Hard-core scattering for N-body systems *

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Abstract

We prove propagation properties (maximal and minimal velocity bounds) for pseudo-resolvents associated to N-body hamiltonians with short-range potentials that are infinite on a star-shaped domain centred at the origin. Motivated by the fact that the invariance principle holds for usual N-body systems, we define the cluster wave operators in terms of pseudo-resolvents and prove that they exist and are asymptotically complete. For any cluster decomposition a, these operators intertwine the hard-core pseudo-selfadjoint hamiltonians corresponding to the pair of pseudo-resolvents R, R_a, and equal the Abel operators constructed in terms of hamiltonians.

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1 Introduction

One of the most important goals in scattering theory is the study of the asymptotic behavior (when $t \to \pm\infty$) of $e^{-itH}\psi$, where ψ is an arbitrary state from the orthogonal complement of the space of eigenvectors of the hamiltonian H. More precisely, we are interested in finding a family $\{H_a\}$ of selfadjoint operators, with simpler (and known) spectral and evolution properties, such that for any state ψ a family of vectors $\{\psi_a^\pm\}$ should exist, for which the convergences:

$$\|e^{-itH}\psi - \sum_a e^{-itH_a}\psi_a^\pm\| \stackrel{t\to\pm\infty}{\longrightarrow} 0 \tag{1.1}$$

be satisfied. If this takes place, then we say that the system is *asymptotically complete*.

The particularity of the N-body hamiltonians is that they are a sum of a differential operator (with excellent dispersion properties) and a perturbation that does not vanish (when $|x| \to \infty$) along certain directions of the configuration space X. This makes us think that, if asymptotic completeness holds for such systems, then $e^{-itH_a}\psi_a^\pm$ should be asymptotically localized within some cones centred on the classical trajectories.

These geometrical ideas allowed V.Enss to show that (1.1) is true for three-body quantum systems with potentials that decay slightly faster than the Coulomb interaction. After that, asymptotic completeness for N-body short-range quantum systems has been proved in 1987 by I.Sigal and A.Soffer (ref. [SiSo1]), and in the following years, many people tried to simplify or to extend their proof for more complicated many-body problems. Indeed, there was first an effort of making the theory more "readable", done by J.Derezinski in [De1]. Then, G-M.Graf, using jointly ideas from the earlier works of V.Enss ([E1] - [E4]) and from [SiSo1] but also from more recent papers of Sigal and Soffer (like [SiSo3] and especially [SiSo2]), succeeded in giving in [G] a remarkable time-dependent-like proof of the quoted result, which differed from the previous proofs by several (important) aspects. We shall emphasize only the fact that in [G], some of the main propagation properties have been obtained without the use of the Mourre estimate, i.e. independently of an intimate knowledge of the spectral properties of the hamiltonian. These properties where sufficient for showing the existence of the cluster wave operators but not for their completeness. Indeed, for the last result, a propagation property involving jointly a time-dependent localisation in position and a localisation in the total energy was needed, and for proving it a good knowledge of the spectrum of the N-body hamiltonian was crucial. Actually, this is the only place where Graf invoques the Mourre estimate (in order to obtain (local) positivity for the commutator of the hamiltonian with the generator of the dilations group) and by this means he eliminates the decay hypothesis imposed in [SiSo1] on the second derivative of the potential. Further, using refined results on the Mourre theory due to W.Amrein, Anne Boutet de Monvel and V.Georgescu (see ref. [ABG1] and also [BGM] for optimality) we have shown in [If], on the lines of [G], that no condition on the derivatives of the potential was needed in order to prove completeness for the Agmon-type systems. Moreover, since locally the potentials where allowed to be as singular as the the kinetic energy permits it, the question of the validity of a statement on asymptotic completeness for much singularly perturbed systems (as the hard-core N-body quantum systems) arises naturally. Indeed, the interest for such problems is rather old, going back e.g. to the works of W.Hunziker (ref. [Hu]) and especially of D.W.Robinson, P.Ferrero and O. de Pazzis (see ref. [R], [FPR]), where, under rather restrictive assumptions on the geometry of the potentials (spheric symmetry, the supports of the singularities where cylinders centred on the subspaces of the relative movement of the clusters) and on the forces (repulsivity), the absence of the singular continuous spectrum and the existence and the completeness of the wave operators corresponding to the *elastic* channel have been established. But this is, of course, a very simplified case, because even if the problem was posed in an N-body context, the above hypotheses transformed it in an one-channel scattering problem. Very recently, Anne Boutet de Monvel, V. Georgescu and A. Soffer using both the locally conjugate operator method and an algebraic approach (which appears naturally in the N-body context), have succeeded in giving in [BGS] a complete spectral analysis for this type of highly singular hamiltonians. More precisely, it is proven that under quite reasonable smoothness conditions imposed on the border of the supports of the singularities, the generator of the dilations group is (in some weak sense) conjugated to the hard-core N-body hamiltonian, which proves to be sufficient to obtain a limiting absorption principle (even in an optimal form). Then, absence of the singular-continuous part of the spectrum and local decay follow in a standard way. Our task is to continue this work by studying the scattering properties of these systems.

We shall begin by describing the geometrical particularities of the configuration space X (an euclidian space) related to the N-body problem. Let us denote by $\mathscr{L}$ a finite partially ordered index set and demand it to be a lattice. Take then a family $\{X^a\}_{a\in\mathscr{L}}$ of subspaces of X such that $X^{\sup\{a,b\}} = X^a + X^b$, and $\{0\}$ and X correspond to $\min\mathscr{L} \equiv a_{\min}$ and $\max\mathscr{L} \equiv a_{\max}$ respectively. In the usual N-body situation, X is the space of the configurations of the set of N particles relative to the center of mass coordinate system, $\mathscr{L}$ is the lattice of partitions of the set $\{1, \ldots, N\}$, X^a is the subspace of X consisting of the configurations which describe the internal motion of the clusters (fragments) of the partition a and, finally, X_a (the orthogonal of X^a in X with respect to a well chosen scalar product) can be identified with the space of configurations of the relative motions of the clusters. Let us denote for any $a \in \mathscr{L}$ by π^a the orthogonal projection on X^a. Further, according to S.Agmon (see [Ag]) a N-body type hamiltonian is defined as the sum between the (positive) Laplace-Beltrami operator Δ and a family $\left\{\tilde{V}(a)\right\}_{a\in\mathscr{L}}$ of operators which factorize as $\tilde{V}(a) = \tilde{V}^a \circ \pi^a$. Here we considered the simplest case when $\tilde{V}^a$ is the operator (in $\mathscr{H}(X^a)$) of multiplication by a function having a good decay at infinity in all directions of X^a. Although this assumption is currently used in many of the papers dedicated to this subject, in the more recent ones it is shown that the same results can be obtained if $\tilde{V}^a$ is a reasonable differential operator. As for the hard-core systems, the physical picture of clusters formed by particles that cannot get arbitrarily close one to each other is modelled by (positive) singularities of the potentials, having as supports compacts of X^a; of course, a short and long range part can be added to these singular potentials. A precise definition is obtained by using a limiting process, in which the hard-core hamiltonian H is seen as a limit, in strong resolvent sense, of the family of self-adjoint operators in $\mathscr{H}(X)$:

$$H_\alpha = \Delta + \sum_{a\in\mathscr{L}} (V(a) + \alpha\chi(a)) = \tilde{H} + \alpha \sum_{a\in\mathscr{L}} \chi^a \circ \pi^a. \tag{1.2}$$

We have denoted by $\tilde{H}$ a standard N-body hamiltonian which, under natural assumptions on the symmetric operators V^a, it becomes a selfadjoint, bounded from below operator with form-domain the Sobolev space of order one $\mathscr{H}^1(X)$. Notice that, when tending to infinity, the parameter $\alpha \geq 0$ will increase the value of the cylindrically supported perturbation $\chi(a)$, where χ^a denotes the operator of multiplication with the characteristic function of a compact K^a from X^a.

It has been proven (in [BGS], Lemma 3.7 and Proposition 3.8) that for *each* z in the complement in $\mathbb{C}$ of $[\inf\sigma(\tilde{H}), \infty)$, the limit $R(z) \equiv \lim_{\alpha\to\infty}(z - H_\alpha)^{-1}$ exists in the strong sense in $B(\mathscr{H}^{-1}, \mathscr{H}^1)$ and in the norm of each of the Banach spaces $B(\mathscr{H}^s, \mathscr{H}^t)$ for $-1 \leq s \leq t \leq 1$ and $t - s < 2$. Actually, $R(z)$ is a *self-adjoint pseudo-resolvent family*, i.e. it satisfies the first resolvent identity and $R(z^*) = R(z)^*$. It is known (see [HP]) that the closure in $\mathcal{H}$ of $\operatorname{Ran} R(z)$ is a *proper* subspace of $\mathscr{H}$ (let us denote it by $\mathscr{H}_\infty$) which does not depend on z, and which coincides with the closure of the domain of a self-adjoint operator H for which $R(z)\,|_{\mathscr{H}_\infty} = (z-H)^{-1}$ and $R(z)\,|_{\mathscr{H}\ominus\mathscr{H}_\infty} = 0$. We will call H a *pseudo-selfadjoint* operator on $\mathscr{H}$, and this will be the hamiltonian modelling a hard-core Agmon-type problem. Nevertheless, we shall refer to its spectral properties as to those of the selfadjoint operator H which acts in the proper subspace $\mathscr{H}_\infty$.

We emphasize that one of the difficulties in studying the spectral and scattering properties of such operators is the fact that they are not densely defined. Moreover, we cannot say a priori that hamiltonians constructed as above factorize in the same tensor product form as those from the family $\{H_\alpha\}$. Actually this holds. Indeed, as in the usual N-body problem, using the limiting process described before, it is possible to construct for each $a \in \mathscr{L}$ a pseudo-selfadjoint hamiltonian H^a, which corresponds to the hard-core problem relative to the sublattice $\mathscr{L}_a = \{b \in \mathscr{L} \mid b \leq a\}$. On the other hand, it is shown that for any $a \in \mathscr{L}$, the family of resolvents $\{R_{a,\alpha}\}_{\alpha\geq 0}$ of the (genuine) selfadjoint operators

$$H_{a,\alpha} = H^a_\alpha \otimes_a 1 + 1 \otimes_a (\pi_a \nabla)^2$$

tends in norm in $B(\mathscr{H}^{-1}(X), \mathscr{H}^\theta(X))$, for any $\theta < 1$, to a pseudo-resolvent R_a (to which corresponds the pseudo-selfadjoint subhamiltonian H_a) whenever the convergence $H^a_\alpha \to H^a$ takes place in the norm-resolvent sense in $B(\mathscr{H}^{-1}(X^a), \mathscr{H}^\theta(X^a))$. This implies

$$H_a = H^a \otimes_a 1 + 1 \otimes_a \Delta_a, \tag{1.3}$$

where Δ_a denotes the Laplace-Beltrami operator in $\mathscr{H}(X^a)$, and where the above tensor product sum is an operator defined in the proper subspace $\overline{D(H^a)} \otimes \mathscr{H}(X_a)$ of $\mathscr{H}(X)$, with operator domain $D(H_a) = D(H^a) \otimes \mathscr{H}^2(X_a)$.

Then, the set of *thresholds* of H is defined as

$$\tau(H) = \bigcup_{a \in \mathscr{L}\setminus\{a_{\max}\}} \sigma_p(H^a), \tag{1.4}$$

whereas the set of *critical values* of H, denoted by $\mathcal{C}(H)$ will be the union of $\tau(H)$ with $\sigma_p(H)$, the point spectrum of H.

The main ingredient for the study of the spectral and scattering properties of the hamiltonian is the Mourre estimate, which states (see (1.5) below) local positivity for the commutator of H with the generator of the dilations group A, the last one being defined as:

$$A = \frac{1}{2}(P \cdot Q + Q \cdot P) = \frac{1}{2} \sum_{j=1}^{\dim X} (P_j Q_j + Q_j P_j),$$

where Q_j is the operator of multiplication by the coordinate x_j (w.r.t. some orthonormal basis of X) and $P_j \equiv -i\frac{\partial}{\partial x_j} = \mathcal{F}^* Q_j \mathcal{F}$, with $\mathcal{F}$ the Fourier transform. Actually, this estimate stress *strict* positivity for the lower semicontinuous function $\rho^A_H : \mathbb{R} \to (-\infty, +\infty]$ on an open set of $\mathbb{R}$, where for all $\lambda \in \mathbb{R}$:

$$\rho^A_H(\lambda) \overset{def}{=} \sup\{\mu \in \mathbb{R} \mid \exists f \in C_0^\infty(\mathbb{R};\mathbb{R}),\ f(\lambda) \neq 0 \text{ s.t. } f(H)[iH, A]f(H) \geq \mu f(H)^2\}. \tag{1.5}$$

In [ABG2], [BG3], [Ma], an extensive study of this function is made. It is also important to point out that under the assumption of strong-C^1 regularity of the hamiltonian w.r.t. A, even if H is a pseudo-selfadjoint operator, the identity:

$$[A, R(z)] = R(z)[A, H]R(z) \tag{1.6}$$

is valid for any $z \in \mathbb{C} \setminus \sigma(H)$. We refer to §5 from [BGS] for the precise definitions of the above commutators and for the proof of the Mourre estimate in the context of the hard-core N-body systems. In fact, the result we needed and that we will intensely use is the strict positivity of $\rho_R^A(\lambda)$ on the set $\mathbb{R} \setminus \mathcal{C}(R)$.

Another difficulty arising from the fact that the limit of $\{H_\alpha\}$ is only pseudo-selfadjoint in $\mathscr{H}$ comes from the way we have to interpret the limit of $\{e^{itH_\alpha}\}$ and, correspondingly, the way we have to define the cluster wave operators. It is not a trivial fact (see [Da]) that, for any $a \in \mathscr{L}$, the family of evolution groups generated by $\{H_{a,\alpha}\}$ has a limit, but this limit exists only on $\mathscr{H}_{a,\infty}$, the closure of $\mathrm{Ran}R_a$ in $\mathscr{H}$. Moreover, taking into account the inclusion $\mathscr{H}_{b,\infty} \subseteq \mathscr{H}_{a,\infty}$, true for any $a, b \in \mathscr{L}$ with $a \le b$, and the Theorem 3.23 (ii) from [Da], we see that the domain of the limit $\lim_{\alpha \to +\infty} e^{itH_\alpha}$ is, a priori, included in the range of $\lim_{\alpha \to +\infty} e^{itH_{a,\alpha}}$ even when this one is applied to the vectors of $\mathscr{H}_\infty$.

As for the cluster wave operators, in the usual N-body context (i.e. for any α finite) they are defined as

$$\Omega_{a,\alpha}^{\pm} \equiv \Omega^{\pm}(H_\alpha, H_{a,\alpha}; \mathcal{E}_{a,\alpha}) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_\alpha} e^{-itH_{a,\alpha}} \mathcal{E}_{a,\alpha}. \tag{1.7}$$

Notice that since $\mathcal{E}_{a,\alpha} = E_{pp}(H_\alpha^a) \otimes_a 1$ commutes with $H_{a,\alpha}$ (which has purely absolutely continuous spectrum) and also with bounded functions of $H_{a,\alpha}$, as a consequence of the Hilbert space isomorphisms

$$\mathscr{H}(X) \cong L^2((X_a, d\xi_a); \mathscr{H}(X^a)) \cong \int_{X_a}^{\oplus} \mathscr{H}(X^a)\, d\xi_a$$

(see [BGS], [Ge]), the limits (1.7) can be seen as wave operators with identificator $\mathcal{E}_{a,\alpha}$ (see [BaWo]). Moreover, for any $z \in (-\infty, \inf \sigma(\tilde{H}))$, the equality $E_{pp}(H_\alpha^a) = E_{pp}(R_\alpha^a(z))$ is true, so the identification operator is the same for the wave operators constructed in terms of hamiltonians and for those constructed in terms of resolvents. For the hard-core case, these identificators are no more equal, and we shall use the notation $\mathcal{E}_a$ for $E_{pp}(H^a) \otimes_a 1$, which projects $\mathscr{H}$ into $\mathscr{H}_\infty$. Then $\mathcal{E}_a = E_{H_a}(\mathbb{R})\, \mathcal{E}_a = E_{R_a}(\mathbb{R} \setminus \{0\})\, \mathcal{E}_a$, where $E_A(\Delta)$ denotes the spectral measure of the operator A on $\Delta \subset \mathbb{R}$.

It appears that we should try to redefine the hard-core wave operators in terms of known objects. This is also suggested by the invariance principle, which is true in the usual N-body case at least for the admissible function $(z - .)^{-1}$ (see [BaWo] for definitions), with z chosen as above. Indeed, suppose first that *both* strong limits $\Omega^{\pm}(H_\alpha, H_{a,\alpha}; \mathcal{E}_{a,\alpha})$ and $\Omega^{\pm}(R_\alpha, R_{a,\alpha}; \mathcal{E}_{a,\alpha})$ *exist*. Then, the corresponding absolute abelian limits exist also, and they are equal to the strong ones (see Corollary 6.14 in [BaWo]). We can thus use the *weak* form of the invariance principle (see Theorem 11.25 in [BaWo]) for this pair of operators, in order to get their *equality*, and thus the equality of the strong limits also.

It seems thus quite natural to *define* the wave operators corresponding to the hard-core case in terms of resolvents, and to prove existence and asymptotic completeness for $\Omega^{\pm}(R, R_a; \mathcal{E}_a) \equiv \Omega_a^{\pm}$.

As we shall see in the next sections, the way we do this uses an algebraic framework which is proper also for the usual N-body resolvents, so, in what follows, we will automatically prove not only the existence of $\Omega_a^{\pm}$, but also the existence of $\Omega^{\pm}(R_\alpha, R_{a,\alpha}; \mathcal{E}_{a,\alpha})$, who was previously taken as hypothesis for the weak invariance principle in the case of usual N-body systems.

Obviously, the intertwining property $\Omega_a^{\pm} = E_R(\mathbb{R} \setminus \{0\})\, \Omega_a^{\pm}$ is valid, and since for all $a \in \mathscr{L}$, $(z - H_a)R_a = E_{H_a}(\mathbb{R})$, $\Omega_a^{\pm}$ are partial isometries with final domain $\mathscr{H}_\infty$, which intertwine the pair H, H_a on the closure of the range of R_a. The connection between $\Omega_a^{\pm}$ and the wave operators defined in terms of hamiltonians as limits of

$$\mathcal{W}_a(t) = e^{itH} E_H(\mathbb{R})\, e^{itH_a} \mathcal{E}_a,$$

is made with the strong form of the invariance principle. Indeed, let $\mathcal{W}_a^{\pm}$ be the absolute abelian limit of $\mathcal{W}_a(t)$, i.e.

$$\lim_{\varepsilon\to+0}(\pm\varepsilon)\int_0^{\pm\infty} e^{\pm\varepsilon t}\,\|\mathcal{W}_a(t)\psi-\mathcal{W}_a^{\pm}\psi\|^2\,dt=0. \tag{1.8}$$

Then, in the Appendix 6.2 we will show that the existence of $\Omega_a^{\pm}$ implies existence of $\mathcal{W}_a^{\pm}$ and their equality. This shows that the strong limits of $\mathcal{W}_a(t)$, if they exist, they are partial isometries with final domain $\mathscr{H}_\infty$. Actually, this can be shown also directly using a reasonement similar to that described in the proof of Lemma 4.1.

Finally, we would like to remark that despite their boundedness, the resolvents are not always comfortable objects to work with, mainly because of their non-local character. This feature becomes critical when one wants to prove, on the lines of [G], [De2], [Z], the following propagation theorem, that we consider as being the main result of the paper.

Theorem 1.1 *Let $a\in\mathscr{L}$ be arbitrarily chosen. If $\theta\in C_0^\infty(\mathbb{R}\setminus\mathcal{C}(R_a))$ and if $J\in C_0^\infty(X)$ has its support localised sufficiently close to the origin and outside any of the subspaces of the family $\{X_b\}_{b\in\mathscr{L}\setminus\mathscr{L}_a}$, then the estimate:*

$$\int_1^\infty \frac{dt}{t}\|J\big(\tfrac{Q}{t}\big)\,\theta(R)\,e^{-itR}\psi\|^2\le C\|\psi\|^2 \tag{1.9}$$

is true for some positive constant C and for all $\psi\in\mathscr{H}(X)$.

In Section 5 a more precise form of this theorem is stated and proved, and it is also shown that as a consequence of a particular case of it, the *asymptotic completeness* statement:

$$\sum_{a\in\mathscr{L}\setminus\{a_{\max}\}}\Omega_a^{\pm}(\Omega_a^{\pm})^*=E_c(H) \tag{1.10}$$

is valid.

Let us pass in review the contents of the following sections. In the following section we expose (following [BG1,3,4] and [BGS]) the algebraic framework related to an N-body type problem, putting emphasis on the results we need in our paper. We also briefly remind the construction of the partition of unity in the configuration space due to G-M.Graf and list some of the properties of the vector field attached to this family of functions. Section 3 is devoted to the proofs of those propagation estimates that can be deduced without the use of the Mourre estimate. It is shown in Section 4 that the existence of the limits $\Omega_a^{\pm}$ is a consequence of these estimates. Finally, Section 5 is mainly devoted to the proof of the Theorem 1.1.

