Non–Degenerate "Arnold Diffusion"

Luigi Chierchia*

Dipartimento di Matematica

Università di Roma Tre

Via Corrado Segre 2, 00146 Roma (Italy)

Internet: luigi@matrm3.mat.uniroma3.it

April 1996

Abstract

The Hamiltonian $H \equiv \frac{1}{2}|I|^2 + \varepsilon V(\varphi;\varepsilon)$ with $V \equiv \cos\varphi_4 + \varepsilon^8 \sum_{j=1}^{3} \cos(\varphi_j + \varphi_4)$ and $I \in \mathbb{R}^4$, is shown to exhibit "Arnold diffusion" (or, more precisely, "Arnold instability") on each energy surface, providing the first example of "Arnold diffusion" for one–parameter perturbations of non–degenerate ("steep"), completely integrable Hamiltonian systems.

Contents

*__Acknowledgments__ It is a pleasure to thank C. Falcolini and C. Liverani for lasting and fruitful discussions. I am also indebted with C. Falcolini for precious help on the computational aspects related to this paper. This work is part of the research program of the European Network on: "Stability and Universality in Classical Mechanics", # ER–BCHRXCT940460.

1 Introduction and Main Result

0. The term "Arnold diffusion" is usually (and rather vaguely) used in connection with the instability of "action–variables" occurring "near resonances" in high dimensional Hamiltonian systems which are small perturbations of "completely integrable systems[1]". In this introduction I will stick to such tradition, but in § 2 I shall propose a mathematical definition for "Arnold diffusion" (which to avoid confusion, will be called "Arnold instability[2]") and formulate a conjecture ("subconjecture") weaker than that formulated in 1963 by Arnold.

In 1963 V.I. Arnold conjuctured that: "The typical case in higher–dimensional problems is *topological instability*: through an arbitrary small neighbourhood of any point there pass phase trajectories along which the slow variables[3] drift away from the initial values by a quantity of order 1"; compare [**?**], page 189 (from which the citation is taken) and, of course, Arnold's fantastic four–page–long paper [**?**], which started all this "diffusion businness". The proof or the (less likely) disproof of this conjecture is matter for the next millennium (while the proof of the subconjecture of § 2 might be matter of years), but by now there start to be a few concrete examples exhibiting "Arnold diffusion" (i.e. examples of nearly–integrable systems with *some* phase trajectories whose action–variables, near resonances, drift away by $O(1)$ from their initial values "following long chains of heteroclinic connections"); compare point 2 of this introduction.

However, notwithstanding the above mentioned (few) examples of and the (very) extensive references to "Arnold diffusion" both in the mathematical and in the physical literature, until now there are *no* examples of *one*–parameter analytic perturbations of *non–degenerate* ("steep"), *completely* integrable Hamiltonian systems proved to exhibit "Arnold diffusion". The purpose of this paper is to fill this gap.

1. (Formulation of the result) Consider the eight–dimensional phase space ("four degrees of freedom") $\mathcal{M} \equiv \mathbb{R}^4 \times \mathbb{T}^4$, where $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$, endowed with the *standard symplectic form*

$$\sum_{j=1}^{4} dI_j \wedge d\varphi_j \ , \quad (I_1, ..., I_4) \equiv I \in \mathbb{R}^4 \ , \quad (\varphi_1, ..., \varphi_4) \equiv \varphi \in \mathbb{T}^4 \ .$$

[1]A "Completely integrable system" is a (analytic) Hamiltonian system whose phase space is foliated by invariant tori on which the flow is quasi–periodic; a "quasi–periodic motion" is a trajectory conjugated to a linear flow on a torus: $\theta \in \mathbb{T}^N \to \theta + \omega t \in \mathbb{T}^N$ where $\mathbb{T}$ is $\mathbb{R}/(2\pi\mathbb{Z})$ and ω (the "frequency vector") is a rationally independent vector in $\mathbb{R}^N$.

[2] In fact, one of the problems with "Arnold diffusion" is its definition! The term was introduced, in the physical literature, by Chirikov and Zaslavskij ["Stochastic instability of non–linear oscillations", Usp. Fiz. Nauk *105*, no. 1, 2–40 (1971)] and, to the best of my knowledge, there exists no mathematical definition of it. In [**?**] there have been introduced several notions (such as "diffusion paths", "paths open for diffusion" etc.) in the context of "a priori unstable systems", which roughly speaking, are nearly–integrable systems whose integrable part is *not* completely integrable (in the sense specified above) but it generates also (partially) hyperbolic motions. The definition given in § 2 below is a slight modification of the definition given in [**?**].

[3]Or "action–variables".

Let $H \equiv H(I, \varphi; \varepsilon)$ denote the one–parameter family of Hamiltonian functions[4]

$$H = \frac{1}{2}|I|^2 + \varepsilon V(\varphi; \varepsilon) \ , \qquad V \equiv \cos\varphi_4 + \varepsilon^8 \sum_{j=1}^{3} \cos(\varphi_j + \varphi_4) \ , \tag{1.1}$$

where $|I|^2 \equiv I \cdot I \equiv \sum_{j=1}^4 I_j^2$. Denote by ϕ_H^t the Hamiltonian flow generated by[5] H.

Theorem 1.1 *Given $E_2 > E_1 > 0$, there exist two positive numbers ε_0 and d_0 such that, for each $E_1 \le E \le E_2$ and for $0 < |\varepsilon| \le \varepsilon_0$, one can find initial data (I^0, φ^0) on the energy level $H^{-1}(E)$ satisfying*

$$\sup_{t \ge 0} \left| I(t) - I^0 \right| > d_0 \ , \qquad \text{where} \quad \Big(I(t), \varphi(t)\Big) \equiv \phi_H^t(I^0, \varphi^0) \ .$$

The initial value I^0 depends continuously on $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

In other words on each energy surface (bounded away from zero and ∞), for $\varepsilon \ne 0$ small enough, there exist phase points (continuously depending on ε) whose action variables drift away by $O(1)$ from their initial values.

In fact, the "unstable trajectories" of this Theorem (as in Arnold's example [**?**]) evolve near long "chains of heteroclinic connections". Such connections are trajectories lying in the *transverse intersections* of invariant Lagrangian manifolds ("whiskers") which are asymptotic to different *three–dimensional invariant tori*. These tori depend continuously on ε and for $\varepsilon = 0$ correspond to *double resonances*[6]. For $\varepsilon \ne 0$ these "doubly–resonant tori" deform into three–dimensional tori (while the fourth dimension "becomes hyperbolic") on which the Hamiltonian flow is (analytically) conjugated to the linear flow

$$(\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3 \to \ (\theta_1 + \alpha_1 t, \theta_2 + \alpha_2 t, \theta_3 + |\varepsilon|^{\frac{1}{2}} \alpha_3 t) \ ,$$

where the frequency vector $(\alpha_1, \alpha_2, |\varepsilon|^{\frac{1}{2}} \alpha_3)$ is rationally independent and the numbers α_i are of order 1.

