = \sum_{|\underline{k}|<K} 2\,\underline{k}^2$, one finds:

$$\sigma = 2\overline{D}_K\,(\alpha_i + \alpha_e) + \overline{\alpha}_i - \overline{\alpha}_e = 2\overline{D}_K\alpha + \overline{\alpha}_i - \overline{\alpha}_e \tag{1.8}$$

where $\overline{\alpha}_i, \overline{\alpha}_e$ are suitably defined: *e.g.*

$$\overline{\alpha}_e = \frac{\sum_{\underline{k}} \underline{k}^4 \overline{\underline{g}}_{\underline{k}} \cdot \underline{\gamma}_{\underline{k}}}{\sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}_{\underline{k}}|^2} - 2\frac{\big(\sum_{\underline{k}} \underline{k}^2 \overline{\underline{g}}_{\underline{k}} \cdot \underline{\gamma}_{\underline{k}}\big)\big(\sum_{\underline{k}} \underline{k}^4 \underline{\gamma}_{\underline{k}}^2\big)}{\big(\sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}_{\underline{k}}|^2\big)^2} \tag{1.9}$$

so that $\sigma \simeq 2\overline{D}_K\alpha$.

We call $\sigma_{ns}, \varepsilon_{ns}$ the time averages of σ and of $\rho\nu L^3 \sum_{\underline{k}} |\underline{k}|^2 |\underline{\gamma}_{\underline{k}}|^2$ (total vorticity, see (1.1)) with respect to the distribution μ_{ns}. As mentioned we expect, on the basis of dimensional analysis, that $\varepsilon_{ns} \sim \varepsilon_k = \sqrt{|\underline{g}|^3 L}$, see [LL]).

Correpondingly we consider the solutions of the GNS equations with total vorticity $\varepsilon_{gns} \equiv \varepsilon = \varepsilon_{ns}$ and call σ_{gns} the average of σ with respect to μ_{gns}.

The H–theorem of Ruelle, [R3], tells us that $\sigma_{gns} \ge 0$, and $\sigma_{gns} > 0$ if the distribution μ_{gns} is not proportional to the volume measure. Here we shall suppose that this is the case if $\underline{g} \ne \underline{0}$: at least if the force field is such that it generates a chaotic motion (*i.e.* a motion with at least one positive Lyapunov exponent) for all but a set of zero volume initial data. In this situation:

Conjecture ("equivalence of dynamical ensembles"): Given $\varepsilon \equiv \varepsilon_{gns}$ suppose that the parameter ν in the NS equation is adjusted so that $\varepsilon_{ns} = \varepsilon_{gns}$ then the averages $\sigma_{gns}, \sigma_{ns}$ of σ with respect to μ_{gns} and with respect to μ_{ns} and, in fact, the averages of all (reasonable) observables coincide, up to corrections $O(R^{-1})$, $R = \varepsilon^{1/3} L^{4/3} \nu^{-1}$.

We recognize in this conjecture a statement very analogous to the familiar statements on the equivalence of thermodynamic ensembles, with the thermodynamic limit replaced by the limit $R \to \infty$ of infinite Reynolds number. The paper [ZJ] contains various statements that can be seen as steps towards the formalization given above of the general conjecture.

It can be weakened very much for the purposes of possible applications: we state it in the above very strong form to stress an extreme thought. Applications will be envisaged elswhere.

On heuristic grounds, it would be justified if one did accept that the entropy creation rate reaches its average on a time scale that is fast compared to the hydrodynamical scales. The coefficient $\alpha \simeq (2\overline{D}_K)^{-1}\sigma$, see (1.8), would be confused with its average $\langle\alpha\rangle_{gns}$ and *identified with the viscosity* constant ν.

In this way the GNS and the NS equations would be equivalently good: both being the macroscopic trace of two *equivalent microscopic dissipation mechanisms*: one explicitly specified by the gaussian constraint of constant dissipation and the other *unspecified but phenomenologically modeled* by a constant viscosity.

The interest of the conjecture is that it allows us to deduce properties of the "usual" NS equation from properties of the GNS equation: the latter, being reversible, can be studied via the *chaotic hypothesis* of [GC1], [GC2] which leads to non trivial predictions like the fluctuation theorem of [GC1].

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