2 The algebraic and geometric frameworks

As we have already mentioned in the first section, for any $a\in\mathscr{L}$ the direct sum $X^a\oplus X_a$ determines a canonical isomorphisms of Hilbert spaces $\mathscr{H}(X)\cong\mathscr{H}(X^a)\otimes\mathscr{H}(X_a)\cong L^2(X_a;\mathscr{H}(X^a))$. Let us denote by $\mathbf{K}(X^a)$ the $C^\star$-algebra of compact operators on $\mathscr{H}(X^a)$ and let $\mathbf{T}(X_a)$ be the $C^\star$-algebra naturally associated to the translation group in $B(\mathscr{H}(X_a))$, i.e. the norm-closure (in $B(\mathscr{H}(X_a))\equiv B(X_a)$) of the $\star$-subalgebra of operators of the form $f(\pi_aP)=\mathcal{F}_{X_a}^*f(\pi_aQ)\mathcal{F}_{X_a}$, with $f\in C_\infty(X_a)$. Then, the norm-closure in $B(X)$ of the linear space generated by the operators $S\otimes_aT$ which correspond (through the first of the above isomorphisms) to $S\otimes T\in\mathbf{K}(X^a)\otimes\mathbf{T}(X_a)$, will be a $C^\star$-subalgebra of $B(X)$, named the algebra of a-semicompact operators and denoted by:

$$\mathscr{T}(a)=\mathbf{K}(X^a)\otimes_a\mathbf{T}(X_a). \tag{2.1}$$

Further, with the aid of the family $\{\mathscr{T}(b)\}_{b\in\mathscr{L}}$ the vector space sum

$$\mathscr{T}_a = \sum_{b\in\mathscr{L}_a} \mathscr{T}(b) \tag{2.2}$$

(with $\mathscr{L}_a = \{b \in \mathscr{L} \mid b \leq a\}$) is constructed for each $a \in \mathscr{L}$. It is shown (see [BG1,3]) that the above sum is direct in the topological sense and that the canonical projections $\mathcal{P}(b) : \mathscr{T} \to \mathscr{T}(b)$ (which assign to any $T \in \mathscr{T} \equiv \mathscr{T}_{a_{\max}}$ an unique $T(b) \in \mathscr{T}(b)$) are norm-continuous and satisfy $\mathcal{P}(b)[T^*] = T(b)^*$. We will refer to $\mathscr{T}(b)$ as to the *b-connected component* of $\mathscr{T}$. Moreover, for any $a \in \mathscr{L}$ the projection $\mathcal{P}_a \overset{def}{=} \sum_{b\in\mathscr{L}_a} \mathcal{P}(b)$ is a $\star$-homomorphism between $\mathscr{T}$ and its $C^\star$-subalgebra $\mathscr{T}_a$. For any $a \in \mathscr{L} \setminus \{a_{\max}\}$ a concrete expression for $\mathcal{P}_a$ was given by W.N.Polyzou (see (6) in [Po], and also [PSS], [BG1,3]). Notice that using the Möbius inversion formula (see Theorem 4.18 in [Ai]) we can retrieve any of the operators $T(a) \in \mathscr{T}(a)$ as a weighted sum of elements of the family $\{T_b\}_{b\in\mathscr{L}_a}$, the weight being the Möbius function $\mu(b, a)$.

The second important property of the family of algebras of semicompact operators is its *graduation* with respect to the semi-lattice $\mathscr{L}$, which means that for any $a, b \in \mathscr{L}$, the inclusion

$$\mathscr{T}(a)\, \mathscr{T}(b) \subseteq \mathscr{T}(\sup\{a, b\}) \tag{2.3}$$

is valid. We resume al these properties by saying that $\mathscr{T}$ is an *$\mathscr{L}$-graded $C^\star$-algebra.*

Let us go back now to the pseudo-selfadjoint operators H defined in Section 1. According to [BG3], we shall say that H is *affiliated to the $C^\star$-algebra $\mathscr{T}$* iff *all* the realisations of the $\star$-homomorphism $\phi : C_\infty(\mathbb{R}) \to B(X)$ belong to $\mathscr{T}$. But since the Stone-Weierstrass theorem ensures the existence of a bijection between the $\star$-homomorphisms $\phi : C_\infty(\mathbb{R}) \to B(X)$ and pseudo-resolvents (such that for any $f \in C_\infty(\mathbb{R})$ we get $\phi(f)\,|_{\mathscr{H}_\infty} = f(H)$ and $\phi(f)\,|_{\mathscr{H}\ominus\mathscr{H}_\infty} = 0$), and since H does not depend on the z from the pseudo-resolvent w.r.t. which it has been defined, an equivalent definition for the affiliation of H to $\mathscr{T}$ is: $R(z) \in \mathscr{T}$ for *some* $z \in \mathbb{C}\backslash\sigma(H)$. Remark that if H is affiliated to $\mathscr{T}$ then any of the members of the family $\{H_a\}_{a\in\mathscr{L}}$ are affiliated to the elements of $\{\mathscr{T}_a\}_{a\in\mathscr{L}}$ respectively. This is due to the fact that given two $C^\star$-subalgebras $\mathscr{T}$ and $\tilde{\mathscr{T}}$ of $B(X)$, to any pseudo-selfadjoint operator H affiliated to $\mathscr{T}$ it corresponds through the $\star$-homomorphism $\mathcal{P} : \mathscr{T} \to \tilde{\mathscr{T}}$ an unique pseudo-selfadjoint operator $\mathcal{P}[H]$, which is affiliated to $\tilde{\mathscr{T}}$ and for which $\mathcal{P}[\phi] = \tilde{\phi}$ is true on all $C_\infty(\mathbb{R})$. In [BGS] several affiliation criterions are given and it is also proved that the hard-core N-body hamiltonian is affiliated to the N-body algebra $\mathscr{T}$. As a consequence of this, a HVZ theorem is obtained for this context.

We are particularly interested in establishing the extension of the first resolvent identity to the context of $\mathscr{L}$-graded $C^\star$-algebras. For the case of the usual N-body systems, this is called the Weinberg-Van Winter equation (see ref. [Po], [BG1]). Actually, as we shall see below, borrowing the algebraical-combinatorial technique of deducing such kind of identity, it is possible to give (firstly for N-body hamiltonians $\tilde{H}$ with bounded perturbations), for any $a, b \in \mathscr{L}$, a precise meaning to the difference $\tilde{R}_a - \tilde{R}_b$ and to the a-connected component of $\tilde{R} \in \mathscr{T}$, in terms of a sum of regularising operators. These results will be important in the effort of getting useful commutation relations between hard-core pseudo-resolvents and multiplication operators.

Let us begin by introducing some special subsets of the lattice $\mathscr{L}$. We shall call a totally ordered subset $\mathscr{C}$ a *maximal chain in $\mathscr{L}$*, iff for any $a, b \in \mathscr{C}$, $b < a$, for which there is no $c \in \mathscr{C} \setminus \{a, b\}$ such that $b < c < a$, there is also no other $d \in \mathscr{L} \setminus \{a, b\}$ such that $b < d < a$. We define then the rank of the finite lattice $\mathscr{L}$ as $\max\{\operatorname{card}\mathscr{C} \mid \mathscr{C} \subseteq \mathscr{L},\ \mathscr{C} \text{ maximal}\} \equiv |\mathscr{L}|$. In the N-body case, $\mathscr{L}$ is the lattice of partitions of a set of N elements, for which $|\mathscr{L}| = N$. Since $\mathscr{L}$ is union of maximal chains $\mathscr{C}$, any $a \in \mathscr{L}$ will belong to at least one of those which satisfy $\max\mathscr{C} = \max\mathscr{L}$. Then, by convention, the *rank* of a in $\mathscr{L}$ is taken to be the number

$|a|_{\mathscr{L}} \equiv \max\{\mathrm{card}\,\mathscr{C} \mid \max\mathscr{C} = \max\mathscr{L},\, \min\mathscr{C} = a,\, \mathscr{C} \text{ maximal}\}$. This notion allows us to define for any $1 \leq n \leq |\mathscr{L}| \equiv N$, the n-th *level* of $\mathscr{L}$ as $\mathscr{L}(n) = \{b \in \mathscr{L} \mid |b|_{\mathscr{L}} = n\}$. We thus have $\mathscr{L} = \bigsqcup_{n=1}^{N} \mathscr{L}(n)$, where $\bigsqcup$ stands for 'disjoint union of sets'. Finally, we introduce for some arbitrarily fixed $a \in \mathscr{L}$ and any $b \in \mathscr{L}_a$, two special subsets of the lattice $\mathscr{L}_a$:

$$\begin{aligned} \mathscr{L}_a^b \equiv (\mathscr{L}_a)^b &= \{c \in \mathscr{L}_a \mid c \geq b\} && (2.4) \\ \mathscr{L}_{ba} &= \{c \in \mathscr{L} \setminus \mathscr{L}_b \mid \sup\{b, c\} = a\}. && (2.5) \end{aligned}$$

It is clear that $\mathscr{L}_a^b$ is a lattice in $\mathscr{L}_a$ and that $\mathscr{L}_{ba}$ is a bilateral ideal in $\mathscr{L}_a$. Notice also that, if $\mathscr{L}^a$ denotes $\mathscr{L}_{a_{\max}}^a$, then for all $a \in (\mathscr{L} \setminus \mathscr{L}^b) \cup \{b\}$ we have $\mathscr{L}_{ba} = \emptyset$ and for all $a \in \mathscr{L}^b \setminus \{b\}$, we have $\{a\} \subseteq \mathscr{L}_{ba}$. Let us put the sign $\not\sim$ between two elements of the lattice $\mathscr{L}$ whenever they are incomparable. The proof of the following identity will give us more intuition on the sets introduced at (2.5):

$$\mathscr{L}_a \setminus \mathscr{L}_b = \bigsqcup_{c \in \mathscr{L}_a^b \setminus \{b\}} \mathscr{L}_{bc}. \tag{2.6}$$

Indeed, for any $c \in \mathscr{L}_a^b \setminus \{b\}$, since for all $d \in \mathscr{L}_a^b \setminus \{c\}$ we have $\sup\{d, b\} = d \neq c$, we get $\mathscr{L}_a^b \cap \mathscr{L}_{bc} = \{c\}$. Secondly, to each of these c corresponds in $\{d \in \mathscr{L}_a \mid d \not\sim b\}$ the remainder of the set $\mathscr{L}_{bc}$, which is disjoint of any of the sets $\mathscr{L}_{b\tilde{c}}$ corresponding to another $\tilde{c}$ of $\mathscr{L}_a^b \setminus \{b\}$. For, supposing the existence of some $d \in \mathscr{L}_{bc} \cap \mathscr{L}_{b\tilde{c}}$, $\{b, d\}$ will form a pair having (in the case $c \not\sim \tilde{c}$) no least upper bound in $\mathscr{L}$. But this would mean that $\mathscr{L}$ is not a lattice. Finally, in order to prove the inclusion $\subseteq$ in (2.6) notice that we have already shown that the r.h.s. of (2.6) includes $\mathscr{L}_a^b \setminus \{b\}$, and that for any $d \in \mathscr{L}_a$, $d \not\sim b$, we have $\sup\{b, d\} \in \mathscr{L}_a^b$ and $d \in \mathscr{L}_{b,\sup\{b,d\}}$. The inverse inclusion is trivial.

Further, corresponding to the set $\mathscr{L}_{ba}$ we construct the following bilateral ideal of $\mathscr{L}_a$:

$$\mathscr{T}_{ba} = \sum_{c \in \mathscr{L}_{ba}} \mathscr{T}(c), \tag{2.7}$$

and denote by $\mathcal{P}_{ba}$ the canonical projection of $\mathscr{T}$ onto it. Then, the resolvent identity

$$\tilde{R}_a - \tilde{R}_b = \tilde{R}_b(\tilde{H}_a - \tilde{H}_b)\tilde{R}_a$$

can be written with the aid of (2.6) as

$$\tilde{R}_a - \tilde{R}_b = \sum_{c \in \mathscr{L}_a} \tilde{R}_b \mathbf{I}_{bc} \tilde{R}_a,$$

where $\mathbf{I}_{bc}$ is the sum over $\mathscr{L}_{bc}$ of the symmetric operators $\tilde{H}(d)$ which all, except $\Delta = \tilde{H}_{a_{\min}} = \tilde{H}(a_{\min})$ (who is affiliated to $\mathscr{T}_{a_{\min}}$) belong to $\mathscr{T}(d)$ respectively. Then, iterating the above formula and taking into account that, as a consequence of the definition of $\mathscr{L}_{bc}$, $\mathbf{I}_{bc} = 0$ if $c \leq b$ or $c \not\sim b$, we get $\tilde{R}_a - \tilde{R}_b = \sum \tilde{R}(\mathscr{C})$, with

$$\tilde{R}(\mathscr{C}) = \tilde{R}_b \mathbf{I}_{bb_1} \tilde{R}_{b_1} \mathbf{I}_{b_1 b_2} \tilde{R}_{b_2} \ldots \mathbf{I}_{b_{n-1} b_n} \tilde{R}_{b_n} \tag{2.8}$$

and where the sum is taken over all the (not necessarily maximal) chains $\mathscr{C}$ of $\mathscr{L}_a$ having the g.l.b. b and as l.u.b. any of the elements of $\mathscr{L}_a$. Moreover, using an inductive argument, it is shown (see Lemma 3.11 in [BGS]) that for any chain $\mathscr{C}$, $\tilde{R}(\mathscr{C}) \in \mathscr{T}_{\min\mathscr{C}\max\mathscr{C}}$. This, together with

$$\mathcal{P}_{ba}\left(\sum_{d \in \mathscr{L} \setminus \mathscr{L}_a} \mathcal{P}(d) + \sum_{d \in \mathscr{L}_b} \mathcal{P}(d)\right) = 0$$

shows that $\mathcal{P}_{ba}[\tilde{R}]$ can be computed as:

$$\tilde{R}_{ba} = \mathcal{P}_{ba}[\tilde{R}_a - \tilde{R}_b] = \sum \tilde{R}(\tilde{\mathscr{C}}) \tag{2.9}$$

where this time the sum is performed over all the chains $\tilde{\mathscr{C}} \subset \mathscr{L}$ with $\min \tilde{\mathscr{C}} = b$ and $\max \tilde{\mathscr{C}} = a$. Notice that in the case $b = a_{\min}$ we have $\mathscr{T}_{a_{\min}a} = \mathscr{T}(a)$ and thus $\tilde{R}(a) = \tilde{R}_{a_{\min}a}$, which tells us that the a-connected component of the hard-core resolvent is a regularising operator on all $\mathscr{H}(X)$, given by the sum (without repetitions) $\sum \tilde{R}(\tilde{\mathscr{C}})$ performed over all the chains of the sublattice $\mathscr{L}_a$.

We are now prepared to state two important consequences of the properties described above, that we will use throughout the paper. They where proved in [BGS] for hard-cores, by extension from the particular usual N-body context, with the aid of the limiting procedure. The first one, is the following commutation relation (see Proposition 3.13 in [BGS])

$$[g, R(a)] = \sum_{b \in \mathscr{L}_a} R(b)[g, \Delta]R_{ba} + R(a)[g, \Delta]R_a, \tag{2.10}$$

true for any multiplication operator with functions $g \in C^2(X)$ having bounded derivatives of first and second order (we are particularly interested in the case when g is the identity function). Notice that (2.10), (2.9) and (2.8) show that the multiple commutators $[\ldots[R, Q], Q], \ldots Q]$, usually denoted by $\mathbf{ad}_Q^k(R)$, are in $B(\mathscr{H}(X))$ for any finite order k.

The second result (Theorem 6.5 in [BGS]) states, for any $a \in \mathscr{L}$, decay of the a-connected component of the hard-core resolvent along all the directions of X^a, with the same rate as that imposed by the hypothesis on the non-hard-core part of the potentials. Since in this paper we are interested only in short-range N-body potentials, the only decay condition we impose to the functions V^a (see (1.2) and the comments following it) is: for any $a \in \mathscr{L} \setminus \{a_{\min}\}$, there is some $\mu > 1$ such that $\langle Q^a \rangle^\mu V^a$ belongs to $B(\mathscr{H}^1(X^a), \mathscr{H}^{-1}(X^a))$. We used $\langle x \rangle$ as abbreviation of $\sqrt{1 + x^2}$. Then, the precise result we need states:

$$\langle Q^a \rangle^\mu R(a) \in B(\mathscr{H}^{-1}(X), \mathscr{H}^1(X)). \tag{2.11}$$

In the rest of this section we shall briefly review (following [G] and [De2]) those geometric particularities of the configuration space X which are specific to the N-body problem. Indeed, the need to put in evidence some privileged directions for the non-propagation, which are closely related to the particular structure of the potentials, suggests the division of X into disjoint sets, all except one being neighborhoods (cone or semi-cylinder -shaped) of these directions. Moreover, a smooth vector field is constructed on X (by convolution or averaging), from the first distributional derivative of the convex, locally Lipschitz application $\varrho : X \to \mathbb{R}$

$$\varrho = \frac{1}{2} \max_{a \in \mathscr{L}} \left\{ |\pi_a(.)|^2 + \nu_a \right\}. \tag{2.12}$$

In the above definition each of the parameters ν_a belongs to some interval $(\nu_{|a|}, \tilde{\nu}_{|a|})$ with positive bounds conveniently chosen, such that for any $1 \le n < m \le N$ the inequalities $\nu_m < \tilde{\nu}_m < \nu_n < \tilde{\nu}_1$ be satisfied. In this sense, ϱ can be seen also as a function of the argument $\boldsymbol{\nu} \in \prod_{a \in \mathscr{L}} (\nu_{|a|}, \tilde{\nu}_{|a|})$ having each of its coordinates double indexed, firstly by the number of the level of $\mathscr{L}$ to which each element a belongs and secondly by the number assigned to this element within the level. The total number of coordinates will thus be $\operatorname{card} \mathscr{L}$.

The identity (2.12) tells us that ϱ is an almost everywhere differentiable function on X, having classical derivative a.e. defined on X which coincides with ϱ', the distributional derivative of ϱ.

Actually, $\varrho' \in L^\infty_{\rm loc}(X)$. Moreover, it can be shown (see Theorem 4.1.7 in [Hö]) that the second distributional derivative of ϱ is a measure on X, taking as values *positive* matrices from $B(X)$.

In order to see in which way is the vector field ϱ' a distortion of the identic mapping on X, we make the connection between (2.12) and the 'privileged directions' X_a by the means of a finite, disjoint, a.e. -covering $\{\Xi_a\}_{a\in\mathscr{L}}$ of X, defined by:

$$\begin{aligned}\Xi_a &= \bigcap_{b\neq a}\left\{x\in X \mid |\pi_a(x)|^2+\nu_a > |\pi_b(x)|^2+\nu_b\right\} \qquad (2.13)\\ &= \left\{x\in X \mid |\pi_a(x)|^2+\nu_a > \max_{b\neq a}\{|\pi_b(x)|^2+\nu_b\}\right\}.\end{aligned}$$

This allows us to calculate explicitly ϱ (and its derivatives) with the aid of the partition of unity $\{J_a\}_{a\in\mathscr{L}}$ subordinated to the above open a.e. -covering of X. Indeed,

$$\varrho'(x) = \sum_{a\in\mathscr{L}} J_a(x)\pi_a(x) \qquad (2.14)$$

holds as X-valued distributions and a.e. as functions. Moreover, since ϱ'' is a positive measure,

$$\varrho''(x) \geq \sum_{a\in\mathscr{L}} J_a(x)\pi_a \qquad (2.15)$$

is satisfied for *all* $x\in X$.

In order to obtain smooth partition of unity and vector field, we construct them from J_a, resp. ϱ, either as in [G], [De2], by convolution with a C_0^∞-function φ (supported in a neighborhood of the origin in X, and satisfying $\int\varphi(x)\,dx = 1$, $\int x\varphi(x)\,dx = 0$), or, as in [Ya], by an averaging process, performed on the cartesian product of intervals $\prod_{n=1}^{N}\prod_{i=1}^{\mathrm{card}\mathscr{L}(n)}(\nu_{ni},\tilde\nu_{ni})$ previously defined, like in:

$$g(x) = \int_{\mathbb{R}^{\mathrm{card}\mathscr{L}}} \tilde\varphi(\boldsymbol{\nu})\, G(x,\boldsymbol{\nu})\, d\boldsymbol{\nu}.$$

Here $\tilde\varphi$ is a positive C_0^∞ function supported in the above product set, with $\|\tilde\varphi\|_{L^1} = 1$, and in our case $G(.\,,\boldsymbol{\nu})$ will be replaced by J_a and ϱ respectively (whose dependence on $\boldsymbol{\nu}$ is given in (2.13), resp. (2.12)). We will denote the new, smooth objects by j_a and r respectively, and it is not difficult to show that all the properties previously enumerated (especially (2.14) and (2.15)) hold for them also. It is also shown that if Id denotes the identity function, then the mappings $r' - Id$, $r''\cdot Id - r'$ and r''' are bounded in the L^∞ norm.

In Appendix 6.1, properties of some subsets of X which play an important role in the spectral and scattering analysis, like the open cones (defined for any $0\leq d<1$)

$$\Gamma_a(d) = \left\{x\in X \mid \min_{b\in\mathscr{L}\setminus\mathscr{L}_a}|\pi^b(x)| = \mathrm{dist}(x,\cup_{b\in\mathscr{L}\setminus\mathscr{L}_a}X_b) > d|x|\right\}, \qquad (2.16)$$

or the "cells" $\overset{\circ}{X}_a = X_a\setminus\bigcup_{b\in\mathscr{L}\setminus\mathscr{L}_a}X_b$ are given, and the relation between them is studied. In order to get an intuitive image of the link between $\overset{\circ}{X}_a$ and the characteristic function J_a of the set Ξ_a, let us mention only the fact that if the argument of J_a is multiplied by a positive parameter γ, then the support of $J_a(\gamma.)$ will tend to coincide with $\overset{\circ}{X}_a$ when $\gamma\to\infty$.