[4]In [**?**] similar models (with two parameters) were considered: compare the "even class" of §9 in [**?**].

[5]That is, $\Big(I(t), \varphi(t)\Big) = \phi_H^t(I^0, \varphi^0)$ solves the Hamilton equations

$$\dot{I} = -\partial_\varphi H \ , \qquad \dot{\varphi} = \partial_I H \ , \qquad I(0) = I^0 \ , \qquad \varphi(0) = \varphi^0 \ ,$$

where ∂_φ and ∂_I denote the gradients with respect to φ and I: $\partial_\varphi H = \left(\frac{\partial H}{\partial \varphi_1}, ..., \frac{\partial H}{\partial \varphi_4}\right)$, $\partial_I H = \left(\frac{\partial H}{\partial I_1}, ..., \frac{\partial H}{\partial I_4}\right)$ and dot denotes differentiation with respect to the time t.

[6]A "resonance" for a completely integrable system with Hamiltonian $H_0(I)$ is a set of action–variables I defined by relations like

$$\partial_I H_0(I) \cdot n = 0 \tag{1.2}$$

for some non–vanishing integer vector n. The double resonance referred to in the text is given by $\{I \in \mathbb{R}^4 \ : I_3 = I_4 = 0\}$ corresponding to the "independent" relations (1.2) with $n = (0, 0, 1, 0)$ and $n = (0, 0, 0, 1)$.

In the proof of Theorem 1.1 it will be crucial to have a *quantitative* control of the above mentioned transversality and of the distance of (the projection onto action–space of) "neighbourhing persistent invariant tori". Intuitively speaking: one may (easily) show the existence of transverse heteroclinic connections provided the distance of neighbourhing tori is much smaller then the transversality measure. Using techniques developed in [**?**], I shall prove that near the above mentioned double resonances there are $O(1)$–zones of invariant three–dimensional tori such that: (1) the "maximal holes" between two of such invariant tori have a diameter of $O(|\varepsilon|^{146})$; (2) the (stable and unstable) whiskers of these tori intersect with a transversality measure of $O(|\varepsilon|^{145/2})$. These estimates together with a standard Implicit Function Theorem imply immediately the existence of *heteroclinic* connections and, in particular, the possibility of "joining", through a *chain* of heteroclinic connections, *any* two tori in the above mentioned $O(1)$–zones (which means that such chains connect tori whose projection onto action–space ia a quantity of order one). By a result of [**?**] (§ 8) these heteroclinic chains turn out to be "transition chains" (which means that arbitrarly close to the chains there pass phase trajectories or, borrowing the language from hyperbolic dynamical systems, that the pseudo–orbits formed by the union of the heteroclinic connections of a chain may be shadowed by actual orbits).

2. (Comparing with some known results) (i) As already mentioned, the main source of ideas, in the present context, comes from Arnold's paper [**?**]. The *mechanism* that he describes (certainly more important than the result) is beyond all analytical proofs of "diffusion". I tried to capture the main feature of Arnold's mechanism in the definitions given in § 2 below.

There are, however, deep differences between Arnold's model[7] and the example considered here. First of all, Arnold's model is, so to speak, "doubly nearly–integrable": it contains *two* perturbative *independent* parameters ε and μ. If ε is set to be 0, you get a completely integrable system in the sense described above. But if $\varepsilon \neq 0$ and $\mu = 0$ you get a different type of "integrable system", namely, a system carrying, besides quasi–periodic motions, also *partially hyperbolic trajectories* [corresponding to the separatrix of the pendulum $\frac{1}{2}I_3^2 + \varepsilon(\cos\varphi_3 - 1)$]. In particular such integrable system ($\varepsilon \neq 0$, $\mu = 0$) cannot be described by global "action–angle" variables. Arnold's proof relies heavely on these facts and it is perturbative in μ at $\varepsilon > 0$ fixed (and if you try to estimate the value μ_0 mentioned in footnote 7 you get $\mu_0 \sim \exp(-1/\varepsilon^a)$ for some $a \geq 1/2$). Informally, we may say that Arnold's "original difusion" is "degenerate" in the sense that the limiting integrable system is singular (since it cannot be described by global action–angle variables[8]).

[7] In [**?**] Arnold considers the Hamiltonian

$$H = I_1 + \frac{1}{2}(I_2^2 + I_3^2) + \varepsilon(\cos\varphi_3 - 1) + \varepsilon\mu\,(\cos\varphi_3 - 1)(\cos\varphi_1 + \sin\varphi_2)\ ,$$

and proves that, for any $\varepsilon > 0$ and any $d_0 > 0$, there exist a $\mu_0(\varepsilon, d_0) > 0$ such that, for all $0 < |\mu| < \mu_0$ there exist trajectories $(I(t), \varphi(t))$ with $|I_2(T) - I_2(0)| > d_0$ for a suitable $T > 0$.

[8] Of course Arnold's model exhibit "diffusion" if μ is taken to be $\sim \exp(1/\varepsilon^a)$ and such a model

A second (minor) problem with Arnold's example is related to the choice of the perturbation (i.e. to the term proportional to μ), which is done in such a way so as to preserve, for $\mu \neq 0$, *all* the partially hyperbolic, three–dimensional invariant tori of the system with $\mu = 0$. But, even though such a behaviour is highly non–generic, giving up certain regions of phase space, one may treat "general perturbations" by means of KAM theory[9].

(ii) In the remarkable paper [?] Xia proves action instability for a class of three-body problems. From a technical point of view his instability proof is similar to that of Arnold in the following sense. Three–body problems contain many perturbative parameters (ratio of masses, eccentricities, etc.) and are "multiply nearly–integrable" in the sense mentioned above. In [?] "diffusion" is proved to exist when a certain parameter is kept fixed and positive (introducing partially hyperbolic motions) while an independent parameter is taken arbitrarly small. However, I believe that for three–body problems it is possible to prove "Arnold diffusion" *even* if the various parameters are related by power laws. Such opinion is based on the presence in three–body problems of certain degeneracies ("proper degeneracy") common in many celestial mechanical problems (compare next item).

(iii) The first *one*–parameter analytic example proved (in [?]) to exhibit "Arnold diffusion" is the *D'Alembert problem in Celestial Mechanics*[10]. The proof in [?], similarly to

---

would be a one–parameter perturbation of a completely integrable system. It is clear however that this is a rather cheap way out: first of all, from the physical point of view, it makes very little sense to have two small parameters related by an exponential law (while it makes a lot of sense to consider many–parameter problems where the parameters are related by *power* laws); furthermore, having an analytic system depending on parameters only in a C^∞ (rather than analytic) way might appear rather strange.

[9]Compare [?], paragraphs 1 through 8. For other generalizations of some of Arnold's ideas see [?] and [?]. In [?] the problem of the instability of elliptic equilibria in the Hamiltonian C^∞–category is considered (needless to say that looking at C^∞ generic results, where, for example, compactly supported perturbations are allowed, is quite different than working out techniques to *detect* "diffusion" in concrete analytical examples). Recently Arnold's example has been revisited from the variational point of view in [?].

