Escape Orbits for Non-Compact Flat Billiards

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Abstract

It is proven that, under some conditions on f, the non-compact flat billiard $\Omega = \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+;\ 0 \leq y \leq f(x)\}$ has no orbits going *directly* to $+\infty$. The relevance of such sufficient conditions is discussed.

1 Introduction

Let f be a smooth function from $\mathbb{R}_0^+$ to $\mathbb{R}^+$, bounded, vanishing when $x \to +\infty$. No integrability assumptions are given. Now construct a plane billiard table in the following way: $\Omega = \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+;\ 0 \leq y \leq f(x)\}$ as is shown in figure 1. Imagine to have a dimensionless particle moving freely within Ω and reflecting elastically at its boundary.

Some interest seems to be focused recently on these (see [1, 5, 8, 9]) and other billiard systems over non-compact manifolds (intriguing and related examples include [3, 4, 7]). In this contest an issue regards whether and how

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a trajectory may leave any compact region of the configuration space. The study of this is of theoretical interest in itself, as well serving as a possible starting point for the investigation of the ergodic properties of non-compact dynamical systems of which not much seems to be known. Plus, and this is the main motivation for recent work, it has a direct link with the search for *quantum chaos* in the quantized version of these systems. Specifically, unbounded orbits represent the other side of the coin compared to periodic orbits. The distribution of the latter provides information on the spectrum of the Laplace-Beltrami (i.e. Hamiltonian) operator on the given manifold via the celebrated Gutzwiller trace formula or similar asymptotic expansions [2, 6]. Being able to "count" periodic orbits means gaining information on the aysmptotics of the quantum energy levels. This has suggested the idea of *coding* periodic orbits, i.e. associating an infinite string of symbols to each periodic orbit, possibly in a bijective way. [1] Understandably, unbounded trajectories represent a nuisance for such a coding. Finally, it is known that the heat kernel expansion is sensitive to finite or infinite cusps [6]. One would like to relate this fact to the presence of orbits eventually falling into the cusp.

The question we ask ourselves here is the following: are there any orbits of our billiard in Ω which go *directly* to infinity, i.e. maintaining for all times a positive x-component of their velocity? To fix the notation let us call such trajectories *escape orbits*. Of course there is always one of these: it is the orbit which lies on the horizontal semi-axis. We call it the *trivial orbit*.

We are going to show that no other orbits can share this property, provided we require some conditions on f. The question was first touched on by A.M.Leontovich [2] in 1962 ([1], Theorem 2) where – even though he was searching for oscillating unbounded orbits – he obtained the above result for *eventually convex* f's, i.e. convex in a neighborhood of ∞. This is the same result we find much more recently in J.L.King's review [8]. It can be explained easily, at least for billiard tables of finite area. In that case the cusp has asymptotically a vanishing measure. Now, if we have an escape orbit, then, due to the hyperbolicity of the flow in that region of the phase space, we can find a non-zero measure set of escape orbits, that is which go into the cusp, with an obvious contradiction. [3] In fact we can always fix, as the

[1] There are quite a few works on that. [4] has good analogies with the present paper.

[2] To my knowledge.

[3] This argument is described in [8] as "A gallon of water won't fit inside a pint-sized cusp".

initial point of our escape orbit, a point $(x_0, y_0) = (x_0, f(x_0))$ on the upper boundary of Ω, far enough to lie in the region where f in convex. The initial velocity will have an x-component $v_x > 0$. Now it is easily seen that every other set of initial conditions $x_0' \geq x_0$ and $v_x' \geq v_x$ (provided v and v' have the same modulus) would lead to a new escape orbit, due to the dispersing effect of the convex upper wall. The same argument may be used to deduce that an infinite cusp on the Poincaré's disc does not allow non-trivial trajectories to collapse into it, which is implicitly stated in [4].