3 Propagation properties

We begin by the so-called maximal velocity bound theorem (see [SS1-3], [G], [So]):

Proposition 3.1 *Let g be a $C_0^\infty(X\setminus\{0\})$ scalar function. Then there is a constant $\lambda>0$ (which cannot be made arbitrarily small) and a positive constant C such that:*

$$\int_1^\infty \frac{dt}{t}\|g\Big(\tfrac{Q}{\lambda t}\Big)\, e^{-itR}\psi\|^2 \le C\|\psi\|^2 \tag{3.1}$$

Proof: Let us choose the propagation observable of the form:

$$\Phi = h\Big(\tfrac{Q}{\lambda t}\Big)$$

and take h to be a $C_0^\infty(\mathbb{R})$ radial function, constant in a neighborhood of the origin and equal to zero at infinity. Then the computation of the Heisenberg derivative of Φ with respect to the approximating hard-core resolvent R_α gives:

$$\mathrm{D}_{R_\alpha}\Phi = \frac{1}{2\lambda t}R_\alpha(P\cdot h' + h'\cdot P)R_\alpha - \frac{1}{t}\frac{Q}{\lambda t}\cdot h' \tag{3.2}$$

where the dot means scalar product between two vector operators. We have also used the convention according to which, whenever no confusion is possible, we will omit the arguments of the multiplication operators with (eventually time-dependent) functions. In order to simplify the aspect of some rather complicated formulas, we will keep this convention in force along all this paper. Denote now by $\tilde{h}$ the scalar function $Id\cdot h'$ and by $\dot{h}(r)\equiv \frac{d}{dr}h(r)\le 0$. An obvious calculation shows that for all $x\in X$:

$$\left\{\begin{array}{l} h'(x) = \omega\dot{h}(|x|) \\ \tilde{h}(x) = |x|\dot{h}(|x|), \end{array}\right. \tag{3.3}$$

where ω denotes the unit vector $x|x|^{-1}$. Finally, take $g(r)=\Big(-\dot{h}(r)\Big)^{1/2}$ and notice that g is a radial $C_0^\infty(X\setminus\{0\})$ function. Then, (3.2) becomes:

$$\begin{aligned}\mathrm{D}_{R_\alpha}h \;&=\; \frac{1}{t}\left|\frac{Q}{\lambda t}\right|g^2 - \frac{1}{t}\,g\,\frac{R_\alpha(P\cdot\omega+\omega\cdot P)R_\alpha}{2\lambda}\,g \\ &\quad -\frac{1}{2\lambda t}\left([R_\alpha P, g]\cdot\omega g R_\alpha + \mathrm{h.c.}\right) \\ &\quad -\frac{1}{2\lambda t}\left(gR_\alpha P\cdot\omega\,[g,R_\alpha] + \mathrm{h.c.}\right).\end{aligned} \tag{3.4}$$

The last two terms above are of order $O(t^{-2})$ uniformly with respect to α because of the obvious equality

$$[R_\alpha, g] = R_\alpha\left[\frac{P^2}{2}, g\right]R_\alpha$$

and of the fact that for all $\alpha\ge 0$, $R_\alpha\in B\left(\mathscr{H}^{-1},\mathscr{H}^1\right)$.

Denoting $\langle e^{-itR_\alpha}\psi, \,.\, e^{-itR_\alpha}\psi\rangle$ by $\langle.\rangle_{t,\alpha}$, we estimate the expectation value of the first two terms from the r.h.s. of (3.4) as follows:

$$\begin{aligned}\frac{1}{t}\left\langle\left|\frac{Q}{\lambda t}\right|g^2\right\rangle_{t,\alpha} \;&\ge\; \frac{1}{t}\inf_{x\in\operatorname{supp} g}|x|\;\|g\,e^{-itR_\alpha}\psi\|^2 \\ \frac{1}{\lambda t}\langle gR_\alpha(P\cdot\omega+\omega\cdot P)R_\alpha g\rangle_{t,\alpha} \;&\le\; \frac{1}{t}\frac{\|R_\alpha(P\cdot\omega+\omega\cdot P)R_\alpha\|}{\lambda}\,\|g\,e^{-itR_\alpha}\psi\|^2.\end{aligned}$$

Replacing the above inequalities in (3.4) we get

$$\left(\inf_{x\in\operatorname{supp} g}|x| - \frac{1}{\lambda}\sup_{\alpha\geq 1}\|R_\alpha\|^2_{-1,0}\right)\int_1^\infty \frac{dt}{t}\|g\Big(\frac{Q}{\lambda t}\Big)\, e^{-itR_\alpha}\psi\|^2 \leq \tilde{C}\|h\|_\infty\,\|\psi\|^2.$$

This shows that if λ is chosen (with respect to the support of g) such that

$$\lambda > \sup_{\alpha\geq 1}\|R_\alpha\|^2_{-1,0}\ \sup_{x\in\operatorname{supp} g}|x|^{-1}, \tag{3.5}$$

applying the usual Fatou lemma (for the integral over t) yields:

$$\begin{aligned}\int_1^\infty \frac{dt}{t}\|g\Big(\frac{Q}{\lambda t}\Big)\, e^{-itR}\psi\|^2 &= \int_1^\infty \frac{dt}{t}\ \liminf_{\alpha\to\infty}\|g\Big(\frac{Q}{\lambda t}\Big)\, e^{-itR_\alpha}\psi\|^2 \\ &\leq \liminf_{\alpha\to\infty}\int_1^\infty \frac{dt}{t}\|g\Big(\frac{Q}{\lambda t}\Big)\, e^{-itR_\alpha}\psi\|^2 \leq C\|\psi\|^2\end{aligned}$$

so the proposition is proved. □

Let us make some comments on this first a priori result. From the point of view of the physical interpretation, it is rather clear why the greatest lower bound of the support of g is important: the quantum system cannot delocalise arbitrarily fast in time since its energy R is a bounded, decreasing function. There is also the special case of the so called "tails" (scattering states describing the system already localised at infinity at finite times) which have non null asymptotic probability and thus have to belong to the orthogonal complement of the range of the projector g. Then, a brutal cutoff introducing a least upper bound for $\operatorname{supp} g$ (as in the hypothesis of the above Proposition) would avoid the problem caused by these "tails" but will make us loose the information about those states describing a system with really large asymptotic velocities and low energy. This shows that the above result is far from being optimal. Nevertheless, Sigal and Soffer established in [SiSo3] a finer one, which holds for states belonging to a dense set of vectors, in which the upper limiting cutoff is eliminated and where the norm in the r.h.s. of (3.1) is taken in a weighted Lebesgue space. But this result is no more an abstract nonsense, since a localisation in the complement of the set of the critical values of the hamiltonian is needed and the Mourre estimate has been used in order to prove it.

Let us notice also that the lower bound established in (3.5) for the value of λ is not optimal. One could have proved the Proposition directly, without using the approximating family $\{R_\alpha\}_{\alpha\geq 0}$ and obtaining the best λ possible, but this requires the result (2.10) concerning commutators with hard-core resolvents (see the way it is used in the proof of the following Proposition). Moreover, we shall see that not all the results we need for proving asymptotic completeness can be deduced by working with a sequence of approximating hamiltonians, the algebraic properties of the N-body (hard-core) resolvents being crucial in the proof of the furthercomming theorems.

Proposition 3.2 ***i)*** *Let $f \in C_0^\infty(X)$ be constant around the origin on a region with interior diameter not too small (i.e. proportional to the maximal velocity bound). Then there exists $\delta > 0$ (depending on the short-range part of the hard-core potential) such that for all $a \in \mathscr{L}$ and all $\psi \in \mathscr{H}(X)$ one has:*

$$\int_1^\infty \frac{dt}{t}\|j_{a,\delta}^{1/2}\left([iR, Q_a] - \frac{Q_a}{t}\right) f\Big(\frac{Q}{t}\Big)\ e^{-itR}\psi\|^2 \ \leq C\|\psi\|^2, \tag{3.6}$$

where $j_{a,\delta} = j_a\left(\frac{Q}{t}t^\delta\right)$ denotes the multiplication operator with the smoothed characteristic function of the set Ξ_a (see (2.13)).

ii) *Moreover, if* $\Gamma_a(0) = X \setminus \bigcup_{b\not\leq a} X_b$*, then the following estimate is true for all function* J *in* $C_0^\infty(\Gamma_a(0))$ *with support not greater than that of* f*:*

$$\int_1^\infty \frac{dt}{t} \| \left| [iR, Q_a] - \frac{Q_a}{t} \right| J\!\left(\tfrac{Q}{t}\right) e^{-itR}\psi\|^2 \leq C\|\psi\|^2. \tag{3.7}$$

Proof: *i)* Let us first introduce a notation concerning only the (vector or scalar) operators of multiplication with $C^p(X)$ functions g: g_β will stand for $g(.\,t^{\beta-1})$ if $\beta > 0$. Notice that this is consistent with the meaning of $j_{a,\delta}$ and if $(g')_\beta$ will be denoted by g'_β, then by iteration we get

$$g_\beta^{(\alpha)} \equiv (g^{(\alpha)})_\beta = t^{(1-\beta)|\alpha|}(g_\beta)^{(\alpha)}$$

for any multiindex α with $|\alpha| \leq p \leq \infty$.

Further, denote by A_δ the operator:

$$\begin{aligned} A_\delta &= \frac{1}{2}\sum_{a\in\mathscr{L}} ([iR, Q_a]\cdot Q_a\, j_{a,\delta} + \text{h.c.}) \\ &= \frac{1}{2}\frac{t}{t^\delta}\left([iR, Q_a]\cdot r'_\delta + r'_\delta \cdot [iR, Q_a]\right) \end{aligned} \tag{3.8}$$

where $r'_\delta = \sum_{a\in\mathscr{L}} \pi_a j_{a,\delta}$ is the smooth vector field introduced in Section 2. Using this operator we shall construct the propagation observable:

$$\Phi_\delta = f\left(\frac{A_\delta}{t} - \frac{Q^2}{2t^2}\right) f \tag{3.9}$$

(the central part of Φ_δ will occasionally be called S_δ). Then, we calculate as usual the Heisenberg derivative of Φ_δ and get:

$$\mathrm{D}_R\Phi_\delta = 2Re\ (\mathrm{D}_R f)\, S_\delta f + f\,(\mathrm{D}_R S_\delta)\, f \tag{3.10}$$

The term we need in the conclusion of the proposition (part *i)*) will be furnished by the second term in the r.h.s. of the above equality, while the other terms will be proved as being integrable in t on $[1,\infty)$. Let us begin with the simplest one:

$$\int_1^\infty dt\, \langle \mathrm{D}_R\Phi_\delta\rangle_t \leq 2\sup_{t\geq 1} |\langle f\, S_\delta\, f\rangle_t| \tag{3.11}$$

where the notation $\langle . \rangle_t$ stands for $\langle e^{-itR}\psi,\ .\, e^{-itR}\psi\rangle$. Since S_δ is obviously uniformly bounded in time on the support of f, it yields the integrability of $\mathrm{D}_R\Phi_\delta$.

Let us now pass to the second term in the r.h.s. of (3.10) and calculate:

$$\mathrm{D}_R S_\delta = \left[iR, \frac{A_\delta}{t}\right] - \left[iR, \frac{Q^2}{2t^2}\right] + \frac{\partial}{\partial t}\left(\frac{A_\delta}{t}\right) + \frac{1}{t}\frac{Q^2}{2t^2}\,. \tag{3.12}$$

We shall estimate each of this terms separately:

$$\begin{aligned} \frac{1}{t}[iR, A_\delta] &= \frac{1}{2}\sum_{b\in\mathscr{L}} \left([iR,[iR,Q_b]]\cdot \frac{Q_b}{t} j_{b,\delta} + \text{h.c.}\right) \\ &\quad + \frac{1}{2t^\delta}\left([iR, Q]\cdot [iR, r'_\delta] + \text{h.c.}\right). \end{aligned} \tag{3.13}$$

The last line in the above equality can be computed by the repeated use of the commutation relation (2.10) (notice also that according to it, the first double sum in (3.14) below is precisely $[iR, Q]$), as follows:

$$\begin{aligned}
[iR, r'_\delta] &= \sum_{b\in\mathscr{L}} [iR(b), r'_\delta] \\
&= \sum_{b,c\in\mathscr{L}} \left(R(c)\left[i\frac{P^2}{2}, r'_\delta\right] R_{cb} + R(b)\left[i\frac{P^2}{2}, r'_\delta\right] R_b \right) \\
&= \frac{1}{t} t^\delta\, r''_\delta \cdot \sum_{b,c\in\mathscr{L}} \left(R(c)PR_{cb} + R(b)PR_b\right) \qquad (3.14)\\
&\quad -\frac{i}{2t^2} t^{2\delta} \sum_{b,c,d\in\mathscr{L}} R(d)G_\delta\left(R_{dc}PR_{cb} + R_{db}PR_b\right) \\
&\quad -\frac{i}{2t^2} t^{2\delta} \sum_{b,c\in\mathscr{L}} R(c)\left(r'''_\delta + G_\delta R_c P\right) R_{cb} \\
&\quad -\frac{i}{2t^2} t^{2\delta} \sum_{b\in\mathscr{L}} R(b)\left(r'''_\delta + G_\delta R_b P\right) R_b,
\end{aligned}$$

where G_δ has been used as a shorthand for $P \cdot r'''_\delta + r'''_\delta \cdot P$. Then, because of the boundedness of r''' (in the L^∞-norm) and the fact that also P is bounded by the resolvents, the last three terms of the above relation will give rise to an $O\left(t^{-2(1-\delta)}\right)$ contribution; as for the first one, it can be minorated with the aid of

$$r''_\delta(x) \geq \sum_{a\in\mathscr{L}} j_{a,\delta}(x)\pi_a \qquad (3.15)$$

(which is true for all $x \in X$ as $B(X)$ -valued measures on X, and for all δ). Finally, we get:

$$\begin{aligned}
\frac{1}{t}[iR, A_\delta] &\geq \frac{1}{2}\sum_{a\in\mathscr{L}} \left([iR, [iR, Q_a]] \cdot \frac{Q_a}{t} j_{a,\delta} + \text{h.c.} \right) \\
&\quad + \frac{1}{t}\sum_{a\in\mathscr{L}} [iR, Q_a] \cdot j_{a,\delta}\, [iR, Q_a] + O\left(t^{-2+\delta}\right) \qquad (3.16)
\end{aligned}$$

We pass now to the second term from the r.h.s. of (3.12): using the fact that the family $\{j_{a,\delta}\}_{a\in\mathscr{L}}$ forms a partition of unity for any $\delta \geq 0$ we get:

$$\begin{aligned}
\left[iR, -\frac{Q^2}{2t^2}\right] &= -\frac{1}{2t}\sum_{a\in\mathscr{L}} \left(\frac{Q_a}{t} \cdot j_{a,\delta}\, [iR, Q_a] + \text{h.c.} \right) \\
&\quad -\frac{1}{2t^{1+\delta}} \sum_{a\in\mathscr{L}} \left(\frac{Q^a}{t} t^\delta \cdot j_{a,\delta}\, [iR, Q^a] + \text{h.c.} \right). \qquad (3.17)
\end{aligned}$$

The last term here is $O\left(t^{-1-\delta}\right)$ because all the components of $\frac{x^a}{t} t^\delta$ are bounded on the support of $j_{a,\delta}$; in other words, if Id denotes the identity function, the last term above equals

$$-\frac{1}{2t^{1+\delta}}\left([iR, Q] \cdot (r' - Id)_\delta + \text{h.c.}\right)$$

and from the definition of the vector field we know that $r' - Id$ is in $L^\infty(X)$.

In a similar manner, computing the third term of the r.h.s. of (3.12) one obtains:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{A_\delta}{t}\right) &= -\frac{1}{2t^{1+\delta}}([iR,Q]\ \cdot r'_\delta + \text{h.c.}) \\ &\quad +\frac{\delta-1}{2t^{1+\delta}}\left([iR,Q]\ \cdot\ (r''\cdot Id - r')_\delta + \text{h.c.}\right). \end{aligned} \tag{3.18}$$

Since $r''\cdot Id - r'$ is also in $L^\infty(X)$, the last term above gives also an $O\left(t^{-1-\delta}\right)$ contribution. Finally, using the same trick, we compute

$$\frac{1}{t}\frac{Q^2}{2t^2} = \frac{1}{t}\sum_{a\in\mathscr{L}}\frac{Q_a}{t}\cdot j_{a,\delta}\,\frac{Q_a}{t} + O\left(t^{-(1+2\delta)}\right) \tag{3.19}$$

and summing up (3.16) to (3.19) one obtains the estimate:

$$\begin{aligned}\mathrm{D}_R S_\delta \ &\geq\ \frac{1}{t}\sum_{a\in\mathscr{L}}\left([iR,Q_a]-\frac{Q_a}{t}\right)\cdot j_{a,\delta}\left([iR,Q_a]-\frac{Q_a}{t}\right) \\ &\quad +\frac{1}{2}\sum_{a\in\mathscr{L}}\left([iR,[iR,Q_a]]\cdot\frac{Q_a}{t}\,j_{a,\delta} + \text{h.c.}\right) \\ &\quad +O\left(t^{-2+\delta}\right)+O\left(t^{-1-\delta}\right)+O\left(t^{-1-2\delta}\right). \end{aligned} \tag{3.20}$$

It remains to show that the second term from the r.h.s. of this inequality is integrable w.r.t. t when taken in the mean value $\langle f\,.\,f\rangle_t$. Let us remark first that the product of the double commutator $[iR,[iR,Q_a]]$ with some multiplication operators with functions of argument $\frac{x}{t}$ will appear quite frequently in this paper. We will prove below that if these functions are supported outside the subspaces X_b for all $b\in\mathscr{L}\setminus\mathscr{L}_a$ then the desired decay in t of the terms containing such products will be ensured. Roughly speaking, this is due to the good decay along certain directions (depending on the given $b\in\mathscr{L}$) of the b-connected component of our "hamiltonian" (i.e. the resolvent operator). Let us notice also that since the b-connected component of the N-body algebra $\mathscr{T}$ is (a semicompact operator algebra) of the form $\mathscr{T}(b) = \mathbf{K}(X^b)\otimes_b\mathbf{T}(X_b)$, its elements will commute with $P_a \equiv 1\otimes_a P_{X_a}$. Hence, taking into account that $\mathscr{T}$ is direct sum over $\mathscr{L}$ of $\mathscr{T}(b)$, we get:

$$\begin{aligned}[iR,[iR,Q_a]]\ j_{a,\delta} &= \sum_{b\in\mathscr{L}} R\,[iR(b),P_a]\,R\,j_{a,\delta} \\ &= \sum_{b\in\mathscr{L}\setminus\mathscr{L}_a} R\left(R(b)P_a - P_aR(b)\right)\left(ij_{a,\delta}R-[iR,j_{a,\delta}]\right). \end{aligned} \tag{3.21}$$

Using the commutation formula (2.10) we will be able to compute $[iR,j_{a,\delta}]$ as

$$\begin{aligned}[iR,j_{a,\delta}] &= \sum_{c\in\mathscr{L}}[iR(c),j_{a,\delta}] \\ &= \frac{t^\delta}{t}j'_{a,\delta}\sum_{c,d\in\mathscr{L}}\left(R(d)PR_{dc}+R(c)PR_c\right)+O\left(t^{-2+2\delta}\right) \\ &= \frac{t^\delta}{t}j'_{a,\delta}[iR,Q]+O\left(t^{-2+2\delta}\right), \end{aligned} \tag{3.22}$$

so (3.14) becomes:

$$[iR,[iR,Q_a]]\, j_{a,\delta} \;=\; \sum_{b\in\mathscr{L}\setminus\mathscr{L}_a} RR(b)\left\{(ij_{a,\delta}P_a+t^{\delta-1}j'_{a,\delta})R+t^{\delta-1}j'_{a,\delta}P_a[iR,Q]\right\}$$

$$-\sum_{b\in\mathscr{L}\setminus\mathscr{L}_a} RP_aR(b)\left(ij_{a,\delta}R+t^{\delta-1}j'_{a,\delta}[iR,Q]\right)+O\left(t^{-2+2\delta}\right). \tag{3.23}$$

The r.h.s. of the above equality has the advantge that it contains only products of $R(b)$ with the functions $j_{a,\delta}$ and $j'_{a,\delta}$. Notice also that for all $b\in\mathscr{L}\setminus\mathscr{L}_a$ there are strictly positive constants $C_{a,b}$ such that

$$\langle x^b\rangle \geq |x^b| \equiv |x^b_{a_{\min}}| \geq |x^b_a| \geq C_{a,b}$$

on the support of j_a. This shows that $\left(\langle x^b\rangle t^{\delta-1}\right)^{-\mu} j_{a,\delta} \leq C_{a,b}{}^{-\mu} j_{a,\delta}$ which joint to (2.11), gives the desired decay in t of the r.h.s. of (3.23) like e.g. in:

$$RR(b)j_{a,\delta}P_aR = t^{-\mu(1-\delta)}R\,\underbrace{R(b)\langle Q^b\rangle}_{O(1)}\,\underbrace{\left(\frac{\langle Q^b\rangle}{t}t^{\delta}\right)^{-\mu} j_{a,\delta}}_{O(1)}\,P_aR, \tag{3.24}$$

provided that $0<\delta<1$ satisfies the supplementary condition $\delta<1-\frac{1}{\mu}$.

