Anisotropic random walks and the asymptotically one-dimensional diffusions on the abc-gaskets

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Abstract

A new class of fractals, abc-gaskets, is defined and asymptotically one-dimensional diffusion processes are studied on them. The class contains the Sierpiński gasket as well as infinitely many fractals which lack non-degenerate fixed points of renormalization maps (hence are not in the class of nested fractals). The lack of non-degenerate fixed points implies that the "standard" diffusions are degenerate on such fractals and the standard construction of (non-degenerate) diffusion processes fails.

The asymptotically one-dimensional diffusion, in contrast to the standard diffusion, is constructed on any abc-gasket by means of an unstable degenerate fixed point. To this end, the generating functions for numbers of steps of anisotropic random works on the abc-gaskets are analyzed, according to the line of authors' previous studies, and relevant scaling factors are calculated explicitly. In addition, a general strategy of handling random walk sequences with more than one parameters for the construction of asymptotically one-dimensional diffusion is proposed.

Key words: diffusion process, random walk, finitely ramified fractal, branching process, renormalization group.

Running title: Asymptotically one-dimensional diffusions

1 Introduction

The purposes of this paper are to define a new class of fractals called abc-gaskets and to describe the essential part of the construction of asymptotically one-dimensional diffusions on *abc*-gaskets according to the program of [7]. Our conclusion is Theorem 4.4 at the end of Section 4. The result was announced in [7] without proof.

A diffusion process in general can be viewed as a "continuum limit" of random walks on appropriate lattices on a base space and usual constructions of diffusion processes are essentially based on the existence of *non-degenerate fixed points* of renormalization maps. [A renormalization map is a certain procedure mapping random walks on a lattice to random walks on a similar coarser lattice. It essentially means taking a marginal distribution. See Section 3 for details.] The constructions of standard self-similar diffusion processes on the Sierpiński gasket [11, 3] and on the nested fractals [12] belong to this picture. Although it might seem straightforward to generalize the construction to any simple variants of the Sierpiński gasket, there are in fact simple examples of finitely ramified fractals that do not admit the above construction because of the absence of non-degenerate fixed points. Such examples are found in the class of *abc*-gaskets.

Intuitively saying, the *abc*-gasket is constructed as follows: assemble $a+b+c$ triangles into a block as fig. 1 and successively replace each triangle by a block reduced to the same size as the triangle. [Since by a technical reason, the *abc*-gasket must be a complete metric space, we have to take a closure with respect to some topology. For the presice definition, see Section 2.] The numbers a, b, and c are arbitrary positive integers and in case of $a=b=c=1$ the above construction coincides with that of the Sierpiński gasket.

The necessary and sufficient condition that the renormalization map has a non-degenerate fixed point is

$$a^{-1} < b^{-1} + c^{-1}, b^{-1} < c^{-1} + a^{-1}, c^{-1} < a^{-1} + b^{-1}. \tag{1.1}$$

[See Section 3]. Without this condition the standard method for constructing diffusion processes does not apply. In other words, the "standard" diffusion is degenerate in the sense that paths are almost surely bound in a single horizontal line through the starting point of the paths. [This means that as the base space changes its fractal structure within the class of *abc*-gaskets, the diffusion processes undergo a kind of phase transition.] It, however, should be noticed that the asymptotically one-dimensional diffusion process [7, 1, 2] can be constructed on any *abc*-gasket by means of *an unstable degenerate fixed point*.

The asymptotically one-dimensional diffusion is described as follows. Suppose that we are given a degenerate diffusion process on an *abc*-gasket. Let us perturb this situation by giving *infinitesimal* probabilities for off-horizontal jumps in a microscopic scale. Then we have two possibilities: either the perturbative effect is observed in the macroscopic scale or gradually vanishes. As is shown in Section 3, the criterion for the perturbative effect to survive in the macroscopic scale is

$$a^{-1} < b^{-1} + c^{-1}, \tag{1.2}$$

which is satisfied when b/a and c/a are not large. We call the resulting process *the asymptotically one-dimensional diffusion process*. The condition (1.2) ensures at least one of the degenerate fixed points being unstable. [See Section 3.] Then, we find a trajectory of the renormalization map starting at and going out of the infinitesimal neighborhood of the unstable degenerate fixed point. By tracing this trajectory in the inverse direction, we can define a sequence of anisotropic random walks converging to a non-degenerate diffusion on the *abc*-gasket.