We are going to relax the hypothesis for the non-existence result to hold: asymptotic hyperbolicity is not a necessary condition at all. We may allow f to have flex points and abrupt negative variations (see, for instance, figure 2c). The proofs are presented in the next section, while examples of non-convex f's fulfilling our hypotheses are discussed in the last section in order to understand what the new assumptions actually mean and how far they can be pushed.

2 The result

Theorem 1 *Consider the plane billiard table generated as above by the function f defined on $\mathbb{R}_0^+$), twice differentiable, positive, bounded, such that*

$$f(x) \searrow 0 \text{ as } x \to +\infty. \tag{H1}$$

Also, for x sufficiently large,

$$f'(x) < 0, \tag{H2}$$

Then, under either one of these assumptions:

$$\limsup \frac{f'(x)}{f(x)} < 0; \tag{A1}$$

or

$$\lim_{x \to +\infty} f'(x) = 0 \text{ and } \limsup_{x \to +\infty} \frac{f'' f}{f'}(x) < +\infty; \tag{A2}$$

no orbits but the trivial one are escape orbits.

REMARK. The assumption about the convexity of f is contained in (A2): in fact if $f'' \geq 0$, then necessarily $f' \nearrow 0$ and $f''f/f' \leq 0$.

PROOF. Suppose, contrary to our goal, we have a non-trivial escape orbit: let us fix without loss of generality an initial point in a neighborhood of $+\infty$ where all the asymptotic hypotheses hold. For instance, (H2) would do, and (A2), if this is the case, would be read as

$$f'(x) \geq -\varepsilon \text{ and } \frac{f''f}{f'}(x) < k_1; \tag{1}$$

for some $\varepsilon > 0$. Also, for some consistency of notation let us suppose the initial point is a bouncing point on the upper wall, i.e. $(x_0, y_0) = (x_0, f(x_0))$.

Using the notations of figure 1 we have the fundamental relation:

$$\tan\theta_{n+1}(x_{n+1} - x_n) = f(x_n) + f(x_{n+1}). \tag{2}$$

With a bit of geometry, looking at the same picture, we get

$$\theta_{n+1} = \theta_n + \pi - 2\alpha_n = \theta_n + 2\delta_n,$$

calling $\delta_n = -\arctan(f'(x_n)) > 0$. This summarizes to

$$\theta_n = \theta_1 + 2\sum_{k=1}^{n-1}\delta_k.$$

Thus $\{\theta_n\}_{n\geq 1}$ is an increasing sequence. Since we have assumed the particle never go backwards, then $\theta_n < \pi/2$ for all $n \geq 1$, so $\theta_n \nearrow \theta_\infty \in [\theta_1, \pi/2]$. Hence $\tan\theta_n \geq \tan\theta_1 =: k_2 > 0$. From this, the monotonicity on f, and (2) we have

$$x_{n+1} - x_n \leq \frac{2}{k_2} f(x_n). \tag{3}$$

What stated above implies that $\sum_k \delta_k < +\infty$. Therefore $\delta_k \to 0$. As a consequence, we see that $\exists k_3 \in]0, 1[$ such that $\delta_k =\geq k_3 \tan\delta_k = k_3|f'(x_k)|$. If we are able to prove that

$$-\sum_{k=0}^{\infty} f'(x_k) = +\infty, \tag{4}$$

that inequality implies that $\sum_k \delta_k$ cannot converge, creating a contradiction which finishes the proof.