It remains thus only to show that $|\langle(\mathrm{D}_R f)\, S_\delta f\rangle_t|$ is integrable in t. For this, we choose an orthonormal basis in X and index by $k=1,\dots,\dim X$ the components of the vector operators as $Q,\nabla f$ in this basis; actually, we shall denote by f'_k the components of ∇f and notice that because of the choice we made on f all of them are negative scalar functions. Denote also by $\tilde f_k$ the $C_0^\infty(X\setminus\{0\})$-function $\sqrt{-f'_k}$ and by χ_k a smoothed characteristic function of $\operatorname{supp} f$ verifying $\chi_k\tilde f_k=\tilde f_k$. Then, commuting $\tilde f_k$ through $[iR,Q]$ and S_δ towards right we get:

$$(\mathrm{D}_R f)\, S_\delta f = \frac{1}{t}\sum_k \tilde f_k\,\underbrace{\chi_k\left(\frac{Q_k}{t}-[iR,Q_k]\right)\chi_k S_\delta f}_{O(1)}\,\tilde f_k + O\left(t^{-2}\right) \tag{3.25}$$

where the $O(t^{-2})$ contribution is brought by the remainders of order two or by the (already) $O(t^{-1})$ terms containing double commutators of the form $[\,[iR,Q],\tilde f\,]$. This shows that there are positive constants C_k and $\tilde C$ depending on the support of f (and thus on the maximal velocity bound) such that

$$\int_1^\infty dt\,|\langle(\mathrm{D}_R f)\, S_\delta f\rangle_t| \leq \sum_k C_k\int_1^\infty \frac{dt}{t}\|f\big(\tfrac{Q}{t}\big)\; e^{-itR}\psi\|^2+\tilde C\|\psi\|^2,$$

which proves the statement *i)* of the Proposition via the maximal velocity bound theorem (Proposition 3.1).

We pass now to the proof of *ii)*: let us fix arbitrarily an a in $\mathscr{L}$ and *assume* first (and prove later) that for some J with support as in the hypothesis (i.e. depending on the chosen a) there is a $T>0$ such that for all $t>T$ the equality

$$J(\tfrac{\cdot}{t}) = J(\tfrac{\cdot}{t})\; f^2(\tfrac{\cdot}{t})\sum_{b\in\mathscr{L}_a} j_b(\tfrac{\cdot}{t}t^{\delta}) \tag{3.26}$$

is valid. We stress that this is sufficient for showing that (3.7) is a consequence of (3.6). Indeed, notice first that (changing the letter a by b) an equivalent inequality for (3.6) is

$$\sum_{k=1}^{\dim X_b} \int_1^\infty \frac{dt}{t} \left\langle \left([iR, Q_k] - \frac{Q_k}{t} \right) f^2 j_{b,\delta} \left([iR, Q_k] - \frac{Q_k}{t} \right) \right\rangle_t \le C \|\psi\|^2 .$$

Then, since $b \in \mathscr{L}_a$ means $X_a \le X_b$, the sum of positive terms in the r.h.s. of the inequality (3.27) below will be performed over a bigger index set than that from the l.h.s.. Hence, using our assumption,

$$\begin{aligned} &\int_T^\infty \frac{dt}{t} \left\langle \left([iR, Q_a] - \frac{Q_a}{t} \right) \cdot J \left([iR, Q_a] - \frac{Q_a}{t} \right) \right\rangle_t \\ &= \sum_{k=1}^{\dim X_a} \int_T^\infty \frac{dt}{t} \left\langle \left([iR, Q_k] - \frac{Q_k}{t} \right) J f^2 \sum_{b \in \mathscr{L}_a} j_{b,\delta} \left([iR, Q_k] - \frac{Q_k}{t} \right) \right\rangle_t \\ &\le \|J\|_\infty \sum_{b \in \mathscr{L}_a} \sum_{k=1}^{\dim X_b} \int_1^\infty \frac{dt}{t} \left\langle \left([iR, Q_k] - \frac{Q_k}{t} \right) f^2 j_{b,\delta} \left([iR, Q_k] - \frac{Q_k}{t} \right) \right\rangle_t . \end{aligned} \quad (3.27)$$

It remains to prove the assumption (3.26). Notice that since the family $\{j_{b,\delta}\}_{b \in \mathscr{L}}$ forms a partition of unity it suffices to show that if $\operatorname{supp} J \subset \operatorname{supp} f$ then for all $t > T$, $x \in \operatorname{supp} J$ implies $x \notin \bigcup_{b \in \mathscr{L} \setminus \mathscr{L}_a} \operatorname{supp} j_{b,\delta}$. But this becomes obvious if one thinks at $j_{b,\delta}$ as to a smoothed characteristic functions of semi-cylinders centred on the sets $\overset{\circ}{X}_b$ which shrink around these sets when $t \to \infty$ (with the "velocity" t^δ) and if one takes also into account that the complementary set for $\Gamma_a(0)$ is precisely $\bigsqcup_{b \in \mathscr{L} \setminus \mathscr{L}_a} \overset{\circ}{X}_b$. $\square$

Remark: In order to avoid some of the cumbersome calculations we made when estimating the terms related to the commutator $[iR, A_\delta]$ in the previous proof, we could work with the approximating resolvent R_α (instead of the hard-core resolvent R) and obtain estimates which are uniform with respect to the parameter α. Nevertheless, there will be a difficulty related to the terms containing the commutator $[iR_\alpha, P_a]$ for which the deeper result furnished by the Theorem 6.5 in [BGS] has to be used. But this result concerns the hard-core resolvents, so we will have to pass first to the limit $\alpha \to \infty$ under the integral $\int_1^T dt$ for some $T > 1$, getting (for any $a \in \mathscr{L}$)

$$\begin{aligned} &\int_1^T \frac{dt}{t} \| j_{a,\delta}^{1/2} \left([iR, Q_a] - \frac{Q_a}{t} \right) f\!\left(\tfrac{Q}{t}\right) e^{-itR} \psi \|^2 \\ &\le \sum_{b \in \mathscr{L}} \int_1^\infty dt \left| \left\langle R\, [iR, P_b] \left(it^{\delta-1} j'_{b,\delta} RP - j_{b,\delta} \right) R \frac{Q_b}{t} f \right\rangle_t \right| + O(1) \|\psi\|^2 , \end{aligned}$$

and finally we will let T go to ∞ and apply the quoted theorem as in the previous proof (relation (3.24)).

Proposition 3.3 *If a is an arbitrary element of $\mathscr{L}$ and if $J \in C_0^\infty(\Gamma_a(0))$ then there is a positive constant C such that*

$$\int_1^\infty \frac{dt}{t} \| \left| [iR, Q_a] - \frac{Q_a}{t} \right|^{\frac{1}{2}} J\!\left(\tfrac{Q}{t}\right) e^{-itR} \psi \|^2 \le C \|\psi\|^2 \quad (3.28)$$

for all $\psi \in \mathscr{H}(X)$.

Before starting the proof let us introduce some notations that we shall use along all the rest of the paper: T_a will stand for $[iR,Q_a]-\frac{Q_a}{t}$ and if A is an unbounded (eventually time-dependent) operator then its time-dependent regularisation $\left(|A|^2+t^{-4\beta}\right)^{1/2}$ will be denoted by $\langle A\rangle_\beta$.

Proof: First of all we shall reduce a little the context in which the proposition has to be proved by assuming that J belongs also to a particular class of functions that Derezinski denoted by $\mathscr{F}$ and defined as being formed by $C_0^\infty(X)$-functions f having the property that for any $a\in\mathscr{L}$ there is a neighborhood $\mathcal{V}_a$ of the subspace X_a such that $f=f\circ\pi_a$ in $\mathcal{V}_a$. Obviously, this tells us that on some neighborhood of each X_a, the border of the support of each function in $\mathscr{F}$ will be perpendicular to X_a. It is also easy to check that $\mathscr{F}$ is a $\star$-algebra which separates the points of X and is dense in $C_0^\infty(X)$ in the L^∞-norm. It will be thus sufficient to prove the theorem for $J\in C_0^\infty(\Gamma_a(0))\cap\mathscr{F}$.

Since $\Gamma_a(0)=\bigsqcup_{b\in\mathscr{L}_a}\overset{\circ}{X}_b$, any compact set of $\Gamma_a(0)$ (and in particular the support of J) can be partitioned in ε-neighborhoods of a finite number of points from $\overset{\circ}{X}_b$ for some $b\in\mathscr{L}_a$. The arbitrary (but fixed) choice we make for J will determine a choice for the diameter of these neighborhoods and for the number n_b of those which are centred on points from $\overset{\circ}{X}_b$. Correspondingly, one shall have a partition of unity on the support of J constructed with the aid of a family $\{j_{k_b}\mid k_b=1,\dots,n_b,\ b\in\mathscr{L}_a\}$ and satisfying on X:

$$\sum_{b\in\mathscr{L}_a} J\circ\pi_b\sum_{k_b=1}^{n_b} j_{k_b}^2=J. \tag{3.29}$$

Notice that as a consequence of this choice we will also have $\nabla J=(\pi_b\nabla)J$ on the support of each j_{k_b}.

Secondly, it is clear that the estimate

$$\int_1^\infty\frac{dt}{t}\|\langle T_a\rangle_\beta^{1/2}\,J\!\left(\tfrac{Q}{t}\right)\,e^{-itR}\psi\|^2\ \le C\|\psi\|^2 \tag{3.30}$$

implies (3.28), so it will be sufficient to prove the above inequality. In order to do this, let us choose the propagation observable

$$\Phi=J\,\langle T_a\rangle_\beta\,J \tag{3.31}$$

and compute its Heisenberg derivative:

$$\mathrm{D}_R\Phi=2Re\ (\mathrm{D}_RJ)\ \langle T_a\rangle_\beta\,J+J\ (\mathrm{D}_R\langle T_a\rangle_\beta)\ J. \tag{3.32}$$

We will show first that the second term in the r.h.s. of the above equality will furnish the term from the conclusion of the theorem plus some integrable terms. For this, we shall use a particular form of the definition of the fractional power of a positive operator A (see [T])

$$A^\gamma=C\int_0^\infty\omega^{\gamma-1}(\omega+A)^{-1}A\,d\omega \tag{3.33}$$

which holds strongly on its domain for some positive constant C and for $\gamma\in\left(0,\frac12\right]$. Then, an easy computation gives:

$$\begin{aligned}\mathrm{D}_RA^\gamma &= C\int_0^\infty\omega^{\gamma-1}\left\{\left(\mathrm{D}_R(\omega+A)^{-1}\right)A+(\omega+A)^{-1}\mathrm{D}_RA\right\}\,d\omega\\ &= C\int_0^\infty\omega^{\gamma-1}(\omega+A)^{-1}\,(\mathrm{D}_RA)\left(1-(\omega+A)^{-1}A\right)d\omega.\end{aligned} \tag{3.34}$$

In our case, we shall replace A by $\langle T_a\rangle_\beta^2$ and compute its Heisenberg derivative as:

$$\begin{aligned}\mathrm{D}_R\langle T_a\rangle_\beta^2 &= -\frac{2}{t}\left(\langle T_a\rangle_\beta^2+(2\beta-1)t^{-4\beta}\right)\\ &\quad+([iR,[iR,Q_a]]\cdot T_a+\text{h.c.})\,. \end{aligned}\tag{3.35}$$

Then, taking $\gamma=\frac{1}{2}$ and replacing (3.35) in (3.34) we get:

$$\begin{aligned}&\mathrm{D}_R\langle T_a\rangle_\beta=\\ &=-\frac{2C}{t}\left(\langle T_a\rangle_\beta^2+(2\beta-1)t^{-4\beta}\right)\int_0^\infty\omega^{-\frac{1}{2}}\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\left\{1-\langle T_a\rangle_\beta^2\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\right\}d\omega\\ &\quad+C\int_0^\infty\omega^{\frac{1}{2}}\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\{T_a\cdot[iR,[iR,Q_a]]+\text{h.c.}\}\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}d\omega. \end{aligned}\tag{3.36}$$

The integral in the first term of the above equality is $\frac{1}{2}\langle T_a\rangle_\beta^{-1}$ (see [SiSo2] page 132), so (3.36) becomes:

$$\begin{aligned}\mathrm{D}_R\langle T_a\rangle_\beta&=-\frac{1}{t}\langle T_a\rangle_\beta+(1-2\beta)t^{-4\beta-1}\langle T_a\rangle_\beta^{-1}\\ &\quad+2C\,Re\int_0^\infty d\omega\,\omega^{\frac{1}{2}}\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}T_a\cdot[iR,[iR,Q_a]]\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}. \end{aligned}\tag{3.37}$$

Notice that the second term in the r.h.s. above is $O\left(t^{-1-2\beta}\right)$ because of the obvious estimate $\|\langle T_a\rangle_\beta^{-1}\|\leq t^{2\beta}$, while the first one is exactly the term we need for the estimate (3.30). It will suffice thus to show the integrability (w.r.t. t) of the expectation value $\langle J\,.\,J\rangle_t$ of the third term from (3.37). This will be performed as in the proof of the previous theorem, i.e. by taking into account the decay in t of the double commutator $[iR,[iR,Q_a]]$ on the support of J. Indeed, the hypothesis $J\in C_0^\infty(\Gamma_a(0))$ make us sure of the existence, for all $b\in\mathscr{L}\setminus\mathscr{L}_a$, of some strictly positive constants C_b for which

$$\langle Q^b\rangle\geq|Q^b|\geq C_b t$$

on supp $J(\frac{\cdot}{t})$ for all $t\geq 1$. This shows

$$J\left(\frac{\langle Q^b\rangle}{t}\right)^{-\mu}\leq C_b^{-\mu}J,$$

thus once J being brought nearby the double commutator we can apply the same resonement as in (3.21) to (3.24) (take $\delta=0$ for the present case) in order to obtain an $O(t^{-\mu})$ contribution. Nevertheless, besides the supplementary difficulties raised by the commutator of J with the resolvent $\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}$, there is also the problem of the boundedness of the components of T_a (which cannot be pulled out from the integral over ω because they do not commute with the $\langle T_a\rangle_\beta$ from the resolvents). As we shall see, commuting J with $\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}$ gives rise to $O(t^{-1})$ factors (which are obviously not enough for the integrability in t) containing derivatives of J but also operators T_a. The strategy will be to continue to commute J, J' through these resolvents until either $O(t^{-2})$ terms or products of these functions with $[iR,[iR,Q_a]]$ will appear. Then, only the problems of the boundedness of the resulting T_a's and those of the integrability w.r.t. ω on both $[1,\infty)$ and $[0,1)$ will have to be solved. Actually, we shall see that the first two of them are related while in order to solve the third one we will need to "pick" a little part from the good decay we have obtained in the variable t and to convert it in decay in ω. Let us first compute:

$$\begin{aligned}
&\left[\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}, J\right]\\
&=\frac{2}{t}Re\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}T_a\cdot\left(J'\cdot[iR,Q]P_aR+\text{h.c.}+iRJ'R+O\left(t^{-1}\right)\right)\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}. \quad (3.38)
\end{aligned}$$

This allows to give a precise formula for the operator which stands in (3.37) in the r.h.s. of (and in product with) the $R(b)$'s (where $b\in\mathscr{L}\setminus\mathscr{L}_a$) yielding from $[iR,[iR,Q_a]]$, namely:

$$\begin{aligned}
&R\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}J\\
&\quad=\ (JR+J'[R,Q])\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\\
&\quad\quad+\frac{1}{t}J'R\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}(T_a\cdot O(1)+\text{h.c.})\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\\
&\quad\quad+O\left(t^{-2}\right)\left\{1+\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}(T_a\cdot O(1)+\text{h.c.})\right\}\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\\
&\quad\quad+R\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\left(T_a\cdot O\left(t^{-2}\right)+\text{h.c.}\right)\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\\
&\quad\quad-\frac{1}{t}R\left\{T_a\cdot\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}O\left(t^{-1}\right)\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}O(1)\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}+\text{h.c.}\right\}. \quad (3.39)
\end{aligned}$$

Remark that the last three lines from the r.h.s. above are $O(t^{-2})$ while the first two terms apparently have not the desired decay in t; but since J and J' will be next to some $R(b)$, they also will finally bring an $O(t^{-\mu})+O(t^{-1-\mu})$ contribution. Concerning the boundedness of the components of those T_a from above which are taken in scalar product with $O(1)$ terms it will be no danger to ensure it (uniformly in t and ω) with the aid of a square root of the resolvent $\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}$. Nevertheless, this will *not* be the case for the T_a which is taken in (3.37) in scalar product with $[iR,[iR,Q_a]]$, because in order to ensure integrability in ω on $[1,\infty)$ we need a norm of the resolvent $\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}$ on a power strictly superior to $\frac{3}{2}$. We prefer thus to bound it by a $\langle T_a\rangle_\beta^{-1}$ and to commute the remaining $\langle T_a\rangle_\beta$ towards left, next to the J. Finally, using the Schwartz inequality after one has replaced (3.39) in (3.37) we obtain the estimate:

$$\begin{aligned}
&\int_1^\infty dt\int_0^\infty d\omega\,\omega^{\frac12}\left|\left\langle J\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}T_a\cdot[iR,[iR,Q_a]]\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}J\right\rangle_t\right|\\
&\le C\|\psi\|\int_1^\infty dt\,O\left(t^{-\tilde\mu}\right)\|\langle T_a\rangle_\beta J\,e^{-itR}\psi\|\sum_{k=1}^{\dim X_a}\|\langle T_a\rangle_\beta^{-1}T_k\|\int_1^\infty d\omega\,\omega^{\frac12}\sum_{j=0,1,3}\|\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\|^{2+\frac{j}{2}}\\
&\quad+\tilde C\|J\|_\infty\|\psi\|^2\int_1^\infty dt\,O\left(t^{-\tilde\mu}\right)\int_0^1 d\omega\,\omega^{\frac12}\sum_{j=0,2,3}\|\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\|^{3-\frac{j}{2}}, \quad (3.40)
\end{aligned}$$

with C, $\tilde C$ positive constants depending only on J, J' and where $\tilde\mu$ stands for $\min\{2,\mu\}$. Then, the second term in the r.h.s. of (3.40) will be estimated minorizing $\|\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\|$ by $t^{4\beta}$, while for the first one we will take advantge of $\|\left(\omega+\langle T_a\rangle_\beta^2\right)^{-1}\|\le\omega^{-1}$ in order to dominate the r.h.s. of (3.40) by:

$$\|\psi\|\int_1^\infty dt\,O\left(t^{-\tilde\mu}\right)\|\langle T_a\rangle_\beta J\,e^{-itR}\psi\|\int_1^\infty\omega^{-\frac32}\,d\omega+\|\psi\|^2\int_1^\infty dt\,O\left(t^{12\beta-\tilde\mu}\right)\int_0^1\omega^{\frac12}\,d\omega. \quad (3.41)$$

Since the choice of $\beta > 0$ is at our disposal, we shall take it strictly inferior to $\frac{\tilde{\mu}-1}{12}$, which ensures integrability w.r.t. ω in the second term of the above sum. As for the first one, according to the Schwartz inequality, it will be dominated by

$$\|\psi\| \left(\int_1^\infty dt\, O\left(t^{1-2\tilde{\mu}}\right) \right)^{\frac{1}{2}} \left(\int_1^\infty \frac{dt}{t} \|\langle T_a \rangle_\beta J\, e^{-itR} \psi\|^2 \right)^{\frac{1}{2}}, \tag{3.42}$$

whose finiteness is a consequence of Proposition 3.2 (ii) and of the hypothesis $\mu > 1$.

It remains to estimate the integrability of the expectation value $\langle \cdot \rangle_t$ of the first term from the r.h.s. of (3.32). For this, we will act exactly as in in the proof of the Proposition 3.2 (relation (3.25)) but use the partition of unity one has introduced at the beginning of this proof (see (3.29)). We have:

$$\mathrm{D}_R J = \frac{1}{t} \sum_{b \in \mathscr{L}_a} \sum_{k_b=1}^{n_b} \left([iR, Q_b] - \frac{Q_b}{t} \right) \cdot (\pi_b \nabla) J j_{k_b}^2 + \text{Remainder}, \tag{3.43}$$

where the "Remainder" will be shown as being an $O(t^{-2})$ term. Let us for the moment look at the first term above, and estimate:

$$\begin{aligned} \left\langle \frac{1}{t} \sum_{b \in \mathscr{L}_a} \sum_{k_b=1}^{n_b} T_b \cdot J' j_{k_b}^2 \langle T_a \rangle_\beta J \right\rangle_t \le\; & \frac{1}{t} \sum_{b \in \mathscr{L}_a} \sum_{k_b=1}^{n_b} \langle T_b \cdot J' j_{k_b} \left[j_{k_b}, \langle T_a \rangle_\beta \right] J \rangle_t \\ & + \frac{1}{t} \sum_{b \in \mathscr{L}_a} \sum_{k_b=1}^{n_b} \langle [T_b, j_{k_b}] \cdot J' \langle T_a \rangle_\beta j_{k_b} J \rangle_t \\ & + \frac{1}{t} \sum_{b \in \mathscr{L}_a} \sum_{k_b=1}^{n_b} \|J' \cdot T_b j_{k_b} e^{-itR} \psi\| \; \|\langle T_a \rangle_\beta J j_{k_b} e^{-itR} \psi\|. \end{aligned} \tag{3.44}$$

Notice that as a consequence of

$$\|J' \cdot T_b\, j_{k_b}\, e^{-itR} \psi\| \le \sum_{l=1}^{\dim X_b} \|\partial_l J\|_\infty \, \|T_l \langle T_b \rangle_\beta^{-1}\| \; \|\langle T_b \rangle_\beta\, j_{k_b}\, e^{-itR} \psi\|$$

and of the Schwartz inequality (applied to the integral $\int dt$) we can dominate the last line from (3.44) by

$$C \sum_{b \in \mathscr{L}_a} \sum_{k_b=1}^{n_b} \left(\int_1^\infty \frac{dt}{t} \|\langle T_b \rangle_\beta\, j_{k_b}\, e^{-itR} \psi\|^2 \right)^{\frac{1}{2}} \left(\int_1^\infty \frac{dt}{t} \|\langle T_a \rangle_\beta\, J j_{k_b}\, e^{-itR} \psi\|^2 \right)^{\frac{1}{2}},$$

each of the above integrals being $\le C\|\psi\|^2$ by Proposition 3.2 . Then, the commutator $[j_{k_b}, \langle T_a \rangle_\beta]$ will be computed with the aid of (3.33) as:

$$[j_{k_b}, \langle T_a \rangle_\beta] = 2C\, Re \int_0^\infty d\omega\, \omega^{\frac{1}{2}} \left(\omega + \langle T_a \rangle_\beta^2 \right)^{-1} T_a \cdot \left[\left[iR, Q_a \right], j_{k_b} \right] \left(\omega + \langle T_a \rangle_\beta^2 \right)^{-1}. \tag{3.45}$$

The above double commutator gives an $O(t^{-1})$ contribution, so we will continue to estimate the first two terms from the r.h.s. of (3.44) as before (relations (3.40) to (3.42)), the only difference being that $\tilde{\mu}$ will be replaced by 2. This shows that the l.h.s. of (3.44) is integrable in time; the Remainder from (3.43) can be computed with the aid of the Fourier spectral formula, as:

$$\frac{1}{t^2} \int_X dk\, \widehat{J''}(k)\, e^{-\frac{ik}{t} Q} \int_0^1 d\tau \int_0^\tau d\sigma\, e^{-\frac{ik\sigma}{t} Q} \left[[iR, Q], Q \right] e^{+\frac{ik\sigma}{t} Q} \tag{3.46}$$

where $\widehat{J''}$ is a rapidly decreasing function (as Fourier transform of the smooth, compactly supported J'').