In this situation, we observe a tendency that the extreme anisotropy in the microscopic scale disappears in the macroscopic scale and the isotropy is gradually restored [7, 1, 2, 5, 10]. This phenomenon is not observed on regular spaces such as Euclidean spaces and smooth manifolds. It will be a special feature of fractal spaces of which mechanism may be clarified in an appropriate general framework. [For a sketch of the mechanism of this phenomenon, see [1].]

This paper is organized as follows. In Section 2, we give precise definition(s) of the *abc*-gaskets. In Section 3, we formulate the renormalization map for the transition probabilities of random walks, and give a condition that it has no non-degenerate fixed point. Furthermore, we choose a sequence of anisotropic random walks on *abc*-gaskets keeping the consistency which guarantees the construction of the asymptotically one-dimensional diffusion. The core of this paper is Section 4, where we give the estimate on the generating function for numbers of steps of anisotropic random walks on the *abc*-gaskets that are sufficient for the construction of the asymptotically one-dimensional diffusion along the line of [7]. In course of analysis, we obtain explicit values of relevant dimensional constants and scaling factors. The algebraic part of our proof of Proposition 4.2 is computer-aided because of the routine of rather lengthy calculations. Finally, in Appendix, a general strategy for the choice of sequence of random walks with a multi-parameter are given and an open problem is proposed.

We conclude the Introduction by mentioning recent works on related subjects. Firstly we note that there appeared an alternative and a quite general convergence results for branching processes, which are very nicely applicable to the construction of the same diffusion [9]. An alternative and more general construction of the asymptotically one-dimensional (lower dimensional) diffusion on a subclass of nested fractals, together with some detailed studies such as the asymptotic estimates of the t dependences of the (diagonal) heat kernels $p_t(x,x)$ and the homogenization problems also appeared [4]. A characterization of asymptotically one-dimensional diffusions on the Sierpiński gasket by the exit distributions is given in [13]. The notion of *abc*-gaskets and the construction of the asymptotically one-dimensional diffusion are generalized to scale-irregular fractals in [5].

2 The *abc*-gaskets

The construction of *abc*-gaskets is roughly described in the Introduction. However, by a technical reason, it is desired to define the *abc*-gasket as a complete metric space. One way is to define the *abc*-gasket as a closed subset of $\mathbf{R}^2$ and another is to intrinsically construct it in an abstract style. The former may be more intuitive, while the latter may clarify the fact that the construction of the asymptotically one-dimensional diffusion is logically based on the recursive structure of the fractal. We shall describe below the both, all the arguments in the subsequent sections being independent of the choice of definitions.

2.1 The *abc*-gaskets as a subset of $\mathbf{R}^2$

Let us fix a triangle $\triangle_0$ with $a+b+c$ (smaller) triangles $\triangle_i, i = 1, 2, \cdots, a+b+c$ in $\triangle_0$ on a plane as in fig. 1. Triangles $\triangle_i$, $i = 1, 2, \cdots, a+1$, have their horizontal edges on the horizontal edge of $\triangle_0$; triangles $\triangle_i$, $i = a+1, a+2, \cdots, a+b+1$, have their right edges on the right edge of $\triangle_0$; triangles $\triangle_i$, $i = a+b+1, a+b+2, \cdots, a+b+c, 1$, have their left edges on the left edge of $\triangle_0$; and all the triangles $\triangle_i$, $i = 1, 2, \cdots, a+b+c$, have no points in common except for the vertices on edges of $\triangle_0$ (without their end points).

Next, let us consider the affine map φ_i, $i = 1, 2, \cdots, a+b+c$, which respectively maps $\triangle_0$ onto $\triangle_i$. We extend φ_1 to the whole plane (as an affine map) and denote it by the same symbol. We denote the inverse of the map by φ_1^{-1}. Also we denote by R the reflection with respect to the vertical line containing the left most vertex of $\triangle_0$. Let $\tilde{H}_0 = \triangle_0$, as the union of three line segments, and put

$$\tilde{H} = \varphi_1^{-n-1}\left(\bigcup_{i=1}^{a+b+c} \varphi_i \circ \varphi_1^n(\tilde{H}_n)\right), \quad n = 0, 1, 2, \cdots,$$

and $\tilde{H}_\infty = \bigcup_{n=0}^{\infty} \tilde{H}_n$. A pre-*abc*-gasket (of scale N) H_N is then defined by

$$H_N = \varphi_1^N\left(\tilde{H}_\infty\right) \cup R\left(\varphi_1^N\left(\tilde{H}_\infty\right)\right),$$

where we have duplicated the figure using R by a reason clarified later. Now, the *abc*-gasket G is defined to be the closure of $\bigcup_{n=0}^{\infty} H_n$. We denote the vertices of H_N by G_N.