Defining $g := -f'/f > 0$ will greatly simplify our notation. ¿From (3), we obtain, for some constant k_4,

$$-\sum_n f'(x_n) \geq k_4 \sum_n g(x_n)(x_{n+1} - x_n). \tag{5}$$

Case (A1). Obviously (A1) reads $g \geq k_5 > 0$. Hence, since by hypothesis $x_n \to \infty$, (5) gives (4). It may worth remind that (A1) means we have exponential decay for f. In fact, after an integration, we see that $\forall x_2 > x_1 \geq 0$,

$$f(x_2) \leq f(x_1)\, e^{-k_5(x_2-x_1)}.$$

Case (A2). The proof is little more involved here. We use our assumption on the limit of f' to prove the following

Lemma 1 *There exists a constant k_6 such that $\forall n \in \mathbb{N}$*

$$\frac{f(x_n)}{f(x_{n+1})} \leq k_6.$$

Proof. Let us call $\bar{x}_n$ the point in $[x_n, x_{n+1}]$ provided by the Lagrange mean value theorem. We can write

$$\frac{f(x_n)}{f(x_{n+1})} = 1 - \frac{f'(\bar{x}_n)(x_{n+1} - x_n)}{f(x_{n+1})}.$$

Using (3), this is turned into

$$\frac{f(x_n)}{f(x_{n+1})} \left(1 + \frac{2f'(\bar{x}_n)}{k_2} \right) \leq 1,$$

which yields the lemma, since the term in the brackets is positive for n large enough, because of the assumption about the vanishing of f'.

This enables us to prove

Lemma 2 *There exists a constant $k_7 \in]0,1[$ such that, for n sufficiently large, $g(x_n) \geq k_7 \max_{[x_n,x_{n+1}]} g$.*

Proof. Proving the statement is equivalent to proving that we can find a $k_8 > 0$ for which

$$\log g(\tilde{x}_n) - \log g(x_n) \leq k_8,$$

where $\tilde{x}_n$ maximizes g in $[x_n, x_{n+1}]$. Using again the Lagrange mean value theorem, the fact that $\tilde{x}_n - x_n \leq (2/k_2) f(x_n)$ (a consequence of (3)), and the previous lemma, we obtain

$$\begin{aligned}
\log g(\tilde{x}_n) - \log g(x_n) &= \frac{g'}{g}(\hat{x}_n)(\tilde{x}_n - x_n) \\
&\leq \frac{2}{k_2}\left(\frac{f''}{f'} - \frac{f'}{f}\right)(\hat{x}_n)\ f(x_n) \\
&\leq k_9 \left(\frac{f''}{f'} - \frac{f'}{f}\right)(\hat{x}_n)\ f(x_{n+1}) \\
&\leq k_9 \left(\frac{f'' f}{f'} - f'\right)(\hat{x}_n) \leq k_9(k_1 + \varepsilon),
\end{aligned}$$

having used (1) in the last step.

We are now prompted to get (4) in this case, too. Looking at (5) we can write:

$$\begin{aligned}
\sum_n g(x_n)(x_{n+1} - x_n) &\geq k_7 \sum_n (\max_{[x_n, x_{n+1}]} g)(x_{n+1} - x_n) \\
&\geq k_7 \int_{x_0}^{+\infty} g(x)dx = +\infty,
\end{aligned}$$

since $-\int^\infty (f'/f) = -\lim_{x\to\infty}(\log f(x) + \mathit{const}) = +\infty$. This ends the proof of the theorem.

3 Discussion

The obvious news the theorem says, compared to the mentioned condition $f'' > 0$, is the possibility for f' to oscillate, to a certain extent. Dynamically speaking, the change in direction our particle gets every time it bounces against the upper wall ($\delta_k = -\arctan(f'(x_k))$) need not be a monotone sequence. As a matter of fact, (A2) precisely controls the amount of such an oscillation. An example will illustrate the case: for $\alpha > 1, \beta > 0, c > 1$ define $f_1'(x) := -x^{-\alpha}(\sin(x^\beta) + c) < 0$. This means that we define $f_1(x) := -\int_x^\infty f_1'(z)dz$, which makes sense as a convergent integral. Therefore $f_1''(x) = -\beta x^{-\alpha+\beta-1}\cos(x^\beta) + O(x^{-\alpha-1})$, showing that f_1 is not convex. Now, the asymptotic behavior of f_1 and its derivatives is easily extracted to yield

$$\frac{f_1'' f_1}{f_1'}(x) \asymp x^{-\alpha+\beta}.$$

Thus, (A2) holds if, and only if, $\alpha \geq \beta$, meaning that the faster f_1 vanishes the more violent the oscillation of f_1' is allowed to be.