[10]The model is the following. Consider a planet modelled by a homogeneous rotational ellipsoid whose center of mass revolves on a Keplerian orbit of eccentricity μ, the planet being subject to the gravitational attraction of a fixed star occupying one of the foci of the ellipse. If $(\theta_1, \theta_2, \theta_3)$ denote the Euler angles of the planet, the Lagrangian of this system is given by

$$\frac{J_3}{2}(\dot\theta_2 \cos\theta_1 + \dot\theta_3)^2 + \frac{J_1}{2}(\dot\theta_1^2 + \dot\theta_2^2 \sin^2\theta_1) + c\int_{R(t)} \frac{dx}{|x_0(t) + x|} ,$$

where J_3 and $J_2 = J_1$ are the inertia moment of the planet, c is a positive physical constant, $R(t)$ is the region occupied by the planet at time t and $x_0(t)$ is the position of the center of mass of the planet at time t. Using Andoyer variables (see, e.g., [?]) one can find canonical (symplectic) coordinates on $U \times \mathbb{T}^3$ (U being some open set of $\mathbb{R}^3$) so that the dynamics generated by the above Lagrangian is equivalent to the dynamics generated by the Hamiltonian

$$H = \omega I_1 + \varepsilon \frac{I_2^2}{2} + \frac{I_3^2}{2} + \varepsilon\mu\, f(I,\varphi;\mu) , \qquad (I,\varphi) \in U \times \mathbb{T}^3 ,$$

for a suitable analytic function f; here $(1-\varepsilon)$ is essentially the ratio between the polar and the equatorial radii of the planet and $2\pi/\omega$ is the year of the planet. In [?] it is proved that *taking* $\mu = \varepsilon^n$ (for a large the proof presented here, relies on a quantitative comparison of the density of persistent partially hyperbolic tori versus a transversality measure of homo/heteroclinic connections, which turns out to be *polynomial in* ε. The "proper degeneracy" mentioned above (which is crucial for the proof to work) consists in the fact that the integrable limit *does not depend upon all the action variables* and that such degeneracy is removed considering higher orders (in ε) terms. Technically, this degeneracy introduces an "extra time–scale" which allows to prove polynomial transversal intersections. The simplest one–parameter model problem with a properly degenerate limiting integrable system for which "Arnold diffusion" has been proved may be found in[11] [**?**].

(iv) Finally, let me mention the very interesting variational approach to instability of J.N. Mather; see, e.g., [**?**].

3. (Some remarks and open problems) (i) The limiting integrable Hamiltonian $H|_{\varepsilon=0} = \frac{1}{2}|I|^2$ in (1.1) is non–degenerate from many points of view: it is non–degenerate and iso–energetically non–degenerate (in the sense of KAM theory) and, what is more important in this context, it is *steep* in the sense of Nekhoroshev[12].

(ii) The transversal homoclinic intersections, which the proof of Theorem 1.1 relies on, are obtained by "splitting up" certain homoclinic connections. Technically this is done, as in [**?**], through suitable "high order[13] Poincaré–Melnikov–Arnold integrals": see the "whisker calculus" discussed in Appendix **??**. Once more, it is fundamental, for proving "Arnold diffusion", that such splitting is *polynomial in* ε and *not* exponentially small in ε as it happens, instead, in lower dimensional systems, in Arnold's example or as it is expected to happen in general cases *near simple resonances*.

It is fundamental because one wants to construct the above mentioned long chains of heteroclinic connections. Each link of these chains corresponds to heteroclinic orbits asymptotic to neighbourhing invariant tori and, in turn, the "distance" between these tori (better the "distance of the projection onto action–space" of the tori) correspond to a

enough integer n) there exist orbits for which $|I_2(T) - I_2(0)| = O(1)$ for a suitable $T > 0$ (which is actually computed).

[11]In [**?**] I considered the following Hamiltonian

$$H = \varepsilon\Big(\frac{I_1^2}{2} + \frac{I_2^2}{2}\Big) + \frac{I_3^2}{2} + \varepsilon(\cos\varphi_3 - 1) + \varepsilon^n[\cos(\varphi_1+\varphi_3) + \cos(\varphi_2+\varphi_3)]\ ,$$

and proved, for large enough integers n, the existence of orbits such that $|I(T) - I(0)| = O(1)$ for a suitable $T > 0$.

[12]Examples of steep functions are strictly convex functions or, more in general, analytic functions whose restrictions to any affine subspace have only complex–isolated critical points (compare [**?**] §3.4 of chapter 5). If steepness is violated "fast drift" may occur as shown by the following example due to Nekhoroshev. The Hamiltonian $\frac{1}{2}(I_1^2 - I_2^2) + \varepsilon\sin(\varphi_1 - \varphi_2)$ generates the "fast–drifting motions" $\varphi(t) = \varphi^0 + (at + \varepsilon\alpha t^2/2, -at - \varepsilon\alpha t^2/2)$, $I(t) = (a + \varepsilon\alpha t, -a - \varepsilon\alpha t)$ where $\alpha = -\cos(\varphi_1^0 - \varphi_2^0)$. Such motions take place on the resonance $\{I_1 + I_2 = 0\}$. Notice that this example is integrable and that $I_1 + I_2$ is an *integral* of motions (in fact there are no transverse intersections, etc.: in other words these "fast drifts" are very special and have nothing to do with "Arnold diffusion).

[13]"Fourth order", to be precise.

tiny "elementary" drift of the corresponding action–variable. Link after link one builds up a chain connecting tori whose corresponding action–variables are at a distance of order 1. Now, if the splitting in homoclinic connections is exponentially small one needs to detect zones in action–space with a "density of persistent tori" which is also exponentially small (otherwise you will not be able to construct heteroclinic chains connecting action–variables lying at an $O(1)$–distance). It is true that Nekhoroshev's Theorem (see [?]), used in connection with KAM theory, guarantees such "high density zones of persistent tori", however you realize that this strategy leads to a rather touchy comparisons among different kinds of exponentials.

This remark should explain the reason for investigating multiple resonances and for looking up systems with four (or more) degrees of freedom. In fact, in three degrees of freedom (which is the minimal dimension allowed by KAM theory for "Arnold diffusion" to occur; see [?], §3.4 of chapter 5) double resonances correspond to one–dimensional regions of action–variables, which, in turn, *fixing the energy*, reduce to isolated points in action–space. These points, in general, correspond, after perturbation, to isolated invariant tori (and such isolated tori are no good for constructing long chains of heteroclinic connections, which "Arnold diffusion" relies on).

For all these reasons, I regard a proof of "non–degenerate Arnold diffusion" in "general" systems with *three* degrees of freedom as an interesting open problem.

(iii) The exponent 8 in (1.1) (rather than the "more natural" exponent 1) is needed for a "shorter" proof of the asymptotics (for small ε) of the measure of transversal splittings. Needless to say, I believe that 1 is the right value (and work in this direction is in progress in [?]).