2.2 An intrinsic definition of the *abc*-gasket

The pre-*abc*-gasket as a graph

For the convenience of readers, we reproduce the definition of the pre-*abc*-gasket in [6], where it was defined in a more general context.

Denote the set of all positive integers by $\mathbf{N}$, and the set of all non-negative integers by $\mathbf{Z}_+$. Fix $\sigma = (a, b, c) \in \mathbf{N^3}$. Define an equivalence relation $\overset{\sigma}{\sim}$ on $\{0, 1, 2, \cdots, a+b+c-1\} \times \{0, 1, 2\}$, parametrized by σ, by the defining relations

$$\begin{cases} (i,1) \overset{\sigma}{\sim} (i+1,0) & 0 \le i < a, \\ (i,2) \overset{\sigma}{\sim} (i+1,1) & a \le i < a+b, \\ (i,0) \overset{\sigma}{\sim} (i+1,2) & a+b \le i < a+b+c-1, \\ (a+b+c-1,0) \overset{\sigma}{\sim} (0,2). & \end{cases}$$

For $n \in \mathbf{Z}_+$, the *finite* pre-*abc*-gasket at n-th stage construction $\tilde{H}_n$, parametrized by (a, b, c), is a triplet

$$\tilde{H}_n = (V_n, B_n, P_n),$$

of a set of vertices V_n, a set of edges (a set of unordered pairs of vertices) B_n, and a set of three vertices $P_n = \{p_{n0}, p_{n1}, p_{n2}\} \subset V_n$, defined inductively as follows.

$\tilde{H}_0$ is defined by $V_0 = \{0, 1, 2\}$, $B_0 = \{\{0, 1\}, \{1, 2\}, \{2, 0\}\}$, and $P_0 = \{p_{00}, p_{01}, p_{02}\}$, where $p_{0i} = i$, $i = 0, 1, 2$.

Assume that $\tilde{H}_{n-1}$ is defined for an $n \in \mathbf{N}$. Define an equivalence relation $\sim$ on a set of pairs

$$\{(m, v) \mid m \in \{0, 1, 2, \cdots, a+b+c-1\},\ v \in V_{n-1}\},$$

by the defining relation

$$(m, v) \sim (m', v') \ \text{ if and only if } \ v = p_{n-1,i}, \ v' = p_{n-1,j} \text{ for some } i, j, \ \text{ and } (m, i) \overset{\sigma_n}{\approx} (m', j). \tag{2.1}$$

V_n is then defined by

$$V_n = \{(m, v) \mid m = 0, 1, 2, \cdots, a + b + c - 1, \ v \in V_{n-1}\}/\sim \ .$$

Denote the equivalence class of (m, v) by $((m, v))$. B_n is defined by

$$\begin{aligned} B_n &= \{\{((m, v)), ((m, w))\} \mid \\ &\qquad m = 0, 1, \cdots, a + b + c - 1, \ \ \{v, w\} \in B_{n-1}\}, \end{aligned}$$

and $P_n = \{p_{n0}, p_{n1}, p_{n2}\}$ is defined by

$$p_{n0} = ((0, p_{n-1,0})), \ \ p_{n1} = ((a, p_{n-1,1})), \ \ p_{n2} = ((a + b, p_{n-1,2})). \tag{2.2}$$

For each $n \in \mathbf{N}$, there is an injection

$$\iota : \ V_{n-1} \to V_n$$

defined by

$$V_{n-1} \ni v \mapsto ((0, v)) \in V_n.$$

The map ι maps a bond $\{v, v'\} \in B_{n-1}$ to a bond $\{((0, v)), ((0, v'))\} \in B_n$. We can therefore identify (V_{n-1}, B_{n-1}) as a subset of (V_n, B_n). Define a graph $\tilde{H}_\infty = (V_\infty, B_\infty)$ by

$$\tilde{H}_\infty = \bigcup_{n \in \mathbf{Z}_+} (V_n, B_n),$$

with the identification induced by ι assumed.