Finally, the integral over t of the l.h.s. of (3.32) is obviously dominated by

$$\begin{aligned} 2\sup_{t\geq 1}\left|\langle J\,\langle T_a\rangle_\beta\, J\rangle_t\right| &\leq 2\sup_{t\geq 1}\|J\|_\infty\,\|\langle T_a\rangle_\beta\, J\, e^{-itR}\psi\| \\ &= 2\sup_{t\geq 1}\|J\|_\infty\,\{\,\underbrace{\langle\, J\, T_a^2\, J\,\rangle_t}_{O(1)} + t^{-4\beta}\|J\|_\infty^2\,\|\psi\|^2\,\}^{\frac{1}{2}}, \end{aligned}$$

which finishes the proof of the Proposition. □

Remarks: 1) The estimate (3.7) remains valid if one replaces Q_a by any of its components relative to some basis from X. Moreover, this is also true for the sharper estimate (3.28), as a consequence of the implication (see e.g. Proposition 6.2 in [F]):

$$A_k^2 \leq \sum_{j=1}^{n} A_j^2 \implies A_k \leq \left(\sum_{j=1}^{n} A_j^2\right)^{\frac{1}{2}},$$

valid for the set $\{A_j\}_{j=1,\ldots,n}$ of (unbounded) positive self-adjoint operators.

2) Since the propositions 3.2 and 3.3 have been proved for an arbitrary lattice $\mathscr{L}$, they are also true for any sublattices of $\mathscr{L}$. More precisely, let a be an arbitrarily fixed element of $\mathscr{L}$, and denote by $T_{b,a}$ the operator $[iR_a,Q_b]-\frac{Q_b}{t}$. Then, due to (2.2), the estimates (3.7) and (3.28) will also be true if one replaces R by R_a, i.e. for any $b\in\mathscr{L}_a$ and any $J\in C_0^\infty(\Gamma_b(0))$ there is a positive constant C such that:

$$\int_1^\infty \frac{dt}{t}\left(\|\,|T_{b,a}|\;J\!\left(\tfrac{Q}{t}\right)\,e^{-itR_a}\psi\|^2 + \|\,|T_{b,a}|^{\frac{1}{2}}\;J\!\left(\tfrac{Q}{t}\right)\,e^{-itR_a}\psi\|^2\right) \leq C\|\psi\|^2. \tag{3.47}$$

Moreover, taking into account that $b\in\mathscr{L}_a$ implies $\Gamma_b(0)\subseteq\Gamma_a(0)$ and using the reasonement described in the proof of the Proposition 3.2 (relation (3.24)) we see that the difference between $T_{b,a}$ and T_b is of order $O(t^{-\mu+1})$ on the support of J. Then, the obvious inequality $(A+B)^2\leq 2\,(A^2+B^2)$ which holds for any pair of self-adjoint (not necessarily positive) operators A, B, shows that $T_{b,a}$ can be replaced by T_b in the first norm from the above inequality. In what follows we will see that the same is true for the second norm above, i.e. the estimate:

$$\int_1^\infty \frac{dt}{t}\|\,\left|[iR,Q_b]-\frac{Q_b}{t}\right|^{\frac{1}{2}}\,J\!\left(\tfrac{Q}{t}\right)\,e^{-itR_a}\psi\|^2 \leq C\|\psi\|^2 \tag{3.48}$$

holds for all $b\in\mathscr{L}_a$, all $J\in C_0^\infty(\Gamma_b(0))$ and all $\psi\in\mathscr{H}(X)$. Indeed, using the formula (3.33), an easy calculation gives:

$$\begin{aligned} \langle T_b\rangle_\beta-\langle T_{b,a}\rangle_\beta = C&\int_0^\infty d\omega\,\omega^{\frac{1}{2}}\left(\omega+\langle T_{b,a}\rangle_\beta^2\right)^{-1}\left[i\left(R-R_a\right),Q_b\right]^2\left(\omega+\langle T_b\rangle_\beta^2\right)^{-1} \\ &+C\,Re\int_0^\infty d\omega\,\omega^{\frac{1}{2}}\left(\omega+\langle T_{b,a}\rangle_\beta^2\right)^{-1}T_{b,a}\cdot\left[i\left(R-R_a\right),Q_b\right]\left(\omega+\langle T_b\rangle_\beta^2\right)^{-1}. \end{aligned} \tag{3.49}$$

This shows that we only have to commute the J's from the left or from the right through the resolvent $\left(\omega+\langle T_b\rangle_\beta^2\right)^{-1}$ in order to get either products of J, J' with $R-R_a$ or $O(t^{-2})$ terms. The boundedness of the components of $T_{b,a}$ will be ensured exactly like in the proof of the previous proposition (see the comments we made after the equation (3.39)). In this way we show that the above difference is of integrable order, and thus (3.47) implies (3.48).

4 Wave operators

An important feature of the strategy that Sigal and Soffer imagined in order to prove existence of the cluster wave operators and asymptotic completeness for usual N-body short-range systems, is that both of this problems are treated as if they where of the same difficulty. Indeed, it is first proven the existence of the strong limits of $\exp(itH)J_b\exp(-itH_b)$, where $\{J_b\}$ is a family pseudodifferential operators verifying a partition of unity in the phase-space. In our case, we shall take, as in [G], [De2], a time-dependent family of identificators and state:

Proposition 4.1 *If a is an arbitrary element of $\mathscr{L}$ and if J is a $C_0^\infty(\Gamma_a(0))$ function, then the operator domain of the following limits:*

$$W^{\pm}(R, R_a; J) \;=\; \operatorname*{s-lim}_{t\to\pm\infty} e^{itR} J\Big(\frac{Q}{t}\Big)\, e^{-itR_a} \tag{4.1}$$

$$W^{\pm}(R_a, R; J) \;=\; \operatorname*{s-lim}_{t\to\pm\infty} e^{itR_a} J\Big(\frac{Q}{t}\Big)\, e^{-itR} \tag{4.2}$$

is the whole $\mathscr{H}(X)$. Moreover, the statement is true even when J is a bounded continuous function with support in $\Gamma_a(0)$.

Proof: As in the proof of the Proposition 3.3, for showing the first part of the Proposition it will be sufficient to take $J \in C_0^\infty(X)\cap\mathscr{F}$. We shall also use the same partition of unity on the support of J constructed with the aid of the family $\{j_{k_b} \mid k_b = 1,\dots,n_b,\, b\in\mathscr{L}_a\}$.

In order to prove the existence of the limits (4.1) and (4.2) the Cook criterion will be used. More precisely, the computation:

$$\frac{d}{dt}\left\langle \psi, e^{itR_a} J e^{-itR}\psi\right\rangle = \left\langle \psi, e^{itR_a}\left\{\mathrm{D}_R J - iJ\,(R-R_a)\right\} e^{-itR}\psi\right\rangle$$

shows that a sufficient condition for the convergence of $e^{itR_a}Je^{-itR}$ in expectation value on $\psi\in\mathscr{H}(X)$ is

$$\int_1^\infty \left|\left\langle \psi, e^{itR_a}\,\mathrm{D}_R J\, e^{-itR}\psi\right\rangle\right| dt \;+\; \|\psi\|^2 \sum_{b\in\mathscr{L}\setminus\mathscr{L}_a}\int_1^\infty \|R(b)J\|dt \;\le\; C\|\psi\|^2. \tag{4.3}$$

Then this weaker type of convergence yields (in our special case) strong convergence in a standard manner. Reasoning in the same way as in the proof of the Proposition 3.2 (see relation (4.3)) we prove that the integrand in the second term from the r.h.s. of (3.21) is of order $O(t^{-\mu})$ for all $b\in\mathscr{L}\setminus\mathscr{L}_a$. Further, using the formula (3.43) we show that the first term in (4.3) is dominated (modulo some $O(t^{-2})$ contributions) by:

$$\begin{aligned}
&\int_1^\infty \frac{dt}{t}\sum_{b\in\mathscr{L}_a}\sum_{k_b=1}^{n_b}\sum_{l=1}^{\dim X_b} \|T_l\langle T_l\rangle_\beta^{-1}\|\;\|\langle T_l\rangle_\beta^{1/2} j_{k_b}\, e^{-itR_a}\psi\|\;\|\langle T_l\rangle_\beta^{1/2} j_{k_b}(\partial_l J)\, e^{-itR}\psi\| \\
&\le C\sum_{b\in\mathscr{L}_a}\sum_{k_b=1}^{n_b}\sum_{l=1}^{\dim X_b}\left(\int_1^\infty \frac{dt}{t}\|\langle T_l\rangle_\beta^{1/2} j_{k_b}\, e^{-itR_a}\psi\|^2\right)^{\frac12}\left(\int_1^\infty \frac{dt}{t}\|\langle T_l\rangle_\beta^{1/2} j_{k_b}(\partial_l J)\, e^{-itR}\psi\|^2\right)^{\frac12},
\end{aligned}$$

where in the above estimate the Schwartz inequality has been used. Finally let us notice that each of the integrals from the above brackets is $\le C\|\psi\|^2$ as a consequence of the Remark following the Proposition 3.3.

Suppose now $J\in BC(\Gamma_a(0))$. Let χ be the smoothed characteristic function of a neighborhood of the origin in X, and denote by χ_γ the operator of multiplication with $\chi\Big(\frac{\cdot}{\gamma t}\Big)$, where

$\gamma > 0$ is a parameter. Then the product $J\chi_\gamma$ plays the role of the J's from the first part of the proposition, so it will be sufficient to prove that for any positive ε we can choose γ in such a way that

$$\sup_{t\geq 1} \|e^{itR_a} J\,(1-\chi_\gamma)\, e^{-itR}\psi\| < \varepsilon \|\psi\|_{\mathscr{H}_1}$$

for all ψ belonging to the weighted Lebesgue space of order one $\mathscr{H}_1(X)$. Moreover, using the obvious inequality $1-\chi_\gamma \leq Id$, true for γ sufficiently large, we see that a stronger condition is given by

$$\sup_{t\geq 1} \frac{1}{t} \|\,|Q|\, e^{-itR}\psi\| < \frac{\varepsilon\gamma}{\|J\|_\infty} \|\psi\|_{\mathscr{H}_1}\ . \tag{4.4}$$

But (4.4) is a particular case of a result due to Radin and Simon (see Theorem 2.1 in [RaS]). Indeed, all we have to do is to mimic the proof of the quoted theorem (after one has taken as hamiltonian the approximating resolvent R_α) and finally obtain:

$$\|\,|Q|\, e^{-itR_\alpha}\psi\| \leq \|\,|Q|\psi\| + \int_0^t d\tau \left\langle \tilde{P}_\alpha^2 \right\rangle_{t,\alpha}^{1/2}$$

where $\left\{\tilde{P}_\alpha\right\}_{\alpha\geq 0}$ denotes the uniformly bounded family of operators $R_\alpha P R_\alpha$. Applying the Fatou lemma to the above inequality proves (4.4) provided that γ is chosen superior to $\varepsilon^{-1}\|J\|_\infty$.$\square$

Lemma 4.1 *If a is an arbitrary element of $\mathscr{L}$ and if $\mathcal{E}_a$ denotes the projection $E_{pp}(H^a)\otimes_a 1$, then the limits:*

$$\Omega_a^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itR} e^{-itR_a} \mathcal{E}_a \tag{4.5}$$

exist on all $\mathscr{H}(X)$. Moreover, if $a \neq b$ are two elements of $\mathscr{L}$, then the ranges of the corresponding cluster wave operators are orthogonal.

Proof: Since the existence of the operators $W^\pm(R, R_a; J)$ has been proved on $\mathscr{H}(X)$ for all $J \in BC(X)$ with support in $\Gamma_a(0)$, an $\varepsilon/3$ argument shows that in order to prove the existence of (4.5) the following convergence

$$\|\left(1 - J\!\left(\tfrac{Q}{t}\right)\right)\, e^{-itR_a} u\| \stackrel{t\to\pm\infty}{\longrightarrow} 0 \tag{4.6}$$

has to be true for all u belonging to a dense set of $\mathcal{E}_a\mathscr{H}(X)$. The choice we will make for J in (4.6) depends on the vector u.

For this, let us construct the dense set of u's by taking simple tensors of the form $w^a \otimes_a v_a$ where w^a belongs to $E_{pp}(H^a)\mathscr{H}(X^a)$ (and thus is an eigenvector of H^a for the eigenvalue λ^a) and $v_a = g_a(\tilde{P}_a)\, v_a$ with $g_a \in C_0^\infty(\overset{\circ}{X}_a)$ and

$$\tilde{P}_a \equiv \left(z - \lambda^a - \frac{P_a^2}{2}\right)^{-1} P_a \left(z - \lambda^a - \frac{P_a^2}{2}\right)^{-1}. \tag{4.7}$$

As we explained before we shall *choose* $J \in BC(\Gamma_a(0)) \cap \mathscr{F}$ depending on each u by requiring:

$$\begin{cases} \operatorname{supp} J|_{X_a} \supset \operatorname{supp} g_a \\ J|_{X_a} = 1 \text{ on the support of } g_a\,. \end{cases} \tag{4.8}$$

Then, the l.h.s. of (4.6) equals:

$$\|\left(1 - J\!\left(\tfrac{Q}{t}\right)\right) e^{-itR_a} w^a \otimes_a g_a(\tilde{P}_a)\, v_a\| = \|\left(1 - J\!\left(\tfrac{Q}{t}\right)\right) e^{-it(z-\lambda^a - P_a^2/2)^{-1}} w^a \otimes_a g_a(\tilde{P}_a)\, v_a\| \tag{4.9}$$

because of the relation (1.3), which in turn is a consequence of the Theorem 3.10 from [BGS].

Let us take now one more cutoff function $h \in C_0^\infty(X^a)$ with $h = 1$ in a neighborhood of the subspace X_a. The precise choice we will make for the support of h will depend on the choice we make on J and hence on g_a, so for the moment we let the shape of the support of h entirely at our disposal. In what follows we shall denote by $A \lesssim B$ the inequality (between numbers) $A \leq B + o(1)$, where $o(1)$ is defined w.r.t. $t \to \pm\infty$. Then the r.h.s. of (4.9) will be majorized by:

$$\begin{aligned}
&\| \left(1 - J\left(\tfrac{Q}{t}\right)\right) \, e^{-it(z-\lambda^a - P_a^2/2)^{-1}} h\left(\tfrac{Q}{t}\right) \, w^a \otimes_a g_a(\tilde{P}_a) \, v_a\| + \| \left(1 - h\left(\tfrac{Q}{t}\right)\right) \, w^a \otimes_a v_a\| \\
&\quad \lesssim \; \| \left(1 - J\left(\tfrac{Q}{t}\right)\right) \, h\left(\tfrac{Q}{t}\right) \, g_a(\tilde{P}_a) \, e^{-it(z-\lambda^a - P_a^2/2)^{-1}} w^a \otimes_a v_a\| \\
&\quad \leq \; C\| \left(g_a(\tilde{P}_a) - g_a\left(\tfrac{Q_a}{t}\right)\right) \, e^{-it(z-\lambda^a - P_a^2/2)^{-1}} w^a \otimes_a v_a\| \\
&\qquad + \| \left(1 - J\left(\tfrac{Q}{t}\right)\right) \, h\left(\tfrac{Q}{t}\right) \, g_a\left(\tfrac{Q_a}{t}\right) \, e^{-it(z-\lambda^a - P_a^2/2)^{-1}} w^a \otimes_a v_a\|
\end{aligned}$$

where C is a positive constant depending on J and h and where in the asymptotic inequality one has supposed that h has the supplementary property $h = h \circ \pi^a$ so it commutes with the unitary group $\exp\{-it(z - \lambda^a - P_a^2/2)^{-1}\}$.

The first term in the above inequality tends towards zero when $t \to \pm\infty$ as a consequence of the Theorem 7.1.29 of Hörmander (see [Hö]), and thus one has shown that:

$$\| \left(1 - J\left(\tfrac{Q}{t}\right)\right) \, e^{-itR_a} u\| \lesssim \| \left(1 - J\left(\tfrac{Q}{t}\right)\right) \, h\left(\tfrac{Q}{t}\right) \, g_a\left(\tfrac{Q_a}{t}\right) \, e^{-it(z-\lambda^a - P_a^2/2)^{-1}} w^a \otimes_a v_a\|. \tag{4.10}$$

Since $J \in \mathscr{F}$ one can make our final hypothesis on h, namely: the support of h is chosen (in function of J) to be included in the neighborhood of X_a for which $J = J \circ \pi_a$. Then, in the r.h.s. of (4.10) we will have:

$$\left(1 - J\left(\tfrac{Q}{t}\right)\right) \, h\left(\tfrac{Q}{t}\right) \, g_a\left(\tfrac{Q_a}{t}\right) = \left(1 - J\left(\tfrac{Q_a}{t}\right)\right) \, g_a\left(\tfrac{Q_a}{t}\right) \, h\left(\tfrac{Q}{t}\right) = 0$$

because of the second hypothesis in (4.8), and thus the first statement of the lemma is proved.

It remains to show that for arbitrary $a \neq b$ in $\mathscr{L}$ and for any vectors $\varphi, \psi \in \mathscr{H}(X)$ we have $\langle \Omega_a^\pm \varphi, \Omega_b^\pm \psi \rangle = 0$. Actually, it will be sufficient to prove the convergence

$$\langle e^{-itR_a} \mathcal{E}_a \varphi, \, e^{-itR_b} \mathcal{E}_b \psi \rangle \stackrel{|t| \to \infty}{\longrightarrow} 0$$

for φ and ψ belonging to the dense sets previously introduced. This time we don't need any function g_a so we will take it equal to one. But for these vectors, the l.h.s. of the above relation equals:

$$\langle \mathcal{E}_a \varphi, \, e^{-it(z-\lambda^a - P_a^2/2)^{-1}} e^{it(z-\lambda^b - P_b^2/2)^{-1}} \mathcal{E}_b \psi \rangle$$

which tends to zero when t goes to ∞ as a consequence of the Riemann-Lebesgue lemma.□

5 Proof of the minimal velocity theorem

Our main result, the Theorem 1.1, can be stated in a more precise form as follows:

Theorem 5.1 *Let $a \in \mathscr{L}$ and $\varepsilon > 0$ be arbitrarily chosen. Let (J, θ) and $(\hat{J}, \hat{\theta})$ be two couples of functions, with $J, \hat{J} \in C_0^\infty(\Gamma_a(0))$ and $\theta, \hat{\theta} \in C_0^\infty(\mathbb{R})$, such that $\hat{J}, \hat{\theta}$ equal one on the support of J and θ respectively. Suppose moreover that for arbitrary $\lambda \in \mathbb{R}$, one of the following two conditions:*

$$\inf\left\{\left(\rho^A_{R_a}(\lambda) - \varepsilon\right)\mu^2 - \frac{x_a^2}{2} \mid x \in \operatorname{supp} \hat{J} \text{ and } \mu \in \operatorname{supp}\theta\right\} > \sup_{x\in\operatorname{supp}\hat{J}} \sum_{l=1}^{\dim X^a} \| [iR, Q_l] \| \frac{|x_l|}{2} \tag{5.1}$$

$$\left\{\frac{2\inf\mu^2}{\sup\mu^2} \mid \mu \in \operatorname{supp}\theta\right\} \left(\rho^A_{R_a}(\lambda) - \varepsilon\right) > \sum_{l=1}^{\dim X} \|P_l\, \hat{\theta}(R)\| \sup_{x\in\operatorname{supp}\hat{J}} |x_l| \tag{5.2}$$

is satisfied. Then, the estimate:

$$\int_1^\infty \frac{dt}{t} \|J\left(\tfrac{Q}{t}\right)\theta(R)\, e^{-itR}\psi\|^2 \leq C\|\psi\|^2 \tag{5.3}$$

is true for some positive constant C and for all $\psi \in \mathscr{H}(X)$.