(iv) In fact, I believe that the method of proof of Theorem 1.1 should be also good enough for proving "diffusion" in general high dimensional, one–parameter, non–degenerate, nearly–integrable problems (compare the "subconjecture" of § 2 below).

(v) As the reader might by now have guessed the proof of the above Theorem will not imply Arnold's conjecture for the Hamiltonian (1.1), since the regions that we shall be able to connect by heteroclinic chains are *not* arbitrarily placed in action–space. In fact there is *no* example at all for which Arnold's conjecture has been proven. Of course, any improvement on such matter would also be interesting.

2 Arnold Instability, Theorem 2.1 and a "subconjecture"

In this section I propose a mathematical definition for "Arnold diffusion" (renamed "Arnold instability") and formulate a stronger version of Theorem 1.1.

Consider a phase space $\mathcal{M} \equiv U \times \mathbb{T}^N$, with U an open connected set of $\mathbb{R}^N$, endowed with the standard symplectic form $\sum dI_j \wedge d\varphi_j$ and let us consider on it the flow generated by a Hamiltonian[14]

$$h(I,\varphi;\varepsilon) \equiv h_0(I) + \varepsilon h_1(I,\varphi) + \varepsilon^2 h_2(I,\varphi) + \cdots$$

with h real analytic on $\mathcal{M} \times (-\bar\varepsilon, \bar\varepsilon)$ for some $\bar\varepsilon > 0$.

Definition 2.1 (Heteroclinic chains) *A* heteroclinic chain *is a set of $n \geq 1$ orbits*[15] *$z^{(1)}(t)$,...,$z^{(n)}(t)$ together with $(n+1)$ different minimal sets*[16] *$\mathcal{T}^{(1)}$,...,$\mathcal{T}^{(n+1)}$ such that, for all $1 \leq i \leq n$,*

$$\lim_{t\to-\infty} \operatorname{dist}\Big(z^{(i)}(t), \mathcal{T}^{(i)}\Big) = 0 = \lim_{t\to\infty} \operatorname{dist}\Big(z^{(i)}, \mathcal{T}^{(i+1)}\Big) \ .$$

Definition 2.2 (Transition chains) *A heteroclinic chain, $z^{(1)}(t)$,....,$z^{(n)}(t)$, $\mathcal{T}^{(1)}$,...., $\mathcal{T}^{(n+1)}$, is called* a transition chain *if for any $r > 0$ there exist an orbit $z(t)$ and a time $T > 0$ such that*

$$\operatorname{dist}\Big(z(0), \mathcal{T}^{(1)}\Big) < r \ , \quad \operatorname{dist}\Big(z(T), \mathcal{T}^{(n+1)}\Big) < r \ , \quad \sup_{0\leq t\leq T} \operatorname{dist}\Big(z(t), Z\Big) < r \ ,$$

where Z is the closure of the union over all i's of $\{z^{(i)}(t) \colon t \in \mathbb{R}\}$. The sets $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(n+1)}$ are said to be connected by a transition chain.

Definition 2.3 (Arnold instability) *Given $E \in \mathbb{R}$, the system $(\phi_h^t, h^{-1}(E))$ is called* Arnold unstable *if there exist two positive numbers ε_0 and d_0 such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there exist (closed) invariant sets $\mathcal{T}(\varepsilon)$, $\mathcal{T}'(\varepsilon) \subset h^{-1}(E)$ satisfying the following two conditions:*

(i) $\mathcal{T}(\varepsilon)$ and $\mathcal{T}'(\varepsilon)$ are continuous, at $\varepsilon = 0$, in the Hausdorff metric[17] *and if π_I denotes the natural projection over the action variables*[18] *then*

$$\pi_I\Big(\mathcal{T}(0)\Big) = \{\tilde I\} \ , \qquad \pi_I\Big(\mathcal{T}'(0)\Big) = \{\tilde I'\} \ , \qquad \textit{with} \quad |\tilde I - \tilde I'| > d_0 \ ;$$

(ii) for each $0 < |\varepsilon| < \varepsilon_0$ $\mathcal{T}(\varepsilon)$ and $\mathcal{T}'(\varepsilon)$ are connected by a transition chain.

---

[14] The canonical variables (I, φ) are often called *action–angle variables*. The study of the dynamical system $(\phi_h^t, \mathcal{M})$ was considered by Poincaré, in his fundamental [**?**], *le Problème générale de la Dynamique* (Ch. 13, no. 13, Tome I of [**?**]).

[15] An orbit (or "trajectory" or "motion") is a solution $z(t) = (I(t), \varphi(t)) = \phi_h^t(z(0))$, $(z(0) \in \mathcal{M})$, of the Hamilton equations $\dot I = -\partial_\varphi h$, $\dot\varphi = \partial_I h$.

[16] A closed subset of the phase space $\mathcal{M}$ is called *minimal* (with respect to the Hamiltonian flow ϕ_h^t) if it is non–empty, invariant for ϕ_h^t and contains a dense orbit.

[17] The Hausdorff metric δ on closed subsets of some metric space, with metric δ_0, is defined as

$$\delta(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} \delta_0(x,y) \ , \ \sup_{x\in B}\inf_{y\in A} \delta_0(x,y)\} \ .$$

[18] $\pi_I(z) = \pi_I((I,\varphi)) = I$ and, in general, $\pi_\varphi(z) = \varphi$, $\pi_{I_j}(z) = I_j$, $\pi_{\varphi_j}(z) = \varphi_j$ etc.

Theorem 1.1 is an immediate corollary of the following result proved in § 3 below. Let H and E_2, E_1 be as in §1, point 1.

Theorem 2.1 *For $E \in [E_1, E_2]$, the system $(\phi_H^t, H^{-1}(E))$ is uniformly Arnold unstable, "uniformly" meaning that the constants ε_0 and d_0 of Definition 2.3 depend only on E_1 and E_2.*

We are ready to formulate a conjecture, which is the precise content of the expectation mentioned in (iv), point 3 of § 1.

Subconjecture *In general, one–parameter perturbations of completely integrable, non–degenerate, high–dimensional Hamiltonian systems produce Arnold unstable systems.*

3 Proofs of the theorems

In this section we prove Theorem 2.1, which, as mentioned above, implies immediately Theorem 1.1. The proof is divided in several steps. A few (technical) details are reported either in appendices or in footnotes.

We shall give the proof for $\varepsilon > 0$; to treat the case $\varepsilon < 0$ one will simply perform first the trivial change of variables $\varphi_i \equiv \varphi_i' + \pi$, $(i = 1, ..., 4)$. Also, to set the unstable point of the pendulum $\frac{1}{2}I_4^2 + \varepsilon \cos\varphi_4$ in $(I_4, \varphi_4) = (0, 0)$, we shall add to the Hamiltonian the constant $-\varepsilon$ so that, *from now on H is replaced by*

$$H = \frac{1}{2}|I|^2 + \varepsilon(\cos\varphi_4 \; - 1\;) + \varepsilon^9 \sum_{j=1}^{3} \cos(\varphi_j + \varphi_4) \;, \qquad \varepsilon \geq 0 \;. \tag{3.1}$$

Finally, in the following proof, " const " denotes suitable (possibly different) positive constants depending only on ε_0 and E_1, E_2.