Note that with the identification ι, $p_{n0} = p_{00} = 0$ holds for any $n \in \mathbf{N}$. We call $0 \in \tilde{H}_\infty$ the origin, and also use the notation O.

Let $(V_{\infty,R}, B_{\infty,R})$ be the graph defined as (V_∞, B_∞), with (a,c,b) in place of the parameter (a,b,c). Define also an equivalence relation $\overset{R}{\sim}$ by $(+,O) \overset{R}{\sim} (-,O)$. A graph $\tilde{H}'_\infty = (V'_\infty, B'_\infty)$ (pre-abc-gasket as a graph) is defined by

$$V'_\infty = \Big(\{(+,v) \mid v \in V_\infty\} \bigcup \{(-,v) \mid v \in V_{R,\infty}\}\Big) \,/\, \overset{R}{\sim},$$

and

$$\begin{aligned} B'_\infty &= \{\{((+,v)),((+,w))\} \mid \{v,w\} \in B_\infty\} \\ &\quad \bigcup \{\{((-,v)),((-,w))\} \mid \{v,w\} \in B_{R,\infty}\}, \end{aligned}$$

where $((+,v))$ denotes the equivalence class of $(+,v)$. Again, we write O for $((+,O)) = ((-,O)) \in V'_\infty$ and call it the origin.

Metric on the pre-abc-gasket

Let $s \in \{+,-\}$, N and n be integers satisfying $n \geq N \geq 0$, for each $k \in \{n, n-1, \cdots, N+1\}$, m_k be an integer satisfying $0 \leq m_k < a+b+c$, and $i \in \{0,1,2\}$. Then the sequence

$$(s, m_n, m_{n-1}, \cdots, m_{N+1}, N, i) \tag{2.3}$$

determines an element $x \in V'_\infty$ by the sequence of equivalence classes

$$\begin{aligned} &x = ((s,v)),\ v = ((m_n, v_{n-1})),\ v_{n-1} = ((m_{n-1}, v_{n-2})), \\ &v_{n-2} = ((m_{n-2}, v_{n-3})),\ \cdots,\ v_{N+1} = ((m_{N+1}, p_{N,i})). \end{aligned}$$

We take (2.3) as a representation of x and write

$$x = (s, m_n, m_{n-1}, \cdots, m_{N+1}, N, i) \in V'_\infty. \tag{2.4}$$

As a convention, we write $x = (s,n,i)$ with $N = n$ for $x = ((s, p_{ni}))$.

Fix $N \in \mathbf{Z}_+$, and define $G_{-N} \subset V'_\infty$ as a set of vertices $x \in V'_\infty$ which has a representation (2.4). $P_0 = V_0$ implies that each element $x \in V'_\infty$ has a representation of the form (2.3) with

$N = 0$, hence, $G_0 = V'_\infty$. Also (2.2) implies

$$G_0 = V'_\infty \supset G_{-1} \supset G_{-2} \supset \cdots .$$

Let $x \in G_0 \setminus \{O\}$. x may have more than one representations. However, (2.1) implies that for each fixed N,

$$x = (s, m_n, m_{n-1}, \cdots, m_{N+1}, N, i) = (s', m'_n, m'_{n-1}, \cdots, m'_{N+1}, N, i')$$

if and only if $(m_{N+1}, i) \overset{\sigma_{N+1}}{\sim} (m'_{N+1}, i')$ and $s = s'$, $m_k = m'_k$, $k = n, n-1, \cdots, N+2$. Hence there is a one-to-one onto map

$$\tilde{\tau}^*_N : \ G_0 \to G_{-N}$$

defined by

(2.5) $$\tilde{\tau}^*_N(s, m_n, m_{n-1}, \cdots, m_1, 0, i) = (s, m_n, m_{n-1}, \cdots, m_1, N, i)\,.$$

Put

$$E_{-N} = \{\{\tilde{\tau}^*_N(x), \tilde{\tau}^*_N(y)\} \mid \{x, y\} \in B'_\infty\}.$$

Let x and y be elements of G_0. Denote by $path(x, y)$ the collection of finite sequences

$$z = \{z_0 = x, z_1, \cdots, z_\kappa = y\}, \quad \text{for some } \kappa = \kappa_z \in \mathbf{Z}_+,$$

which has a property that for each $i = 0, 1, \cdots, \kappa_z - 1$, $\{z_i, z_{i+1}\} \in E_{-\nu_z(i)}$ for some $\nu_z(i) \in \mathbf{Z}_+$.