Notice that depending on the choice of the pair (J, θ), the above result can be seen either as a maximal or as a minimal velocity bound theorem. This result gives more information than we need in order to prove asymptotic completeness. Indeed, we shall see later that the following corollary, which corresponds to the particular case $a = a_{\max}$ of the above theorem, is sufficient (via a standard induction argument) for the proof of (1.10). Let us denote by $\mathcal{C}(R)$ the set of critical values of the operator R. Then:

Corollary 5.2 *Let $\theta \in C_0^\infty(\mathbb{R} \setminus \mathcal{C}(R))$ and take $J \in C_0^\infty(X)$ with the support sufficiently close to the origin in X. Then,*

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itR} J\left(\tfrac{Q}{t}\right) e^{-itR}\theta(R) = 0. \tag{5.4}$$

Proof: It is sufficient to choose the support of J (depending on how close $\operatorname{supp}\theta$ is to the points from $\mathcal{C}(R)$) such that one among the two conditions (5.1), (5.2) holds. Then (5.3) will be valid, which will imply:

$$\liminf_{t\to\infty} \|e^{itR} J\left(\tfrac{Q}{t}\right) e^{-itR}\theta(R)\psi\| = 0$$

for all $\psi \in \mathscr{H}(X)$. But according to the Proposition 4.1 the full strong limits $W^\pm(R, R; J)$ exist on all $\mathscr{H}(X)$, so they equal zero on all sets $\theta(R)\mathscr{H}(X)$ (whose union over all θ as in the hypothesis, is dense in $\mathscr{H}_\infty$). □

Proof of the Theorem 5.1: Since the theorem should be true for any $a \in \mathscr{L}$, during the whole proof we will fix an arbitrarily chosen a and consider (unless otherwise specified) that all the operators of multiplication with functions are relative to this a (as it where be indexed by a). Then the proof will be performed by induction over the levels $\mathscr{L}_a(n)$ ($n = 1, \ldots, \operatorname{rank}\mathscr{L}_a \equiv N_a$) of the sublattice $\mathscr{L}_a$. Nevertheless, there is a particular case, namely $a = a_{\min}$, for which the proof is simpler and it works exactly in the same manner as for the first step of the inductive process corresponding to the cases $a \in \mathscr{L} \setminus \{a_{\min}\}$.

The induction hypothesis will be called $(\mathbf{P}_{n+1})$ and enounced as follows:

$(\mathbf{P}_{n+1})$: *If for any $b \in \mathscr{L}_a(n+1)$ and any $\varepsilon > 0$ there are two couples of functions (J,θ) and $(\hat{J},\hat{\theta})$ with $J,\hat{J} \in C_0^\infty(\Gamma_b(0))$ and $\theta,\hat{\theta} \in C_0^\infty(\mathbb{R})$, such that $\hat{J},\hat{\theta}$ equal one on the support of J and θ respectively, and if for all $\lambda \in \mathbb{R}$, one of the following two conditions:*

$$\inf\left\{\left(\rho^A_{R_b}(\lambda) - \varepsilon\right)\mu^2 - \frac{x_a^2}{2} \mid x \in \operatorname{supp}\hat{J} \text{ and } \mu \in \operatorname{supp}\theta\right\} > \sup_{x\in\operatorname{supp}\hat{J}} \sum_{l=1}^{\dim X^a} \|\,[iR,Q_l]\,\| \frac{|x_l|}{2} \tag{5.5}$$

$$\left\{\frac{2\inf\mu^2}{\sup\mu^2} \mid \mu \in \operatorname{supp}\theta\right\}\left(\rho^A_{R_b}(\lambda) - \varepsilon\right) > \sum_{l=1}^{\dim X} \|P_l\,\hat{\theta}(R)\| \sup_{x\in\operatorname{supp}\hat{J}} |x_l| \tag{5.6}$$

is fulfilled, then the estimate (5.3) is true.

The first step of the (weak) inductive process, namely the validity of $(\mathbf{P}_{N_a})$, will be given by the following lemma. As we said before, this lemma tells us also that the Theorem 5.1 is true in the particular case $a = a_{\min}$ (see condition (5.7) below).

Lemma 5.1 *Let $b = a_{\min}$ in the hypothesis of $(\mathbf{P}_{n+1})$ or suppose that for the same couples of functions the following strict inequality*

$$\inf\left\{\left(\rho^A_{R_{a_{\min}}}(\lambda) - \varepsilon\right)\mu^2 - \frac{x^2}{2} \mid x \in \operatorname{supp}\hat{J} \text{ and } \mu \in \operatorname{supp}\theta\right\} > 0 \tag{5.7}$$

holds. Then (5.3) holds also.

Proof: Let us remind first that given a vector operator S in $\mathscr{H}(X)$ we shall denote by S_a, S^a the operators $1 \otimes_a (\pi_a S)$, resp. $(\pi^a S) \otimes_a 1$ in $\mathscr{H}(X)$, but we shall not change the notation when the operators $(\pi_a S)$ acting in $\mathscr{H}(X_a)$ and respectively $(\pi^a S)$ in $\mathscr{H}(X^a)$ will be concerned. As stated before, whenever no confusion is possible, we shall not mention the usual time-dependent argument $\frac{x}{t}$ of the operators of multiplication with functions.

Keeping in mind these conventions, let us choose the propagation observable:

$$\Phi = \theta(R)J\left[iR, \frac{(Q^a)^2}{2t}\right] J\,\theta(R), \tag{5.8}$$

and compute as usual its Heisenberg derivative as:

$$\begin{aligned} \mathrm{D}_R\Phi \;&=\; \frac{1}{t}\,Re\,\theta(R)\,(\mathrm{D}_R J)\left[iR,(Q^a)^2\right] J\,\theta(R) \\ &\quad + \theta(R)J\,\left(\mathrm{D}_R\left[iR, frac(Q^a)^2 2t\right]\right)\,J\,\theta(R)\,. \end{aligned} \tag{5.9}$$

Then the obvious estimate

$$\int_1^\infty dt\,\langle \mathrm{D}_R\Phi\rangle_t \le 2\sup_{t\ge1}|\langle\Phi\rangle_t| \le \sum_{l=1}^{\dim X^a} \|\,[iR,Q_l]\,\| \sup_{x\in\operatorname{supp}J}|x_l|\;\|J\|_\infty^2\,\|\theta\|_\infty^2\,\|\psi\|^2 \tag{5.10}$$

shows that we only have to look at the terms from the r.h.s. of (5.9): we shall begin by proving that the first one is integrable w.r.t. t. For this, let us take an orthonormal basis in X and denote by T_k the components of the vector operator $T_{a_{\min}} = T$ in this basis. Since the support of J is compact, there exists a finite family $\{j_i\}$ of C_0^∞-functions with supports included in $\operatorname{supp}\hat{J}$ and satisfying $J = J\sum_i j_i^2$. Then, proceeding as in the proof of the Proposition 3.3 (see relations (3.43) and (3.44)) we get:

$$(\mathrm{D}_R J)\left[iR, \frac{(Q^a)^2}{t}\right] J = \frac{1}{t}\sum_{k=1}^{\dim X}\sum_i j_i \langle T_k\rangle_\beta^{1/2}\ \underbrace{\langle T_k\rangle_\beta^{-1} T_k}_{O(1)}\ B_k\ \langle T_k\rangle_\beta^{1/2} j_i + O\big(t^{-2}\big)\,. \tag{5.11}$$

Since $j_i \in C_0^\infty(\overset{\circ}{X}_{a_{\min}})$ for any i, we can use (3.30) in order to integrate the first term in the r.h.s. above, provided that

$$B_k \overset{\text{not}}{=} \langle T_k\rangle_\beta^{1/2}\,(\partial_k J)\left[iR, \frac{(Q^a)^2}{t}\right] J\langle T_k\rangle_\beta^{-1/2} \tag{5.12}$$

is shown as being the sum between a uniformly bounded (in t) operator and some integrable terms. We have thus to commute $\langle T_k\rangle_\beta^{1/2}$ towards left. Since J and $\partial_k J$ bound the above commutator, we will only have to show the integrability of sums of the form:

$$\frac{1}{t}\left[\langle T_k\rangle_\beta^{1/2}, g\right]\ [iR, Q_l]\,\tilde g + \frac{1}{t} g\left[\langle T_k\rangle_\beta^{1/2},\ [iR, Q_l]\right]\tilde g\,, \tag{5.13}$$

where g, $\tilde g$ denote the operators of multiplication with the functions $\partial_k J$ or $(\partial_k J)\,\pi_l$ and $J\pi_l$ or J respectively (their argument being as usual $\frac{x}{t}$). Using the formula (3.33) with $\gamma = \frac{1}{4}$ and reasoning exactly as in the proof of the Proposition 3.3 (see (3.45)) it can be shown that if β is appropriately chosen w.r.t. μ, the first term from above brings an integrable contribution. Then, in the same manner, the second term of the above sum is computed as:

$$\frac{2C}{t}\,Re\, gT_k\int_0^\infty d\omega\,\omega^{\frac{1}{4}}\left(\omega + \langle T_k\rangle_\beta^2\right)^{-1}\left\{[\,[iR, Q_k]\,, [iR, Q_l]\,] + \left[[iR, Q_l]\,, \frac{Q_k}{t}\right]\right\}\left(\omega + \langle T_k\rangle_\beta^2\right)^{-1}\tilde g.$$

It is clear that the second term in the curly bracket brings an $O(t^{-2})$ contribution and, for $\beta > 0$ sufficiently small, it will still be integrable w.r.t. t after the integration in ω is performed. As for the first one, the formula:

$$\frac{1}{t}[\,[iR, Q_k]\,, [iR, Q_l]\,] = \frac{1}{2}\left[[iR, [iR, Q_l]\,]\,, \frac{Q_k}{t}\right] + \frac{1}{2}\left[[\,[iR, Q_k]\,, iR]\,, \frac{Q_l}{t}\right] \tag{5.14}$$

shows that we only have to prove:

$$\int_1^\infty dt\int_0^\infty d\omega\,\omega^{\frac{1}{4}}\|gT_k\left(\omega + \langle T_k\rangle_\beta^2\right)^{-1}\frac{Q_l}{t}\,[iR, [iR, Q_k]\,]\left(\omega + \langle T_k\rangle_\beta^2\right)^{-1}\tilde g\langle T_k\rangle_\beta^{-1/2}\| < \infty.$$

The above double commutator can be computed as in the proof of Proposition 3.2, by taking into account that for any $c \in \mathscr{L}$ we have $[P_k, R(c)] = 0$ for all k for which $X_k \subseteq X_c$. Notice that we refer to $\subseteq$ as to a vector space inclusion because X_k could not belong to the lattice $\{X_b\}_{b\in\mathscr{L}}$. We thus have:

$$[\,[iR, Q_k]\,, R] = \sum_{\substack{c\in\mathscr{L}\\ X_k\nsubseteq X_c}} R\,(P_k R(c) - R(c)P_k)\,R.$$

Notice that for all $k = 1, \ldots, \dim X$ the inclusion $X_k \subset X_{a_{\min}}$ shows that the above sum will be performed over a subset of $\mathscr{L}\setminus\{a_{\min}\}$ and since the supports of g, $\tilde g$ are compacts from $\Gamma_{a_{\min}}(0)$ (for all $l = 1, \ldots, \dim X^a$) they are disjoint of any X_c with $c \in \mathscr{L}\setminus\{a_{\min}\}$ so it will be sufficient to commute g, $\tilde g$ towards right or left respectively, in order to obtain either $O(t^{-2})$ terms, or products $gR(c) \sim O(t^{-\mu})$. As in the proof of Proposition 3.3 (see the comments made before (3.40)), there will be a tribute to pay in order to ensure integrability w.r.t. ω, but for a suitable (small, positive) β there will still remain enough decay in t for the convergence of the above double integral.

We pass now to the second term in the r.h.s. of (5.9). Since yet, any of the hypothesis (5.5), (5.6) or (5.7) has been used; in what follows, this term will be estimated in three (somewhat) different ways each of these variants involving the Mourre estimate and only one of the mentioned hypothesis. Let us start by recalling that for any $b \in \mathscr{L}$ the differences $R - R_b$ and $\theta(R) - \theta(R_b)$ and the double commutator $[\,[iR, Q_b]\,, R]$ are of order $O(t^{-\mu})$ on $\operatorname{supp} J \subset \Gamma_{a_{\min}}(0)$. Then the obvious equality

$$\left[iR, \left[iR, \frac{Q_a^2}{2t}\right]\right] = Re\;[iR, [iR, Q_a]\,] \cdot \frac{Q_a}{t} + \frac{1}{t}\,[iR, Q_a]^2 \tag{5.15}$$

shows that:

$$\begin{aligned}
\theta(R)J\,\left(\mathrm{D}_R\left[iR, \frac{(Q^a)^2}{2t}\right]\right)\,J\,\theta(R) &= -\frac{1}{t}\theta(R)J\,\left[iR, \frac{(Q^a)^2}{2t}\right]\,J\,\theta(R)\\
&\quad+\frac{1}{t}JR_{a_{\min}}\theta(R_{a_{\min}})\;[iR_{a_{\min}}, A]\;\theta(R_{a_{\min}})R_{a_{\min}}J\\
&\quad-\frac{1}{t}\theta(R)J\;[iR, Q_a]^2\,J\,\theta(R) + O(t^{-\mu}) + O\left(t^{-2}\right).
\end{aligned}\tag{5.16}$$

We start to prove that (5.5) (taken with $b = a_{\min}$) implies (5.3). Since

$$[iR, Q_a]^2 = T_a^2 + 2Re\,T_a \cdot \frac{Q_a}{t} + \frac{Q_a^2}{t^2}, \tag{5.17}$$

the mean value $\langle . \rangle_t$ of the last term from the r.h.s. of (5.16) dominates the sum:

$$\begin{aligned}
&-\frac{1}{t}\left\{\||T_a|J\,\theta(R)\,e^{-itR}\psi\|^2 + \sum_{l=1}^{\dim X_a}\sup_{x\in\operatorname{supp} J}|x_l|\;\|\langle T_l\rangle_\beta^{1/2}J\,\theta(R)\,e^{-itR}\psi\|^2\right\} + O\left(t^{-2}\right)\,\|\psi\|^2\\
&-\frac{1}{t}\sup_{x\in\operatorname{supp} J} x_a^2\;\|J\,\theta(R)\,e^{-itR}\psi\|^2
\end{aligned}\tag{5.18}$$

in which the first line is integrable in t as a consequence of the propagation theorems 3.2 and 3.3. Further, the first term in the r.h.s. of (5.16) is minorized by:

$$-\frac{1}{t}\sup_{x\in\operatorname{supp} J}\sum_{l=1}^{\dim X^a}\|\,[iR, Q_l]\,\|\;|x_l|\;\|J\,\theta(R)\,e^{-itR}\psi\|^2 \tag{5.19}$$

whereas for the second term from the r.h.s. of (5.16) we apply the Mourre estimate (see def. (1.5) and also (1.6)) in order to dominate it by

$$\frac{2}{t}\left(\rho^A_{R_{a_{\min}}}(\lambda) - \varepsilon\right)\inf_{\mu\in\operatorname{supp}\theta}\mu^2\;\|J\,\theta(R)\,e^{-itR}\psi\|^2 + O(t^{-\mu}) + O\left(t^{-2}\right). \tag{5.20}$$

Then, summing up (5.18) to (5.20) we obtain a lower bound for the mean value of l.h.s. of (5.16):

$$\begin{aligned}
&\left\langle\theta(R)J\,\left(\mathrm{D}_R\left[iR, \frac{(Q^a)^2}{2t}\right]\right)\,J\,\theta(R)\right\rangle_t\\
&\gtrapprox \frac{2}{t}\left\{\left(\rho^A_{R_{a_{\min}}}(\lambda) - \varepsilon\right)\inf_{\mu\in\operatorname{supp}\theta}\mu^2 - \sup_{x\in\operatorname{supp} J}\left(\frac{x_a^2}{2} + \sum_{l=1}^{\dim X^a}\|\,[iR, Q_l]\,\|\;\frac{|x_l|}{2}\right)\right\}\|J\,\theta(R)\,e^{-itR}\psi\|^2
\end{aligned}$$

where $\gtrapprox$ means $\geq$ modulo addition of some terms of integrable order (whenever this type of *equality* will arise, the sign $\approx$ will be used). But since the support of $\hat{J}$ is a dilation of $\operatorname{supp} J$, the hypothesis (5.5) ensures strict positivity of the quantity from the above curly bracket, which proves (5.3).

We pass now to the proof of the implication (5.7) $\Longrightarrow$ (5.3) and compute:

$$\begin{aligned}
\theta(R)J\left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)J\,\theta(R) &= -\frac{1}{t}\theta(R)J\left(Re\,T^a\cdot\frac{Q^a}{t}+\frac{(Q^a)^2}{t^2}\right)J\,\theta(R)\\
&\quad -\frac{1}{t}\theta(R)J\left(Re\,T_a\cdot\frac{Q_a}{t}+T_a^2\right)J\,\theta(R)\\
&\quad +\frac{1}{t}JR_{a_{\min}}\theta(R_{a_{\min}})\,[iR_{a_{\min}},A]\,\theta(R_{a_{\min}})R_{a_{\min}}J\\
&\quad -\frac{1}{t}\theta(R)J\frac{Q_a^2}{t^2}J\,\theta(R)+O(t^{-\mu})+O\left(t^{-2}\right).
\end{aligned}\tag{5.21}$$

Estimating the terms from the r.h.s. above like in (5.18) - (5.20) we see that the l.h.s. of (5.21) dominates (modulo some integrable terms):

$$2\left\{\left(\rho^A_{R_{a_{\min}}}(\lambda)-\varepsilon\right)\inf_{\mu\in\operatorname{supp}\theta}\mu^2-\frac{1}{2}\sup_{x\in\operatorname{supp}J}x^2\right\}\|J\,\theta(R)\,e^{-itR}\psi\|^2$$

in which the curly bracket contains a strictly positive number as a consequence of (5.7). Finally,

$$[iR,Q_a]^2 = T_a^2+Re\,T_a\cdot\frac{Q_a}{t}+\left[iR,\frac{Q_a^2}{2t}\right]\tag{5.22}$$

allows to calculate:

$$\begin{aligned}
\theta(R)J\left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right)J\,\theta(R) &= -\frac{1}{t}\theta(R)J\left[iR,\frac{Q^2}{2t}\right]J\,\theta(R)+O(t^{-\mu})+O\left(t^{-2}\right)\\
&\quad -\frac{1}{t}\theta(R)J\left(Re\,T_a\cdot\frac{Q_a}{t}+T_a^2\right)J\,\theta(R)\\
&\quad +\frac{1}{t}JR_{a_{\min}}\theta(R_{a_{\min}})\,[iR_{a_{\min}},A]\,\theta(R_{a_{\min}})R_{a_{\min}}J
\end{aligned}\tag{5.23}$$

which together with the estimate:

$$\begin{aligned}
\frac{1}{t}\left\langle JR\,\theta(R)\,P\cdot\frac{Q}{t}JR\,\theta(R)\right\rangle_t &\lessapprox \sum_{l=1}^{\dim X}\|P_l\,\hat{\theta}(R)\|\sup_{x\in\operatorname{supp}J}|x_l|\;\|R\,\theta(R)\,J\,e^{-itR}\psi\|^2\\
&\lessapprox \sum_{l=1}^{\dim X}\|P_l\,\hat{\theta}(R)\|\sup_{x\in\operatorname{supp}J}|x_l|\sup_{\mu\in\operatorname{supp}\theta}\mu^2\,\|J\,\theta(R)\,e^{-itR}\psi\|^2
\end{aligned}$$

shows that the l.h.s. of (5.23) is

$$\gtrapprox\frac{1}{t}\left\{2\left(\rho^A_{R_{a_{\min}}}(\lambda)-\varepsilon\right)\inf_{\mu\in\operatorname{supp}\theta}\mu^2-\sum_{l=1}^{\dim X}\|P_l\,\hat{\theta}(R)\|\sup_{x\in\operatorname{supp}J}|x_l|\sup_{\mu\in\operatorname{supp}\theta}\mu^2\right\}\|J\,\theta(R)\,e^{-itR}\psi\|^2.$$

Using the hypothesis (5.6) (written for $b=a_{\min}$) we see again that the above curly bracket is positive, and this finishes the proof of the lemma. □

We begin now the second step of the inductive reasoning, namely one has to prove the implication $(\mathbf{P}_{n+1}) \implies (\mathbf{P}_n)$ for any $n = 1, \ldots, N_a - 1$. Notice that in (5.5) and (5.6) one could have taken $\rho^A_{R_a}(\lambda)$ instead of $\rho^A_{R_b}(\lambda)$, but since $b \in \mathscr{L}_a(n+1)$ implies $\rho^A_{R_a}(\lambda) \leq \rho^A_{R_b}(\lambda)$, $(\mathbf{P}_{n+1})$ would have weaken (anyway, this would not create any disadvantage from the point of view of the final result). For proving the above implication it will suffice to fix arbitrarily $b \in \mathscr{L}_a(n)$ and show that $(\mathbf{P}_n)$ holds for this b. Let thus J, $\hat{J}$ and θ, $\hat{\theta}$ be chosen with respect to this b such that (5.5) or (5.6) be verified. Then Proposition 6.1(iv) shows that for any such J we can construct (as in the proof of Proposition 3.3) a family $\{J_{k_c}\}$ of $C_0^\infty(\Gamma_b(0))$-functions satisfying:

$$J \;=\; J \sum_{c \in \mathscr{L}_b} \sum_{k_c=1}^{n_c} J^2_{k_c} \tag{5.24}$$

$$\operatorname{supp} \hat{J} \;\supset\; \bigcup_{c \in \mathscr{L}_b} \bigcup_{k_c=1}^{n_c} \operatorname{supp} J_{k_c}. \tag{5.25}$$