3.1 The sets $\mathcal{T}(0)$ and $\mathcal{T}'(0)$ and basic sets of frequencies

Fix $E \in [E_1, E_2]$ and let s be a positive integer (the construction on which the proof is based depends on several related constants that we shall leave free for the moment; later we shall take $s = 149$).

There exists $\varepsilon_0 \equiv \varepsilon_0(s) > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$ we may define the *frequency set*

$$\begin{aligned} \Omega \equiv \Omega(\varepsilon, s) \quad &\equiv \quad \{\omega \equiv (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 : \; \frac{1}{2}|\omega|^2 = E \;, \\ &\qquad \omega_2 \in (a, b) \;, \omega_3 \in \left\{ \begin{array}{ll} \{0\} \;, & \text{if } \varepsilon = 0 \\ (c\sqrt{\varepsilon}, 2c\sqrt{\varepsilon}) \;, & \text{if } \varepsilon > 0 \end{array} \right. \;, \quad \text{and} \\ &\qquad |\omega \cdot n| > c\sqrt{\varepsilon} \qquad \forall \; 0 < |n| \leq s \;, n \in \mathbb{Z}^3\} \;, \end{aligned} \tag{3.2}$$

where c is an arbitrary positive number and a, b are suitable positive numbers[19] depending only on s and on E. Notice that[20]

$$\text{area} \quad \text{of} \quad \Omega(\varepsilon, s) \sim \sqrt{\varepsilon} \ . \tag{3.4}$$

Let $0 < \gamma < \varepsilon$ (later γ will be chosen to be a suitable power of ε); let $\tau > 2$ and define

$$\Omega_\gamma \equiv \Omega_\gamma(\varepsilon, s) \equiv \{\omega \in \Omega : \ |\omega \cdot n| > \frac{\gamma}{|n|^\tau} \ , \quad \forall \ 0 \neq n \in \mathbb{Z}^3\} \ . \tag{3.5}$$

Then, if "meas" denotes Lebesgue measure on the sphere of radius $\sqrt{2E}$, one has[21]

$$\text{meas}\,(\Omega \backslash \Omega_\gamma) \leq \ \text{const} \ \gamma \ . \tag{3.6}$$

¿From (3.6) and (3.4) (being $\gamma < \varepsilon$) it follows that

$$\text{meas}\,\Omega_\gamma \sim \sqrt{\varepsilon} \ .$$

¿From the above discussion it follows also that we may adjust a and b (in the definition of the set Ω) so that, for

$$(\omega_1, \omega_2) = (\sqrt{2E - a^2}, a) \ , \quad \text{or} \quad (\omega_1, \omega_2) = (\sqrt{2E - b^2}, b) \ ,$$

(i.e., whenever (ω_1, ω_2) is taken as the "endpoints" of the arc Γ_0 defined in footnote 19) one has

$$|(\omega_1, \omega_2) \cdot n| > \frac{\gamma_0}{|n|^\tau} \ , \qquad \forall \ n \in \mathbb{Z}^2 \backslash \{0\} \ ,$$

where γ_0 is a suitable (ε–independent) positive constant[22].

---

[19] A possible choice of a, b, c can be made as follows. Fix $0 < c < 1$ and let S denote the circle of radius $\sqrt{2E}$ in the (ω_1, ω_2)–plane. Consider on S arcs Γ such that

$$|\omega_1 n_1 + \omega_2 n_2| > 3c\sqrt{\varepsilon} \ , \quad \forall \ 0 < |(n_1, n_2)| \leq s \ , \quad \forall \ (\omega_1, \omega_2) \in \Gamma \tag{3.3}$$

(if $\varepsilon \to 0$ the set of all arcs verifying (3.3) fills up S). Fix $\varepsilon_0 > 0$ small and pick an arc Γ_0, say, in the first quadrant (satisfying (3.3) with ε_0 in place of ε): it is clear that if ε_0 is small enough Γ_0 may be taken of the form $\{(\sqrt{2E - \omega_2^2}, \omega_2) \colon \omega_2 \in (a, b)\}$ with $\sqrt{E}/2 < a < b < \sqrt{E}$. Furthermore, (3.3) will hold on Γ_0 for all $0 \leq \varepsilon \leq \varepsilon_0$ and it is now easy to see that, with this choice of a and b (and possibly decreasing ε_0) the set $\Omega(\varepsilon, s)$ is well defined (and different from the empty set).

[20] Here and in what follows $F \sim f(\varepsilon)$ (with $f(\varepsilon) \neq 0$ for $\varepsilon \neq 0$) means that $\lim_{\varepsilon \to 0} F/f \in (0, \infty)$.

[21] To check (3.6), let S denote the 2–sphere in $\mathbb{R}^3$ of radius $\sqrt{2E}$ and note that $\Omega \backslash \Omega_\gamma \subset \{\omega \in S \colon \exists \ n \in \mathbb{Z}^3 \backslash \{0\}$ for which $|\omega \cdot n| \leq \gamma |n|^{-\tau}\}$. Thus

$$\text{meas}\,(\Omega \backslash \Omega_\gamma) \leq \sum_{n \in \mathbb{Z}^3 \backslash \{0\}} \{\omega \in S : \ |\omega \cdot \frac{n}{|n|}| \leq \frac{\gamma}{|n|^{\tau+1}}\} \leq 2\pi\sqrt{2E} \ \gamma \sum_{n \neq 0} \frac{1}{|n|^{\tau+1}} \ .$$

Hence (3.6) holds if one chooses const $= 2\pi\sqrt{2E_2} \sum_{n \neq 0} \frac{1}{|n|^{\tau+1}}$ (recall that $\tau > 2$).

[22] For example, one could take $\gamma_0 \ll 2\pi\sqrt{2E_2} \sum \frac{1}{|n|^{\tau+1}}$ (compare footnote 21).