For $z \in path(x, y)$ put

$$L(z) = \sum_{i=0}^{\kappa_z - 1} (\min\{a, b, c\} + 1)^{\nu_z(i)}.$$

Then the metric $\tilde{d}$ is defined by $\tilde{d}(x, y) = \inf_{z \in path(x,y)} L(z)$. It is straightforward to see that $\tilde{d}$ is a metric, and, in particular,

(2.6) $$\tilde{d}(x, y) = (\min\{a, b, c\} + 1)^N\,, \quad \{x, y\} \in E_{-N}.$$

In considering (anisotropic) random walks on $\tilde{H}'_\infty$, it is convenient to have the notion of vertex types and edge types [7]. As in Section 3, we classify the vertex set V'_∞ into 6 types, A, B, C, D, E, F, and an edge as an ordered pair of vertices into 18 types, X_y with $X = A, B, C$ and $y = p, q, r, s$, X_y with $X = E, F$ and $y = p, r$, and D_p, D_q. The classification rule is as follows.

(1) The origin O is of type A.

(2) A vertex which has two representations $(s, m_n, \cdots, m_1, 0, i)$ and $(s, m_n, \cdots, m'_1, 0, i')$ for some s, n, i, i', and m_k's, satisfying $(m_1, i) \overset{\sigma}{\sim} (m'_1, i')$ is of type A, B, or C, if $(i, i') = (1, 0)$, $(0, 2)$, or $(2, 1)$, respectively.

(3) Any other vertex with a representation of the form $(s, m_n, \cdots, m_1, 0, i)$ is of type D, E, or F, if $i = 2, 0$, or 1, respectively.

(4) Let $\{x, y\} \in B'_\infty$. Then x has a representation of a form $x = (s, m_n, \cdots, m_1, 0, i)$, and y has that of a form $y = (s, m_n, \cdots, m_1, 0, i')$. If x is of type A then (x, y) as an ordered pair is of type A_p, A_q, A_r, or A_s, if $(i, i') = (0, 2)$, $(1, 2)$, $(0, 1)$, or $(1, 0)$, respectively. If x is of type B then (x, y) is of type B_p, B_q, B_r, or B_s, if $(i, i') = (0, 2)$, $(2, 1)$, $(0, 1)$, or $(2, 0)$, respectively. If x is of type C then (x, y) is of type C_p, C_q, C_r, or C_s, if $(i, i') = (1, 2)$, $(2, 0)$, $(1, 0)$, or $(2, 1)$, respectively. If x is of type D then (x, y) is of type D_p or D_q, if $(i, i') = (2, 0)$ or $(2, 1)$, respectively. If x is of type E then (x, y) is of type E_p or E_r, if $(i, i') = (0, 2)$ or $(0, 1)$, respectively. If x is of type F then (x, y) is of type F_p or F_r, if $(i, i') = (1, 2)$ or $(1, 0)$, respectively.

Inspired by the Sierpiński gasket, we call the edges of types X_r with $X = A, B, C, E, F$, and A_s, the 'horizontal' edges.

The abc-gasket

For $N \in \mathbf{N}$ put $G_N = G_0$. G_N has a graph structure with the edge set

$$E_N \stackrel{\text{def}}{=} B'_\infty. \tag{2.7}$$

Define a metric d_N on G_N by

$$\begin{aligned} d_N(x,y) &= \tilde{d}(x,y)(\min\{a,b,c\}+1)^{-N}\,, \\ & x,y \in G_N. \end{aligned} \tag{2.8}$$

Note that $\tilde{\tau}^*(G_N) = \tilde{\tau}^*(G_0) = G_{-1} \subset G_0 = G_{N+1}$, where the one-to-one map $\tilde{\tau}^*$ is defined in (2.5). We can regard G_N to be a subset of G_{N+1} by identifying G_N with $\tilde{\tau}^*(G_N)$. By induction, we have

$$G_0 \subset G_1 \subset G_2 \subset G_3 \subset \cdots .$$

Let $G_\infty = \bigcup_{N \in \mathbf{Z}_+} G_N$. Using (2.8) and (2.6), we see that if $N' \geq N$

$$d_{N'}(x,y) = d_N(x,y), \quad x,y \in G_N \subset G_{N'}. \tag{2.9}$$

For any x and y in G_∞, define $d(x,y)$ as follows. There exists $N \in \mathbf{Z}_+$ such that $x, y \in G_N$. Then define $d(x,y) = d_N(x,y)$. With (2.9) we see that d is a well-defined metric.