Notice that (5.25) is always possible because of the strict inclusion of $\operatorname{supp} J$ in $\operatorname{supp} \hat{J}$ and that it will be no loss in supposing that $n_b = 1$ (renote J_{k_b} by J_b). Then, since $(\mathbf{P}_{n+1})$ is supposed true, (5.24) allows us to reduce a little the problem by estimating:

$$\begin{aligned}\int_1^\infty \frac{dt}{t} \|J\, \theta(R)\, e^{-itR} \psi\|^2 \;&\leq\; \|J\|^2_\infty \int_1^\infty \frac{dt}{t} \|J_b\, \theta(R)\, e^{-itR} \psi\|^2 \\ &+ \|J\|^2_\infty \sum_{c<b} \sum_{k_c=1}^{n_c} \int_1^\infty \frac{dt}{t} \|J_{k_c}\, \theta(R)\, e^{-itR} \psi\|^2.\end{aligned} \tag{5.26}$$

Then, as a consequence of (5.25) there is a $C_0^\infty(\Gamma_b(0))$-function $\hat{J}_{k_c}$ such that $\hat{J}_{k_c} = 1$ on $\operatorname{supp} J_{k_c}$ and $\operatorname{supp} \hat{J}_{k_c} \subset \operatorname{supp} \hat{J}$. This shows

$$\sup_{x \in \operatorname{supp} \hat{J}} \;\geq\; \sup_{x \in \operatorname{supp} \hat{J}_{k_c}}$$

and since as a consequence of the definition of the function ρ:

$$\rho^A_{R_c}(\lambda) \geq \rho^A_{R_b}(\lambda) \;\text{ for all } c \leq b \text{ and all } \lambda \in \mathbb{R}, \tag{5.27}$$

the assumptions of the type (5.5) and (5.6) in $(\mathbf{P}_n)$ (written for J) are stronger as the same assumptions in any of the $(\mathbf{P}_{n+1})$'s written for J_{k_c}. This shows that the second line of (5.27) is $\leq C\|\psi\|^2$ so it remains only to estimate the first term in r.h.s. of (5.27). For this, let us consider two smooth functions, $\tilde{J}_b$ and f, having the property that for some positive constant C:

$$J_b \leq C\, \tilde{J}_b\, f \tag{5.28}$$

is true on all X. Observe that such functions always exist: it is sufficient to choose them such that the product be a C_0^∞-function and that

$$\operatorname{supp} J_b \subset \operatorname{supp} \tilde{J}_b \cap \operatorname{supp} f. \tag{5.29}$$

We shall, moreover, put some supplementary conditions on $\tilde{J}_b$, namely, we suppose that there is a set called *core*supp $\tilde{J}_b$ (which is centred on the same point of $\mathring{X}_b$ as $\operatorname{supp} J_b$) satisfying:

$$core\operatorname{supp} \tilde{J}_b \;\subset\; \operatorname{supp} \tilde{J}_b \tag{5.30}$$

$$
\begin{aligned}
&core\text{supp}\, \tilde{J}_b \cap \overset{\circ}{X}_b \neq \emptyset && (5.31)\\
&\tilde{J}_b \text{ is constant on } core\text{supp}\, \tilde{J}_b && (5.32)\\
&\tilde{J}_b \in C_0^\infty(\Gamma_b(0)) && (5.33)\\
&\text{supp}\, \tilde{J}_b \subset \text{supp}\, \hat{J}. && (5.34)
\end{aligned}
$$

Notice that (5.34) is allowed because of the definition of $\hat{J}$ and of (5.25). As for f, we suppose that:

$$
\begin{aligned}
&f \in C_0^\infty(X) && (5.35)\\
&\text{supp}\, f \cap \overset{\circ}{X}_b \subset core\text{supp}\, \tilde{J}_b \cap \overset{\circ}{X}_b && (5.36)\\
&f = f \circ \pi_b \text{ on supp}\, \hat{J}. && (5.37)
\end{aligned}
$$

In conclusion, it will be sufficient to prove:

$$
\int_1^\infty \frac{dt}{t} \| f\big(\tfrac{Q_b}{t}\big)\, \tilde{J}_b\big(\tfrac{Q}{t}\big)\; \theta(R)\, e^{-itR}\psi\|^2 \leq C\|\psi\|^2 \tag{5.38}
$$

which will be always verified provided that

$$
\int_1^\infty \frac{dt}{t} \| f([iR,Q_b])\, \tilde{J}_b\big(\tfrac{Q}{t}\big)\; \theta(R)\, e^{-itR}\psi\|^2 \leq C\|\psi\|^2 \tag{5.39}
$$

and that

$$
\int_1^\infty \frac{dt}{t} \left\langle \tilde{J}_b\big(\tfrac{Q}{t}\big) \left\{ f^2([iR,Q_b]) - f^2\big(\tfrac{Q_b}{t}\big) \right\} \tilde{J}_b\big(\tfrac{Q}{t}\big) \right\rangle_t \leq C\|\psi\|^2. \tag{5.40}
$$

In the Appendix 6.4 we prove that estimates of the type (5.40) are consequences of the propagation theorems 3.2 and 3.2, so it remains to prove (5.39). For this, we choose as propagation observable the bounded operator

$$
\Phi = \theta(R)\, f([iR,Q_b])\, \tilde{J}_b\big(\tfrac{Q}{t}\big) \left[iR, \frac{(Q^a)^2}{2t} \right] \tilde{J}_b\big(\tfrac{Q}{t}\big)\, f([iR,Q_b])\, \theta(R). \tag{5.41}
$$

As in the proof of the Lemma 5.1, an inequality of the same type as (5.10) shows that it will suffice to estimate each of the terms yielding from the derivation $\mathrm{D}_R\Phi$. Starting with the one containing $\mathrm{D}_R f([iR,Q_b])$, we shall use the spectral Fourier formula in order to calculate:

$$
[iR, f([iR,Q_b])] = \int_X ds\, \widehat{f'}(s) \int_0^1 d\tau\, e^{is(1-\tau)[iR,Q_b]}\, [iR,[iR,Q_b]\,]\; e^{is\tau[iR,Q_b]}. \tag{5.42}
$$

Then, we commute $\tilde{J}_b$ through $\exp(is\tau[iR,Q_b])$ in order to obtain products of $\tilde{J}_b$, $\tilde{J}_b'$ with the above double commutator (which give $O(t^{-\mu})$ contributions). After all these computations have been performed, we obtain (modulo some $O(t^{-2})$ terms):

$$
\begin{aligned}
[iR, f([iR,Q_b])]\, \tilde{J}_b \;\approx\; & \int_X ds\, \widehat{f''}(s) \int_0^1 i\tau\, d\tau \int_0^1 d\sigma\, e^{is(1-\tau)[iR,Q_b]}\, [iR,[iR,Q_b]\,] \times \\
& \times \left(\tilde{J}_b + \frac{1}{t}\tilde{J}_b' e^{is\tau\sigma[iR,Q_b]}\, [\,[iR,Q_b]\,, Q]\; e^{-is\tau\sigma[iR,Q_b]} \right) e^{is\tau[iR,Q_b]} \qquad (5.43)
\end{aligned}
$$

which shows

$$
\int_1^\infty \frac{dt}{t} \left\langle \theta(R)\; (\mathrm{D}_R f([iR,Q_b])\,)\; \tilde{J}_b \left[iR, \frac{(Q^a)^2}{2t} \right] \tilde{J}_b\, f([iR,Q_b])\, \theta(R) \right\rangle_t \leq C\|\psi\|^2.
$$

We pass now to the proof of the integrability w.r.t. t of the second term resulted from Φ's Heisenberg derivation, i.e. the one containing

$$\mathrm{D}_R \tilde{J}_b = \tilde{J}_b' \cdot T_{a_{\min}} + O\left(t^{-2}\right).$$

Let $\{\tilde{\jmath}_{k_c} \mid c \in \mathscr{L}_b \text{ and } k_c = 1, \ldots, \tilde{n}_c\}$ be a family of $C_0^\infty(\Gamma_b(0))$-functions having supports centred on some points of $\overset{\circ}{X}_c$, satisfying:

$$\tilde{J}_b' = \sum_{c \in \mathscr{L}_b} \sum_{k_c=1}^{\tilde{n}_c} \tilde{\jmath}_{k_c}^2 \, \tilde{J}_b' \tag{5.44}$$

and

$$\operatorname{supp} \tilde{\jmath}_{k_c} \cap X_b = \emptyset \quad \text{for all } c < b. \tag{5.45}$$

Notice also that according to the assumptions (5.36) and (5.32) one can choose the sub-family $\{\tilde{\jmath}_{k_b} \mid k_b = 1, \ldots, \tilde{n}_b\}$ such that:

$$\left(\bigcup_{k_b=1}^{\tilde{n}_b} \operatorname{supp} \tilde{\jmath}_{k_b} \cap \overset{\circ}{X}_b \right) \cap \operatorname{supp} f = \emptyset. \tag{5.46}$$

Hence, denoting by h_{k_c} a $C_0^\infty(\Gamma_b(0))$-function equal to one on the support of $\tilde{\jmath}_{k_c}$ and making use of (5.44) we get:

$$\begin{aligned}
&\left\langle \theta(R)\, f([iR,Q_b]) \left(\mathrm{D}_R \tilde{J}_b\right) \left[iR, \frac{(Q^a)^2}{2t} \right] \tilde{J}_b\, f([iR,Q_b])\, \theta(R) \right\rangle_t \\
&\lessapprox \frac{1}{t} \sum_{c \in \mathscr{L}_b} \sum_{k_c=1}^{\tilde{n}_c} \left\langle \theta(R)\, \tilde{\jmath}_{k_c}\, f([iR,Q_b])\, \underbrace{\tilde{J}_b' \cdot T_{a_{\min}} \left[iR, \frac{(Q^a)^2}{2t} \right] h_{k_c}}_{O(1)}\, f([iR,Q_b])\, \tilde{\jmath}_{k_c}\, \theta(R) \right\rangle_t \\
&\lessapprox \frac{C}{t} \sum_{c \in \mathscr{L}_b} \sum_{k_c=1}^{\tilde{n}_c} \| f([iR,Q_b])\, \tilde{\jmath}_{k_c}\, \theta(R)\, e^{-itR} \psi \|^2 \\
&\lessapprox \frac{C}{t} \sum_{c \in \mathscr{L}_b} \sum_{k_c=1}^{\tilde{n}_c} \| f\!\left(\tfrac{Q_b}{t}\right) \tilde{\jmath}_{k_c}\!\left(\tfrac{Q}{t}\right) \theta(R)\, e^{-itR} \psi \|^2
\end{aligned} \tag{5.47}$$

where in the last step the Appendix 6.4 has been used. Let us prove that the r.h.s. of (5.47) is $\leq C\|\psi\|^2$. For this, observe first that as a consequence of the assumption (5.34) we can choose the family $\{\tilde{\jmath}_{k_c}\}$ such that

$$\operatorname{supp} \tilde{\jmath}_{k_c} \subset \operatorname{supp} \hat{J} \quad \text{for all } c \in \mathscr{L}_b. \tag{5.48}$$

Secondly, (5.46) joint to (5.45) tells us that for all $c \in \mathscr{L}_b$

$$\operatorname{supp}\, (f \tilde{\jmath}_{k_c}) \cap \overset{\circ}{X}_b = \emptyset, \tag{5.49}$$

which together with the inclusion (see (5.48)):

$$\operatorname{supp}\, (f \tilde{\jmath}_{k_c}) \subset \Gamma_b(0) = \bigsqcup_{c \in \mathscr{L}_b} \overset{\circ}{X}_c$$

gives:

$$\operatorname{supp}(f\tilde{\jmath}_{k_c}) \subset \bigsqcup_{c\in\mathscr{L}_b\setminus\{b\}} \overset{\circ}{X}_c \quad \text{for all } c\in\mathscr{L}_b. \tag{5.50}$$

On the other hand, since $b\in\mathscr{L}_a(n)$ is fixed, any $c<b$ will belong to $\bigcup_{i=n+1}^{N_a}\mathscr{L}_a(i)$, or, in other words, for any $c<b$ there is a $\tilde{b}\in\mathscr{L}_a(n+1)$ such that $c\le\tilde{b}$. This shows that (5.50) can be written as:

$$\operatorname{supp}(f\tilde{\jmath}_{k_c}) \subset \bigcup_{\tilde{b}\in\mathscr{L}_a(n+1)} \bigsqcup_{c\in\mathscr{L}_{\tilde{b}}} \overset{\circ}{X}_c \equiv \bigcup_{\tilde{b}\in\mathscr{L}_a(n+1)} \Gamma_{\tilde{b}}(0). \tag{5.51}$$

We would like to apply $(\mathbf{P}_{n+1})$ to any of the functions $f\tilde{\jmath}_{k_c}$. Observe first that another way of writing the hypothesis of $(\mathbf{P}_{n+1})$ (which makes reference to the whole lattice level $\mathscr{L}_a(n+1)$ and not to the elements $\tilde{b}$ from it) is to demand to the support of J to be included in $\bigcup_{\tilde{b}\in\mathscr{L}_a(n+1)}\Gamma_{\tilde{b}}(0)$ and to replace in (5.5) or in (5.6) $\rho^A_{R_{\tilde{b}}}(\lambda)$ by $\min_{\tilde{b}\in\mathscr{L}_a(n+1)}\rho^A_{R_{\tilde{b}}}(\lambda)$. Then, (5.48) tells us that for all $c\in\mathscr{L}_b$ there exists a C_0^∞-function $\hat{\tilde{\jmath}}_{k_c}$ (which equals one on $\operatorname{supp}\tilde{\jmath}_{k_c}$, and has support included in $\operatorname{supp}\hat{J}$), which together with (5.26) and (5.51) allows us to conclude that the hypothesis of $(\mathbf{P}_{n+1})$ written for the product function $f\tilde{\jmath}_{k_c}$ is verified, and thus the r.h.s. of (5.47) is integrable.

It remains to estimate the term resulting from $\mathrm{D}_R\Phi$ which contains the Heisenberg derivative of $\frac{1}{2t}\left[iR,(Q^a)^2\right]$. It will be computed as follows:

$$\begin{aligned}
&\theta(R)\, f([iR,Q_b])\, \tilde{J}_b \left(\mathrm{D}_R\left[iR,\frac{(Q^a)^2}{2t}\right]\right) \tilde{J}_b\, f([iR,Q_b])\, \theta(R)\\
&\quad= \theta(R)\, f([iR,Q_b])\, \tilde{J}_b \left(\frac{\partial}{\partial t}\left[iR,\frac{(Q^a)^2}{2t}\right]\right) \tilde{J}_b\, f([iR,Q_b])\, \theta(R)\\
&\quad -\theta(R)\, f([iR,Q_b])\, \tilde{J}_b \left[iR,\left[iR,\frac{Q_a^2}{2t}\right]\right] \tilde{J}_b\, f([iR,Q_b])\, \theta(R)\\
&\quad +f([iR,Q_b])\, \tilde{J}_b\, \theta(R)\, R\left[iR,\frac{A}{t}\right] R\, \theta(R)\, \tilde{J}_b\, f([iR,Q_b])\\
&\quad -\left[f([iR,Q_b]),\theta(R)\right]\, \tilde{J}_b \left[iR,\left[iR,\frac{Q^2}{2t}\right]\right] \tilde{J}_b\, f([iR,Q_b])\, \theta(R)\\
&\quad +f([iR,Q_b])\, \theta(R)\, \tilde{J}_b \left[iR,\left[iR,\frac{Q^2}{2t}\right]\right] \tilde{J}_b\, \left[f([iR,Q_b]),\theta(R)\right]\\
&\quad +f([iR,Q_b])\, \left[\theta(R),\tilde{J}_b\right] \left[iR,\left[iR,\frac{Q^2}{2t}\right]\right] \tilde{J}_b\, \theta(R)\, f([iR,Q_b])\\
&\quad -f([iR,Q_b])\, \tilde{J}_b\, \theta(R) \left[iR,\left[iR,\frac{Q^2}{2t}\right]\right] \left[\theta(R),\tilde{J}_b\right]\, f([iR,Q_b]).
\end{aligned} \tag{5.52}$$

Since the commutator $[f([iR,Q_b]),\theta(R)]$ is of order $O(t^{-\mu})$ on the support of $\tilde{J}_b$ (which is a compact set of $\Gamma_b(0)$), the forth and the fifth term from the r.h.s. of (5.52) are integrable. The same is true for the last two terms in the above equality. Indeed, notice first that up to addition of some $O(t^{-2})$ contributions they are of the form:

$$\frac{1}{t} f([iR,Q_b])\, \tilde{J}_b' \cdot [\theta(R),Q]\, [iR,[iR,Q]\,] \cdot \frac{Q}{t}\, \tilde{J}_b\, \theta(R)\, f([iR,Q_b]).$$

Remark that there are some components of the above double commutator (namely those relative to some basis in X^b) which are *not* small (in the sense that they do not confer integrable decay in t) on supp $\tilde{J}_b$. Nevertheless, the above term is of the same type as the l.h.s. of (5.47), so we can use the same reasonement in order to show its integrability.

It remains to estimate the first three lines from the r.h.s. of (5.52). We shall proceed as in the proof of the Lemma 5.1, and minorize the first of these terms by:

$$-\frac{1}{t}\sum_{l=1}^{\dim X^a} \| [iR, Q_l] \| \sup_{x\in \operatorname{supp} \tilde{J}_b} |x_l| \; \|\tilde{J}_b\, f([iR,Q_b])\, \theta(R)\, e^{-itR}\psi\|^2 + O\Big(t^{-2}\Big)\, \|\psi\|^2. \tag{5.53}$$

Using the Mourre estimate, the mean value $\langle . \rangle_t$ of the third one is

$$\gtrapprox \frac{2}{t}\Big(\rho^A_{R_b}(\lambda) - \varepsilon\Big) \inf_{\mu \in \operatorname{supp}\theta} \mu^2 \; \|f([iR,Q_b])\, \tilde{J}_b\, \theta(R)\, e^{-itR}\psi\|^2. \tag{5.54}$$

Finally, the second term from the r.h.s. of (5.52) is:

$$\begin{aligned} \approx \; & - Re\,\Big\langle \theta(R)\, f([iR,Q_b])\, \tilde{J}_b\, [iR, [iR, Q_a]] \cdot \frac{Q_a}{t}\, \tilde{J}_b\, f([iR,Q_b])\, \theta(R) \Big\rangle_t \\ & - \frac{1}{t}\Big\langle \theta(R)\, \tilde{J}_b\, f([iR,Q_b])\; [iR, Q_a]^2\; f([iR,Q_b])\, \tilde{J}_b\, \theta(R) \Big\rangle_t, \end{aligned} \tag{5.55}$$

and since $\Gamma_b(0) \subseteq \Gamma_a(0)$ for any $b \in \mathscr{L}_a$, the above double commutator will be of order $O(t^{-\mu})$ on supp $\tilde{J}_b$. The second term of the above sum can be estimated with the aid of

$$[iR, Q_a]^2 = -T_a^2 + 2\, Re\, T_a \cdot [iR, Q_a] + \frac{Q_a^2}{t^2} \tag{5.56}$$

as being (for $\beta > 0$ sufficiently small)

$$\begin{aligned} \gtrapprox \; & -\frac{1}{t} \sup_{x\in \operatorname{supp}\tilde{J}_b} x_a^2 \; \|f([iR,Q_b])\, \tilde{J}_b\, \theta(R)\, e^{-itR}\psi\|^2 \\ & -\frac{2}{t}\sum_{l=1}^{\dim X_a} Re\, \Big\langle \theta(R)\, \tilde{J}_b\, \langle T_l\rangle_\beta^{1/2} f([iR,Q_b])\; \underbrace{\langle T_l\rangle_\beta^{-1} T_l}_{O(1)}\; \underbrace{[iR, Q_l]\;\, f([iR,Q_b])}_{O(1)}\, \langle T_l\rangle_\beta^{1/2}\, \tilde{J}_b\, \theta(R) \Big\rangle_t \\ & + \frac{1}{t}\, \|f\|_\infty^2\, \|\, |T_a|\, \tilde{J}_b\, \theta(R)\, e^{-itR}\psi\|^2. \end{aligned} \tag{5.57}$$

Notice that in order to obtain the second term above one has shown (in the same manner as in the Appendix 6.4) that $\Big[f([iR,Q_b]), \langle T_l\rangle_\beta^{1/2}\Big]$ brings an integrable contribution on the support of $\tilde{J}_b$. Then, the last two lines above are integrable as a consequence of the propagation theorems 3.2 and 3.3, so summing up (5.53), (5.54) and (5.57) we see that the l.h.s. of (5.52) dominates (in the sense of $\gtrapprox$):

$$\frac{1}{t}\left\{ 2\Big(\rho^A_{R_b}(\lambda) - \varepsilon\Big) \inf_{\mu\in\operatorname{supp}\theta} \mu^2 - \sup_{x\in\operatorname{supp}\tilde{J}_b} x_a^2 - \sum_{l=1}^{\dim X^a} \| [iR, Q_l] \| \sup_{x\in\operatorname{supp}\tilde{J}_b} |x_l| \right\} \|f([iR,Q_b])\, \tilde{J}_b\, \theta(R)\, e^{-itR}\psi\|^2.$$

Since the assumption (5.34) together with the hypothesis of $(\mathbf{P}_n)$ ensures strict positivity for the quantity in the above curly bracket, the estimate (5.39) is proven and hence the first implication of the theorem also.