Next, we define *rescaled frequencies*[23]:

$$\Omega_\gamma^* \equiv \Omega_\gamma \times (-\frac{1}{2}, \frac{1}{2}) \text{ , and for } (\omega, \mathring{\ }) \in \Omega_\gamma^* \quad \text{we set } \omega \mathrel{\mathring{\equiv}} (1+\mathring{\ })\omega \ .$$

Thus, if $(\omega, \mathring{\ }) \in \Omega_\gamma^*$ and $n \in \mathbb{Z}^3 \backslash \{0\}$, then, by (3.5), one has

$$|\omega\mathring{\cdot} n| \geq (1+\mathring{\ })\frac{\gamma}{|n|^\tau} \geq \frac{\gamma}{2}\frac{1}{|n|^\tau} \ .$$

Finally, we let the sets $\mathcal{T}(0)$ and $\mathcal{T}'(0)$, appearing in the Definition 2.3 of Arnold instability, be defined as

$$\begin{aligned} \mathcal{T}\,(0) &\equiv \{(I,\varphi) \in \mathcal{M} : (I_1, I_2) = (\sqrt{2E-a^2}, a),\ I_3 = I_4 = 0 = \varphi_4\} \ , \\ \mathcal{T}'(0) &\equiv \{(I,\varphi) \in \mathcal{M} : (I_1, I_2) = (\sqrt{2E-b^2}, b),\ I_3 = I_4 = 0 = \varphi_4\} \ . \end{aligned}$$

For the integrable Hamiltonian

$$H_0 \equiv \frac{1}{2}|I|^2 \ , \qquad (I \in \mathbb{R}^4) \ ,$$

the sets $\mathcal{T}(0)$ and $\mathcal{T}'(0)$ are 3–dimensional *resonant* tori, on which the flow is given by

$$\Big((I_1, I_2, 0, 0), (\varphi_1, \varphi_2, \varphi_3, 0)\Big) \in \mathcal{M} \to \Big((I_1, I_2, 0, 0), (\varphi_1 + I_1 t, \varphi_2 + I_2 t, \varphi_3, 0)\Big) \ .$$

Notice that $\mathcal{T}(0)$ and $\mathcal{T}'(0)$ are invariant but *not* minimal for H_0.

3.2 The KAM construction

Here we construct, for each $(\omega, \mathring{\ }) \in \Omega_\gamma^*$ and for ε small enough, invariant 3–dimensional "whiskered" tori, which are a (non–analytic) continuations of the H_0–invariant tori $\{(I_1, I_2, I_3) = \omega,\ I_4 = 0 = \varphi_4\}$. To do this, introducing an *auxiliary parameter* μ, we consired the *new Hamiltonian*

$$\widetilde{H}(I,\varphi;\varepsilon,\mu) \equiv \frac{1}{2}|I|^2 + \varepsilon(\cos\varphi_4 - 1) + \mu\sum_{j=1}^{3}\cos(\varphi_j + \varphi_4) \ , \tag{3.7}$$

and prove various statements, which will be analytic in μ for $\varepsilon > 0$ fixed. Furthermore, the results will hold for all $\varepsilon > 0$ (small enough) and for $|\mu| \leq \mu_0$ with $\mu_0 > \varepsilon^9$ so that one can set $\mu = \varepsilon^9$ recovering the original Hamiltonian (3.1).

Theorem 3.1 (KAM) *There exists $\varepsilon_0 > 0$ and, for all $0 < \gamma < \varepsilon < \varepsilon_0$, a constant $\mu_0 \equiv \mu_0(\gamma, \varepsilon)$ such that the following statements hold true.*

If $|\mu| \leq \mu_0$, then, for all $(\omega, \mathring{\ }) \in \Omega_\gamma^*$, there exists a unique $\widetilde{H}$–invariant 3–dimensional torus $\mathcal{T}_\mu(\omega, \mathring{\ })$, on which $\phi^t_{\widetilde{H}}$ is (analytically) conjugated to the Kronecker flow $\theta \in \mathbb{T}^3 \to$

[23]This is necessary since we shall eventually work with *fixed energy*.

$\theta+\omega_{\hat{t}}$. *Furthermore, the torus $\mathcal{T}_\mu(\omega,\mathring{\ })$ posses aymptotics manifolds ("whiskers") that may be locally described as follows. One can construct a function*[24]

$$\Big(\theta,(x^+,x^-),\mu,(\omega,\mathring{\ })\Big)\in\mathbb{T}^3\times B^2_{2r}\times B^1_{\mu_0}\times\Omega^*_\gamma\quad\to\quad\zeta_\mu(\theta,x^+,x^-;\omega,\mathring{\ })\in\mathcal{M}\ ,$$

which is C^3 in all its argument and, for $(\omega,\mathring{\ })\in\Omega^*_\gamma$ fixed, is real–analytic on $\mathbb{T}^3_\kappa\times\widehat{B}^2_{2r}\times\widehat{B}^1_{\mu_0}$; here κ is a suitable (ε–independent) number while $r=$ const $\varepsilon^{1/4}$. For $(\omega,\mathring{\ })\in\Omega^*_\gamma$, the torus $\mathcal{T}_\mu(\omega,\mathring{\ })$ and its local stable/unstable whisker $W^{+/-}_{\mu,\mathrm{loc}}(\omega,\mathring{\ })$ are given by[25]

$$\begin{aligned}
&\mathcal{T}_\mu(\omega,\mathring{\ })\equiv\{\zeta_\mu(\theta,0,0;\omega,\mathring{\ }):\ \theta\in\mathbb{T}^3\}\ ,\\
&W^+_{\mu,\mathrm{loc}}(\omega,\mathring{\ })\equiv\{\zeta_\mu(\theta,x^+,0;\omega,\mathring{\ }):\ \theta\in\mathbb{T}^3\ ,|x^+|<2r\}\ ,\\
&W^-_{\mu,\mathrm{loc}}(\omega,\mathring{\ })\equiv\{\zeta_\mu(\theta,0,x^-;\omega,\mathring{\ }):\ \theta\in\mathbb{T}^3\ ,|x^-|<2r\}\ .
\end{aligned}$$

The $\phi^t_{\widetilde{H}}$–dynamics on $W^{+/-}_{\mu,\mathrm{loc}}(\omega,\mathring{\ })$ is given by

$$\phi^t_{\widetilde{H}}\ \zeta_\mu(\theta,x^+,x^-;\omega,\mathring{\ })=\zeta_\mu(\theta+\omega_{\hat{t}},x^+\,e^{-\Lambda t}\ ,x^-\,e^{\Lambda t};\omega,\mathring{\ })\ ,\tag{3.8}$$

for all t such that $(x^+e^{-\Lambda t},x^-e^{\Lambda t})\in B^2_{2r}$; here the "Liapunov exponent" $\Lambda\equiv\Lambda_\mu(\xi;\omega,\mathring{\ })$, which depends on the product $\xi\equiv x^+x^-$, on μ, ω, $\mathring{\ }$, is a C^3 function of $(\xi,\mu,\omega,\mathring{\ })$, and is real–analytic in $(\mu,\xi)\in B^1_{\mu_0}\times B^1_{4r^2}$ for fixed $(\omega,\mathring{\ })\in\Omega^*_\gamma$; moreover $\Lambda_0(0;\omega,\mathring{\ })=\sqrt{\varepsilon}$.

This theorem is essentially a classical KAM result for "codimension 1" invariant partially hyperbolic tori applied after a canonical transformation bringing the system in a suitable normal form[26].