Another example may be interesting to present, to show that there are cases where (A1) holds but (A2) does not. Pick up a $\varphi \in C^\infty(\mathbb{R})$ supported in $]-1/2, 1/2[$ with $\int \varphi = 1 - e^{-1}$. Call, for $k \in \mathbb{N}$,

$$\varphi_k(x) := \varphi((x - k - 1/2)\, e^k),$$

supported in $]k + 1/2 - e^{-k}/2, k + 1/2 + e^{-k}/2[$. Let us now define $h(x) := \sum_{k=0}^{\infty} \varphi_k(x)$. The result is shown in figure 2a. We see that

$$\int_k^{k+1} h(x)dx = \int_k^{k+1} \varphi_k(x)dx = (1 - e^{-1})e^{-k}.$$

Also denote by $H(x) := \int_x^\infty h(z)dz$. Finally, let us introduce $f_2'(x) := -e^{-x} - h(x)$, corresponding to $f_2(x) = e^{-x} + H(x)$. Their graphs are displayed in figure 2b and 2c, respectively. Certainly $f_2' \not\to 0$ and (A2) is not verified. Now, from above we can estimate the value of f_2. In fact $e^{-[x]-1} \leq H(x) \leq e^{-[x]}$ giving $H(x) \leq e^{-x+1}$. Therefore

$$\frac{|f_2'|}{f_2}(x) = \frac{e^{-x} + h(x)}{e^{-x} + H(x)} \geq \frac{e^{-x}}{e^{-x} + e^{-x+1}} = \frac{1}{1+e} \geq 0.$$

That is: (A1) holds as well as the result, in this case.

Is it difficult to say to what extent our theorem is inclusive of the general case or how it can be refined. The point here is that finding a sufficient condition for the non-existence of an escape orbit is much more direct than finding a necessary condition. The shape of f can be *pathological* enough but not in a suitable way that allows a trajectory to go directly to infinity. One thing can be said, though: hypotheses (H1) and (H2) do not suffice and one needs some extra assumption to control a possible wild behaviour of f'. As a matter of fact we may now sketch the construction of a billiard table verifying those hypotheses and having one escape orbit. We will start by first drawing the orbit and then a compatible f.

Consider the polyline shown in figure 3a with $\theta_1 \in]0, \pi/2[$ and $\{y_n\}$ any non-integrable sequence such that $y_n \searrow 0$. If this were an escape orbit then we would have $f(x_n) = y_n$, $f'(x_n) = 0$ and $\theta_n = \theta_1 \; \forall n$. Furthermore $x_{n+1} - x_n = (y_{n+1} + y_n)/\tan\theta_1$ so that $\lim_{n\to\infty} x_n = \infty$, because of our assumption on $\{y_n\}$. Of course any f giving rise to such an orbit cannot satisfy (H2), because of the flat tangent at the bouncing points, but we can slightly modify our picture in order to fit it. Take an integrable sequence $\{\delta_n\}$, $\delta_n > 0$ such that $\theta_\infty := \theta_1 + 2\sum_n \delta_n < \pi/2$. Now modify the trajectory in figure 3a, "shrinking" it in order to have $\theta_n := \theta_1 + 2\sum_{k=1}^{n-1} \delta_k$; keep y_n fixed. The result is drawn in figure 3b. This is again an escape orbit since, due to our choice of θ_∞, the contraction of the little triangles has a lower bound, i.e. $x_{n+1} - x_n \geq (y_{n+1} + y_n)/\tan\theta_\infty$. One can now very easily construct an f which satisfies (H1) and (H2) and whose graph is an upper wall for this trajectory.

This proves our remark.

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