It remains to prove that (5.3) is a consequence of (5.2). This will be performed in the same manner as above, the only difference being that we have to invoque the relation (5.22) instead of (5.56) when one wants to estimate the second term in (5.55). The comments we made after (5.55) show that, up to some integrable terms, the first two lines in the r.h.s. of (5.52) dominate

$$-\frac{1}{t}\left\langle \theta(R)\, f([iR,Q_b])\, \tilde{J}_b \left[iR, \frac{Q^2}{2t}\right] \tilde{J}_b\, f([iR,Q_b])\, \theta(R) \right\rangle_t$$

which, in turn, can be computed as in the proof of the Lemma 5.1 (see the estimate following (5.23)) and thus minorized (modulo $O(t^{-\mu})$) by:

$$-\sum_{l=1}^{\dim X} \|P_l\, \hat{\theta}(R)\| \sup_{x\in \operatorname{supp} \tilde{J}_b} |x_l| \sup_{\mu\in \operatorname{supp}\theta} \mu^2\ \|f([iR,Q_b])\, \tilde{J}_b\, \theta(R)\, e^{-itR}\psi\|^2.$$

This shows, as before, that the l.h.s. of (5.52) dominates

$$\frac{1}{t}\left\{2\left(\rho^A_{R_b}(\lambda)-\varepsilon\right) \inf_{\mu\in\operatorname{supp}\theta}\mu^2 - \sup_{\mu\in\operatorname{supp}\theta}\mu^2 \sum_{l=1}^{\dim X}\|P_l\,\hat{\theta}(R)\| \sup_{x\in\operatorname{supp}\tilde{J}_b}|x_l|\right\} \|f([iR,Q_b])\,\tilde{J}_b\,\theta(R)\,e^{-itR}\psi\|^2$$

where the curly bracket is strictly positive as a consequence of the relation (5.6) corresponding to $(\mathbf{P}_n)$. □

The rest of this section will be devoted to the proof of the statement (1.10). We will use a standard induction reasonement (see [SiSo1], [ABG2], [G], [If]), performed on the levels of some arbitrary lattice $\mathscr{L}$. Let us first denote, for any $a, b \in \mathscr{L}$, $a \le b$, by $\Omega^{\pm}_{a,b}$ the wave operators $\Omega^{\pm}(R_a, R_b; \mathcal{E}_b)$ and notice that they exist as a consequence of the existence of $W^{\pm}(R_a, R_b; J)$, with $\operatorname{supp} J \subset \Gamma_b(0)$. Then, since for the rank one lattice the statement (1.10), written for $\Omega^{\pm}(R_a, R_b; \mathcal{E}_b)$ is trivial, we shall suppose it true for any lattice of rank N, and prove it for all lattices $\mathscr{L}$ of rank $N+1$. Actually, it is sufficient to prove it for any state ψ localised with the aid of a smooth cutoff θ in a compact of $\mathbb{R}\setminus\mathcal{C}(R)$, because, due to the fact that the set of critical values of R is countable, a covering argument (see Proposition 4.2.6 in [ABG2]) will allow us to extend the result on all $\mathbb{R}$. Let us begin by computing, using a conveniently chosen partition of unity in X (e.g. the smooth one constructed in Section 2):

$$e^{-itR}\psi = J_{a_{\max}} e^{-itR}\psi + \sum_{n=2}^{N+1}\sum_{a\in\mathscr{L}(n)} J_a e^{-itR}\psi \,\tilde{\sim}\, \sum_{n=2}^{N+1}\sum_{a\in\mathscr{L}(n)} e^{-itR_a} \sum_{b\in\mathscr{L}_a} \Omega^{\pm}_{a,b}\,\underbrace{(\Omega^{\pm}_{a,b})^* W^{\pm}_a\psi}_{\psi_{a,b}}.$$

In the above modulo $o(1)$ equality $\tilde{\sim}$, $W^{\pm}_a$ stands for the limits in (4.2), and the Corollary 5.2 and the induction hypothesis where used. We thus have:

$$\psi \,\tilde{\sim}\, \sum_{n=2}^{N+1}\sum_{a\in\mathscr{L}(n)} e^{-itR}e^{-itR_a}\sum_{b\in\mathscr{L}_a}\Omega^{\pm}_{a,b}\psi_{a,b} \,\tilde{\sim}\, \sum_{a\in\mathscr{L}\setminus\{a_{\max}\}}\sum_{b\in\mathscr{L}_a} e^{-itR}e^{-itR_b}\mathcal{E}_b\psi_{a,b} \,\tilde{\sim}\, \sum_{a\in\mathscr{L}\setminus\{a_{\max}\}}\sum_{b\in\mathscr{L}_a}\Omega^{\pm}_b\psi_{a,b}$$

which proves $\psi \in \oplus_{b\in\mathscr{L}\setminus\{a_{\max}\}}\Omega^{\pm}_b\mathscr{H}$. □

6 Appendices

6.1 Appendix

Definition 6.1 *Let us define for all $a \in \mathscr{L}$ the following sets:*

i) $\mathscr{L}_a = \{b \in \mathscr{L} \mid b \leq a\}$

ii) $\mathscr{L}^a = \{b \in \mathscr{L} \mid b \geq a\}$

iii) $\overset{\circ}{X}_a = X_a \setminus \bigcup_{b \not\leq a} X_b$

iv) $\Gamma_a(0) = X \setminus \bigcup_{b \not\leq a} X_b$

Notice also the particular cases $a = a_{\min}$, for which $\overset{\circ}{X}_{a_{\min}} = \Gamma_{a_{\min}}(0) = X \setminus \bigcup_{b \not\leq a_{\min}} X_b$ and $a = a_{\max}$, for which $\overset{\circ}{X}_{a_{\max}} = \{0\}$ and $\Gamma_{a_{\max}}(0) = X$. Denoting by the sign $\sqcup$ the disjoint union of sets we have:

Proposition 6.1 *For an arbitrary $a \in \mathscr{L}$,*

i) $\overset{\circ}{X}_a = X_a \setminus \bigcup_{b > a} X_b$,

ii) $\overset{\circ}{X}_a \cap \overset{\circ}{X}_b = \emptyset$ *for all* $a, b \in \mathscr{L}$ *with* $a \neq b$,

iii) $X_a = \bigsqcup_{b \in \mathscr{L}^a} \overset{\circ}{X}_b$,

iv) $\Gamma_a(0) = \bigsqcup_{b \in \mathscr{L}_a} \overset{\circ}{X}_b$.

Remark also that for all $a \in \mathscr{L}$, $\mathscr{L}_a \cap \mathscr{L}^a = \{a\}$ yields $\overset{\circ}{X}_a = \Gamma_a(0) \cap X_a$.

Proof: Remember that we denoted by $\not\sim$ the "incomparability" sign between two elements of the lattice $\mathscr{L}$. Then, on one hand $X_a \setminus \cup_{b \not\sim a} X_b = X_a \setminus \cup_{b \not\sim a} \{X_b \cap X_a\}$, and on the other hand, since $\mathscr{L}$ is sup-stable there is a $c \in \mathscr{L}$ with $c > a$ and $c > b$ such that $X_b \cap X_a = (X^b + X^a)^\perp = (X^c)^\perp = X_c$. This shows the inclusion $X_a \setminus \cup_{b \not\sim a} X_b \subseteq X_a \setminus \cup_{c > a} X_c$ and thus (i) is proved. Let now $a \in \mathscr{L}$ be arbitrarily fixed and take first $b \in \mathscr{L} \setminus \mathscr{L}_a$. Then by Definition 6.1(i), $X_b \cap \overset{\circ}{X}_a = \emptyset$ and $\overset{\circ}{X}_b \subset X_b$ so (ii) is true for these b. For those b belonging to $\mathscr{L}_a \setminus \{a\}$ notice that according to (i) for all $c > b$ we have $X_c \cap \overset{\circ}{X}_b = \emptyset$. But a is one of these c and thus (ii) is proved. In order to prove (iii), suppose as usual a arbitrarily fixed and observe that the rank of an element $b \in \mathscr{L}^a$, denoted by $|b|_{\mathscr{L}^a}$, is generally different of $|b|_{\mathscr{L}}$. Denote also for $1 \leq j \leq \mathrm{rank}\mathscr{L}^a$ the j-th *level* in $\mathscr{L}^a$ as $\mathscr{L}^a(j) \equiv \{b \in \mathscr{L}^a \mid \ |b|_{\mathscr{L}^a} = j\}$ and the union of $\mathscr{L}^a(j)$ with all the levels in $\mathscr{L}^a$ which are below it by $\mathscr{L}^a_j \equiv \{b \in \mathscr{L}^a \mid \ |b|_{\mathscr{L}^a} \geq j\}$. Denote $n \equiv \mathrm{rank}\mathscr{L}^a = |a|_{\mathscr{L}^a}$ and take as induction hypothesis the statement

$$X_a = \bigsqcup_{b \in \mathscr{L}^a_{n-j}} \overset{\circ}{X}_b \cup \bigcup_{l=1}^{n-j-1} \bigcup_{b \in \mathscr{L}^a(l)} X_b \tag{6.1}$$

The first step of the induction is given by

$$X_a = \overset{\circ}{X}_a \cup \bigcup_{b \in \mathscr{L}^a \setminus \{a\}} X_b \tag{6.2}$$

in which the set inclusion $\supseteq$ is ensured by (i). Further, let us suppose that (6.1) is true for some $j < n$. Then, using (ii) one gets:

$$\begin{aligned} X_a &= \bigsqcup_{b \in \mathscr{L}^a_{n-j}} \overset{\circ}{X}_b \cup \bigcup_{b \in \mathscr{L}^a(n-j-1)} (\overset{\circ}{X}_b \cup \bigcup_{c>b} X_c) \cup \bigcup_{l=1}^{n-j-2} \bigcup_{b \in \mathscr{L}^a(l)} X_b \\ &= \bigsqcup_{b \in \mathscr{L}^a_{n-j-1}} \overset{\circ}{X}_b \cup \bigcup_{c \in \mathscr{M}_j} X_c \cup \bigcup_{l=1}^{n-j-2} \bigcup_{b \in \mathscr{L}^a(l)} X_b \end{aligned}$$

where the set $\mathscr{M}_j \equiv \{c \in \mathscr{L} \mid \exists\, b \in \mathscr{L}^a(n-j-1) \text{ such that } c > b\}$ obviously satisfies $\mathscr{M}_j \subseteq \mathscr{L}^a \setminus \bigcup_{l=n-j-1}^{n} \mathscr{L}^a(l)$, thus (iii) is proved.

Finally, (iii) and the definition of $\Gamma_a(0)$ gives:

$$\Gamma_a(0) = \bigsqcup_{b \in \mathscr{L}} \overset{\circ}{X}_b \setminus \bigcup_{b \in \mathscr{L} \setminus \mathscr{L}_a} \bigsqcup_{c \in \mathscr{L}^b} \overset{\circ}{X}_c$$

In order to prove (iv) it will be thus sufficient to show the set inclusion $\mathscr{L}_a \subseteq \mathscr{L} \setminus \{c \in \mathscr{L}^b \mid b \in \mathscr{L} \setminus \mathscr{L}_a\}$, i.e. $\mathscr{L}_a \cap \{c \in \mathscr{L}^b \mid b \in \mathscr{L} \setminus \mathscr{L}_a\} = \emptyset$. Suppose the existence of some $c_0 \in \mathscr{L}$ in this set intersection. Then there is a $b_0 \in \mathscr{L} \setminus \mathscr{L}_a$ such that $c_0 > b_0$, and on the other hand $c_0 \le a$, i.e. $b_0 \in \mathscr{L}_a$. Contradiction. $\square$

6.2 Appendix

We have to check (see [BaWo]) that the existence of the Abel operators Ω_a, which is equivalent to

$$\lim_{T\to\infty} \int_0^T \|(E_H(\mathbb{R}) - \Omega_a)\, e^{-itR_a} \mathcal{E}_a \psi\| = 0,$$

implies the convergence:

$$\lim_{T\to\infty} \int_0^T \|\mathcal{W}_a(t)\psi - \Omega_a\psi\| = 0.$$

But, using twice the intertwining relation for Ω_a, we get for all $\psi \in \mathscr{H}(X)$:

$$\begin{aligned} \|\mathcal{W}_a(t)\psi - \Omega_a\psi\| &= \|E_H(\mathbb{R})\, e^{-itH_a} \mathcal{E}_a\psi - e^{-itH} E_H(\mathbb{R})\, \Omega_a\psi\| \\ &= \|(E_H(\mathbb{R}) - \Omega_a)\, e^{-itH_a} \mathcal{E}_a\psi\|. \end{aligned}$$

Finally, Lemma 1 from [ObWo] can be used in order to get the desired result.

6.3 Appendix

The following Lemma puts in evidence some estimates, *uniform* in $\alpha \ge 0$, concerning the approximating family of hamiltonians $\{H_\alpha\}$. Notice that for any $a \in \mathscr{L}$, (1.2) can be written as $H_\alpha = H_{a,\alpha} + \mathbf{I}_{a,\alpha}$, where $\mathbf{I}_{a,\alpha}$ denotes the sum from the first equality in (1.2), performed only on the set $\mathscr{L} \setminus \mathscr{L}_a$.

Lemma 6.1 *For any $z \ll \inf \sigma(H_\alpha\,|_{\alpha=0})$ and for any $a \in \mathscr{L}$, if $R_{a,\alpha}$ denotes $(H_{a,\alpha} - z)^{-1}$, then there is a constant $C > 0$, independent of α, such that for any $\alpha \ge 0$:*

$$\alpha\, \|R_\alpha^{1/2}\, \chi(a)\, R_\alpha^{1/2}\| + \|R_\alpha^{1/2}\, \mathbf{I}_{a,\alpha}\, R_\alpha^{1/2}\| \le C \tag{6.3}$$

$$\alpha\, \|R_{a,\alpha} \sum_{b \in \mathscr{L} \setminus \mathscr{L}_a} \chi(b)\, R_\alpha\| + \|R_\alpha\, \mathbf{I}_{a,\alpha}\, R_{a,\alpha}\| \le C \tag{6.4}$$

For the proof, notice that the first inequality is a consequence of the hypothesis made on z, and of the obvious identity:

$$1 = R_\alpha^{1/2}\,(H_\alpha - z)\,R_\alpha^{1/2} = \alpha R_\alpha^{1/2}\,\chi(a)\,R_\alpha^{1/2} + R_\alpha^{1/2}\left(H_\alpha\,|_{\alpha=0} + \sum_{b\neq a}\alpha\chi(b)\; - z\right)R_\alpha^{1/2}.$$

Then, using

$$R_\alpha - R_{a,\alpha} = -R_\alpha\,\mathbf{I}_{a,\alpha}\,R_{a,\alpha} = -R_{a,\alpha}\,\mathbf{I}_{a,\alpha}\,R_\alpha$$

we compute

$$\|R_{a,\alpha}\,\mathbf{I}_{a,\alpha}\,R_\alpha\|^2 = \|(R_\alpha - R_{a,\alpha})^2\| \le 2\,\|R_{a,\alpha}\|\,\|R_\alpha\| + \|R_{a,\alpha}\|^2 + \|R_\alpha\|^2 \le C$$

where all the norms are in $B(\mathscr{H})$ and where the uniform boundedness (w.r.t. α) of the family $\{\|R_{a,\alpha}\|_{-1,1}\}$, tells us that C does not depend on α. Finally,

$$\alpha\,\|R_{a,\alpha}\sum_{b\in\mathscr{L}\setminus\mathscr{L}_a}\chi(b)\;R_\alpha\| \le \|R_{a,\alpha}\,\mathbf{I}_{a,\alpha}\,R_\alpha\| + \sum_{b\in\mathscr{L}\setminus\mathscr{L}_a}\|R_{a,\alpha}\langle P\rangle\|\,\|V(b)\|_{1,-1}\,\|\langle P\rangle R_\alpha\|$$

finishes the proof of (6.4). □

Remark that uniform estimates of the type (6.4) are useful when one wants to obtain decay of the difference $R_\alpha - R_{a,\alpha}$ on the support of some time-dependent cutoff $J \in C_0^\infty(\Gamma_a(0))$ (as in the proof of Proposition 4.1). As far as concerns the inequality (6.3), notice that it is of quadratic type, i.e. if χ is of the form $\tilde{\chi}\tilde{\chi}^*$, then it can be written as $\sqrt{\alpha}\,\|R_\alpha^{1/2}\,\tilde{\chi}\| \le C$, which is not enough for showing that the double commutator

$$[iR_\alpha,[iR_\alpha,Q_a]\,] = \sum_{b\in\mathscr{L}\setminus\mathscr{L}_a}\alpha\,R_\alpha^2\,[\chi(b),iP_a]\,R_\alpha^2 = \sum_{b\in\mathscr{L}\setminus\mathscr{L}_a} i\alpha\,R_\alpha^2\,\chi(b)\,P_a R_\alpha^2 + \text{h.c.}$$

is bounded uniformly w.r.t. α. This is one of the numerous reasons which makes the algebraic framework (introduced at the Section 2) indispensable.

6.4 Appendix

We have to prove that for any $a \in \mathscr{L}$ and any $b \in \mathscr{L}^a$, given two functions $J \in C_0^\infty(\Gamma_a(0))$ and $g \in C_0^\infty(X)$ satisfying $g = g\circ\pi_b$ on a neighborhood of the intersection of $\operatorname{supp} J$ with X_b, the estimate

$$\int_1^\infty \frac{dt}{t}\left\langle J\!\left(\tfrac{Q}{t}\right)\left\{g([iR,Q_b]) - g\!\left(\tfrac{Q_b}{t}\right)\right\}J\!\left(\tfrac{Q}{t}\right)\right\rangle_t \le C\|\psi\|^2 \tag{6.5}$$

is true for all $\psi \in \mathscr{H}(X)$.

For showing this, we use the Fourier spectral formula in order to compute the difference:

$$g([iR,Q_b]) - g\!\left(\tfrac{Q_b}{t}\right) = \int_X ds\,\widehat{g'}(s)\int_0^1 d\tau\;e^{is\tau[iR,Q_b]}\,T_b\,e^{is(1-\tau)\frac{Q_b}{t}} \tag{6.6}$$

where $\widehat{g'} \in \mathscr{S}(X)$. We thus have to look, for any $k = 1,\ldots,\dim X_b$, at:

$$\begin{aligned}\left\langle J\,e^{is\tau[iR,Q_b]}\,T_k\,e^{is(1-\tau)\frac{Q_b}{t}}\,J\right\rangle_t \;=\;& \left\langle J\,\langle T_k\rangle_\beta^{1/2}\,\underbrace{e^{is\tau[iR,Q_b]}\,\langle T_k\rangle_\beta^{-1}T_k\,e^{is(1-\tau)\frac{Q_b}{t}}}_{O(1)}\,\langle T_k\rangle_\beta^{1/2}\,J\right\rangle_t\\ &+\left\langle J\,\langle T_k\rangle_\beta^{1/2}\,e^{is\tau[iR,Q_b]}\,\underbrace{\langle T_k\rangle_\beta^{-1}T_k}_{O(1)}\left[\langle T_k\rangle_\beta^{1/2},\,e^{is(1-\tau)\frac{Q_b}{t}}\right]J\right\rangle_t\\ &+\left\langle J\left[e^{is\tau[iR,Q_b]},\,\langle T_k\rangle_\beta^{1/2}\right]\langle T_k\rangle_\beta^{-1/2}\,T_k\,e^{is(1-\tau)\frac{Q_b}{t}}\,J\right\rangle_t.\end{aligned} \tag{6.7}$$

It is clear that the first line above is integrable w.r.t. t as a consequence of Proposition 3.3 (the integrabilities in s and τ are trivial). Further, using the formula (3.33) we compute (as in the proof of Proposition 3.3):

$$\left[\langle T_k\rangle_\beta^{1/2}, e^{is\tau[iR,Q_b]}\right] = 2s\tau Re\, T_k \int_0^\infty d\omega\, \omega^{\frac{1}{4}} \left(\omega + \langle T_k\rangle_\beta^2\right)^{-1} \times$$
$$\times \int_0^1 d\sigma\, e^{is\tau(1-\sigma)[iR,Q_b]} \left[\,[iR,Q_b]\,,T_k\right] e^{is\tau\sigma[iR,Q_b]} \left(\omega + \langle T_k\rangle_\beta^2\right)^{-1}.$$

Hence, we essentially have to decide if for all $k,l = 1,\dots,\dim X_b$, the commutator $\left[\,[iR,Q_l]\,,T_k\right]$ confers on the support of J enough decay in order to ensure integrability with respect to both t and ω. But this has been already shown in the proof of the Lemma 4.1 with the aid of the relation (5.14) (see also the discussion following it), the only difference being that the role of a was played there by $a_{\min}$, so we could take k, l to run from 1 to $\dim X$. Notice also that because of the rapid decay of $\widehat{g'}$ the polynomials in s resulted from the various commutations of J with the unitary groups of the type $\exp(is\tau\sigma\,[iR,Q_b])$ do not influence the integrability in s. Finally, the second term from the r.h.s. of (6.7) will be treated in the same way as the previous one. Actually it is even simpler, because we will not be forced to use the good decay along certain directions of the connected components of R, the basically $O(t^{-\mu})$ decay being replaced by the better $O(t^{-2})$. □

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