We shall need a *quantitative formulation* of the above result, which means to have an *explicit control on the ε–dependence* of the relevant quantities and in particular of μ_0. From standard KAM theory there follows (see also Remarks 3.1 and 3.4 below)

Proposition 3.1 *There exist two (universal) constants $p_1,p_2\geq 1$ such that, if*

$$m\geq 2\ ,\qquad\text{and}\quad m>\frac{p_1+p_2}{s}\ ,\tag{3.9}$$

then in Theorem 3.1 one can take

$$\mu_0=\varepsilon^m\ ,\qquad\text{provided}\quad\gamma\equiv\text{const}\ \varepsilon^{\frac{ms-p_2}{p_1}}\ .\tag{3.10}$$

---

[24] We shall use the following notations: B^n_r and $\widehat{B}^n_r$ denote the open ball in, respectively, $\mathbb{R}^n$ and $\mathbb{C}^n$ of radius r; $\mathbb{T}^n_\kappa$ denotes the complex neighbourhood of the n torus given by $\{z\in\mathbb{C}^n$: s.t. $|\mathrm{Im} z_i|<\kappa\}$; if A is *any* set in $\mathbb{R}^n$ we say that $f\in C^k(A)$ if there exists an open set $U\supset A$ such that $f\in C^k(U)$.

[25] Notice that in [?] it is used the opposite convention, namely, $W^{-/+}$ are denoted in [?] $Y^{+/-}$ (indicating the unstable/stable whisker).

[26] See, e.g., [?] and references therein; see also Remark 3.4 below. A detailed proof may be found in [?], §5

Notice that by choosing s large enough *we can assume that the radius of μ–analyticity is ε^2*:

$$m = 2 \ , \qquad \mu_0 = \varepsilon^2 \ . \tag{3.11}$$

Remark 3.1 I choosed not to include the details of the proofs of Theorem 3.1 and Proposition 3.1 because: 1) the details are straightforward, but (as in all KAM quantitative statements) rather lengthy; 2) the needed KAM techniques are very well established and no new ideas are needed (in particular, a detailed version may be found in § 5 of [**?**]); 3) the essence of the results presented here does not depend on the actual values of p_1 and p_2: different computations of the values of the p_i's (such as, say, those that may be read off [**?**]) would reflect, at most, in changing the value of s (which amounts to consider different instability zones in action–space).

To have concrete numbers in mind, one might take the values (which might come out by carrying out the scheme outlined in Remark 3.4)

$$p_1 = 2 \ , \qquad p_2 = 6 \ . \tag{3.12}$$

Remark 3.2 (On the $\mu = 0$ limit) Here we describe the whiskered tori for $\mu = 0$. As already mentioned above

$$\mathcal{T}_0(\omega, \mathring{}) = \{(I_1, I_2, I_3) = \omega^{\circ}, \ _{I_4=0=\varphi_4 \ , \ (\varphi_1,\varphi_2,\varphi_3)\in\mathbb{T}^3\}} \ _{.In order to describe}$$

ζ_0 (and hence $W_0^{\pm}$) let us consider the canonical transformation[27] $p(x^+, x^-)$, $q(x^+, x^-)$ transforming, in a neighbourhood of the origin, the pendulum $\frac{1}{2}p^2 + \varepsilon[\cos q - 1]$ in *standard hyperbolic normal form*:

$$\frac{1}{2}p^2(x^+, x^-) + \varepsilon[\cos q(x^+, x^-) - 1] = \sqrt{\varepsilon}\ x^+x^- + \varepsilon g\Big(\frac{x^+x^-}{\sqrt{\varepsilon}}\Big) \ , \tag{3.13}$$

where $g(\xi)$ is a function real–analytic on a (complex) ball of radius $\tilde{r}$ (independent of ε) around the origin and p and q are real–analytic in a 2–ball of radius $\tilde{r}\varepsilon^{\frac{1}{4}}$ around the origin[28]. Hence, the function ζ_0 is given by

$$\zeta_0(\theta, x^+, x^-; \omega, \mathring{}) = \Big((\omega; p(x^+, x^-)), (\theta, q(x^+, x^-))\Big) \ , \quad \Lambda_0(\xi; \omega, \mathring{}) = \sqrt{\varepsilon}\Big[1 + \frac{dg}{d\xi}\Big(\frac{\xi}{\sqrt{\varepsilon}}\Big)\Big] \ .$$

[27] I.e., a local real–analytic diffeomorphism of a neighbourhood of the origin such that $dp \wedge dq = dx^+ \wedge dx^-$.

[28] To construct the transformation $p(x^+, x^-)$, $q(x^+, x^-)$, consider, first, the unscaled problem of finding a canonical transformation $p_0(x^+, x^-)$, $q_0(x^+, x^-)$ transforming the ordinary pendulum (in a neighbourhood of the origin) in normal hyperbolic coordinates:

$$\frac{1}{2}p_0^2 + \cos q_0 - 1 = x^+x^- + g(x^+x^-) \ . \tag{3.14}$$

The explicit study of such a problem goes back essentially to Jacobi (compare with Appendix 3 of [?] and references therein). A local description of (p_0, q_0) (without using Jacobi's elliptic integrals) is immediatly obtained by, first, determining a formal (Birkhoff) expansion

$$p_0 = \frac{x^- + x^+}{\sqrt{2}} + \sum_{k\geq 2} \alpha_k(x^+, x^-) \ , \qquad q_0 = \frac{x^- - x^+}{\sqrt{2}} + \sum_{k\geq 2} \beta_k(x^+, x^-) \ ,$$

Remark 3.3 Using the KAM linearization (3.8) it is not difficult to show that the tori $\mathcal{T}_\mu(\omega, \overset{\circ}{})$ are *transition tori* in the sense of [?] (see, e.g., § 8 of [?]). This implies that if $\mathcal{T}_1$,...,$\mathcal{T}_n$ are different KAM tori $\mathcal{T}_\mu(\omega_i, \overset{\circ}{i})$ (contained in the same energy surfaces) such that $W^-_\mu(\omega_i, \overset{\circ}{i})$ intersect transversally $W^+_\mu(\omega_{i+1}, \overset{\circ}{i+1})$ then $\mathcal{T}_1$,...,$\mathcal{T}_n$, together with a suitable set of corresponding connecting orbits (lying in $W^-_\mu(\omega_i, \overset{\circ}{i}) \cap W^+_\mu(\omega_{i+1}, \overset{\circ}{i+1})$), form a *transition chain*.

Remark 3.4 (On the KAM results) A short scheme of proof of Theorem 3.1 and Proposition 3.1 is the following.
(i) The canonical change of variables $I_4 = p(x^+, x^-)$, $\varphi_4 = q(x^+, x^-)$ costructed in Remark 3.2 [plus the identity map in $\bar{I} \equiv (I_1, I_2, I_3)$ and $\bar{\varphi} \equiv (\varphi_1, \varphi_2, \varphi_3)$] transforms the Hamiltonian $\widetilde{H}$ of (3.7) into the Hamiltonian

$$H_1 \equiv \frac{|\bar{I}|^2}{2} + g_1(x^+x^-, \sqrt{\varepsilon}) + \mu f_1(\bar{\varphi}, x^+, x^-; \sqrt{\varepsilon}, \mu) ,$$

where $g_1(x^+x^-, \sqrt{\varepsilon}) = \sqrt{\varepsilon}\ x^+x^- + \varepsilon g\Big(\frac{x^+x^-}{\sqrt{\varepsilon}}\Big)$ [see (3.13)], and f_1 is a linear combination of $\cos\varphi_i$ and $\sin\varphi_i$, with $i \leq 3$, with coefficients which are real–analytic in the product x^+x^-.
(ii) By classical (Poincaré–Birkhoff) perturbation theory, one can find a symplectic transformation, $(x^+, x^-, \bar{I}, \bar{\varphi}) \to (y, x, \bar{I}', \bar{\varphi}')$, μ–close to the identity, conjugating H_1 to a Hamiltonian

$$H_2 \equiv h_2(\bar{I}', yx; \sqrt{\varepsilon}, \mu) + \mu^s f_2(\bar{I}', \bar{\varphi}', y, x, \sqrt{\varepsilon}, \mu) ,$$

where h_2 and f_2 are analytic in all their arguments (except $\sqrt{\varepsilon}$) and h_2 is μ–close to $H_1|_{\mu=0}$. The "small divisors" appearing in such a symplectic transformation[29], have the form $i\bar{I}' \cdot k + \ell\sqrt{\varepsilon}y'x'$ with $k \in \mathbb{Z}^2$, $\ell \in \mathbb{N}$, $|k| + \ell > 0$ and $|k| \leq s - 1$. Thus the domain of definition (and analyticity) of H_2 is given, for $0 < \varepsilon < \varepsilon_0$, by[30] $\Omega \times \mathbb{T}^3 \times B^2_{2r} \times B^1_{\bar{\mu}}$ and $\bar{\mu}$ may be choosen proportional to ε^2.

(with α_k and β_k homogeneous polynomial of degree k) so that (3.14) holds (in the sense of formal power series of x^+ and x^-), and, then, using (say) "Cauchy's majorant method" (see, e.g., [?]) so as to establish the convergence of the power series in ball of radius $\tilde{r}$ around the origin $x^+ = 0 = x^-$. The canonical transformation $p(x^+, x^-)$, $q(x^+, x^-)$ is then obtained by setting

$$p(x^+, x^-) = \sqrt{\varepsilon} p_0\Big(\frac{x^+}{\varepsilon^{\frac{1}{4}}}\ ,\ \frac{x^-}{\varepsilon^{\frac{1}{4}}}\Big)\ ,\quad q(x^+, x^-) = q_0\Big(\frac{x^+}{\varepsilon^{\frac{1}{4}}}\ ,\ \frac{x^-}{\varepsilon^{\frac{1}{4}}}\Big)\ .$$

The number r of Theorem 3.1 can be taken to be

$$r = \frac{1}{4}\tilde{r}\varepsilon^{\frac{1}{4}}\ .$$

[29]The transformation can be found, e.g., by solving up to order μ^{s-1} the Hamilton–Jacobi equation for a generating function of the form $\bar{I}' \cdot \bar{\varphi} + yx^- +$ polynomial in μ of order $(s-1)$ vanishing at $\mu = 0$.
[30]Recall the definition of the set Ω given in (3.2).

(iii) At this point one can use a KAM "fast–converging" scheme (iterating, e.g., a symplectic transformation of the type mentioned in footnote 29 with s replaced by 2) to construct, simultaneously for $\omega_{\mathring{\in}}\Omega^*_\gamma$, invariant tori with flow conjugated to $t \to \omega_{\mathring{t}}$, and their local whiskers, which will be μ^s close to the local whiskers for $H_2|_{\mu=0}$. As it is well known, the smallness requirement (or "KAM condition") necessary in order to carry out the scheme, is of the form

$$\text{const}\,\frac{\mu_0^s\varepsilon^{-p_2}}{\gamma^{p_1}}\ .$$

where μ_0 is a lower bound on the analyticity domain in μ, p_1 and p_2 are constants depending only on the structure of $H_2|_{\mu=0}$. In fact, typically $p_1 = 2$ (reflecting the fact that KAM schemes are "quadratic schemes"). This observation explains the choice of γ in (3.10), once it has been set $\mu_0 = \varepsilon^m$ with $m \geq 2$ (which is needed to perform step (ii) above).

3.3 Extended local whiskers

Here we construct a parametrization (analytic in μ) of the local whiskers covering a region, which, projected over the φ_4 coordinate, contains the closed interval $[-\pi, \pi]$. As it is standard, we shall use the flow to "stretch" the parametrization given in Theorem 3.1.

Let $p(x^+, x^-)$, $q(x^+, x^-)$ be the map described in Remark 3.2, let[31]

$$\bar{z} \equiv \Big(p(0, r), q(0, r)\Big)\ , \tag{3.15}$$

and let T ($\sim \varepsilon^{-\frac{1}{2}} \log \varepsilon^{-1}$) be the (positive) time for which

$$\phi^T_{\text{pend}}(\bar{z}) = (2\sqrt{\varepsilon}, \pi)\ , \tag{3.16}$$

where ϕ^t_{pend} denotes the (bidimensional) flow of the pendulum $\frac{1}{2}p^2 + \varepsilon[\cos q - 1]$. Define

$$\zeta^{\pm}_\mu(\theta, x^+, x^-; \omega, \mathring{\ }) \equiv \phi^{\mp T}_{\widetilde{H}}\zeta_\mu(\theta, x^+, x^-; \omega, \mathring{\ })\ . \tag{3.17}$$

Then $\zeta^\pm_\mu$ has the same smoothness and analyticity properties of ζ_μ, the analogous of (3.8) holds[32] and, furthermore (recalling the notation of footnote 18)

$$\begin{aligned}
&\pi_{\varphi_4}\{\zeta^+_\mu(\theta, x^+, 0; \omega, \mathring{\ }):\ |x^+| < 2r\} \supset [-\pi, \pi]\ , \quad \forall\ \theta\ ,\\
&\pi_{\varphi_4}\{\zeta^-_\mu(\theta, 0, x^-; \omega, \mathring{\ }):\ |x^-| < 2r\} \supset [-\pi, \pi]\ , \quad \forall\ \theta\ .
\end{aligned}$$

[31]Recall that the (x^+, x^-)–analyticity domain of the function ζ_μ in Theorem 3.1 is a ball of radius $2r$ with $r \sim \varepsilon^{\frac{1}{4}}$.

[32] I.e., for all t such that $(x^+e^{-\Lambda t}, x^-e^{\Lambda t}) \in B^2_{2r}$, one has

$$\phi^t_{\widetilde{H}}\ \zeta^\pm_\mu(\theta, x^+, x^-; \omega, \mathring{\ }) = \zeta_\mu(\theta + \omega_{\mathring{t}}, x^+e^{-\Lambda(x^+x^-)t}\ , x^-e^{\Lambda(x^+x^-)t}; \omega, \mathring{\ })\ ,$$