The abc-gasket G is the completion of G_∞ by d.

A subset $G_N \subset G$ has a graph structure with the vertex set G_N and the edge set E_N given by (2.7). We use the notation $H_N = (G_N, E_N)$ to refer to the graph structure, and call it a pre-abc-gasket (of scale N). For $x \in G_N$, we call a vertex $y \in G_N$ an N-neighbor (of x) if $\{x,y\} \in E_N$. We use, for H_N, the notion of vertex types and edge types, A_p, A_r, etc., and the terminology 'horizontal (edge)', in accordance with the corresponding notations for $\tilde{H}'_\infty$.

We also use the notion of a 'unit triangle'. By a unit triangle of H_N (or a unit triangle of scale N) we mean a closure (in G with respect to the metric d) of a set

$$\bigcup_{N'\geq N} \{(s, m_n, \cdots, m_0, m_{-1}, \cdots, m_{-N'+N}, 0, i) \in G_0 \mid i = 0, 1, 2,$$
$$m_{-k} = 0, 1, 2, \cdots, a+b+c-1,\ k = 1, 2, \cdots, N'-N\} \ (\subset G)$$

for some fixed s, m_n, $\cdots$, m_0. $(s, m_n, \cdots, m_0, 0, i) \in G_0 = G_N$, $i = 0, 1, 2$, are defined to be the three vertices of the triangle.

If $a = b = c = 1$, then G is the Sierpiński gasket.

Remark. $\min\{a, b, c\} + 1$ in the definitions of metrics can be replaced by any real ℓ satisfying $1 < \ell \leq \min\{a, b, c\} + 1$. The upper bound implies (2.6) with $\min\{a, b, c\} + 1$ replaced by ℓ. The lower bound implies that there exists $C > 0$ such that if x and y is in a unit triangle of G_N then $d(x, y) \leq C\,\ell^{-N}$.

3 Transition probabilities

Probability laws of random walks on H_n are specified by assigning transition probabilities to each bond (edge) of H_n and the consistency condition between random walks on H_n and H_{n-1} is written in the form of renormalization map.

Renormalization map for transition probabilities

We classify vertices of a pre-*abc*-gasket H_n into 6 groups (type A, B, C, D, E, and F) as fig. 2 according to the possible directions of transitions from each vertex. We refer to the possible directions by the symbols p, q, r, and s, respectively, as fig. 2. As a result, all the bonds of the pre-*abc*-gasket are classified into 18 types which we denote by A_p, A_q, $\cdots$, F_r, respectively.

If $a = 1$, the type D does not exist. For simplicity, we assume $a > 1$, $b > 1$, and $c > 1$. A reader who is interested in the other cases should neglect irrelevant statements and formulae below.

Now let us consider a random walk on H_n. We set the relative probabilities of transitions along bonds of types r, p, and q as $1, \eta$, and ζ, respectively. Namely we assign transition probabilities to 18 types of bonds as Table 1. The transition probability along a bond of type s is set equal to the opposite direction.

We next consider the consistency condition between random walks on H_n and on H_{n-1}. Let us fix a vertex of H_{n-1} and cut the block(s) of H_n to which the vertex belongs. To be specific, we assume that the vertex is of the type A. The vertex A belongs to two blocks (fig. 3). We denote the union of these blocks by K and the vertices of H_{n-1} contained in K by A, A'_p, A'_q, A'_r, and A'_s. Let $\Omega(A,t)$, $t = p, q, r, s$, be the set of all walks on K starting at the vertex A without reaching to any one of the vertices A'_p, A'_q, A'_r, and A'_s before the end at A'_t. Similarly we define the set $\Omega(X,t)$ of walks for $X = B, C, \cdots, F$, and for $t = p, q, r, s$, such that bonds of X_t type exist. Put

$$\Omega(X) = \bigcup_{t=p,q,r,s} \Omega(X,t), \ X = A, B, \cdots, F.$$

We assign a probability to each walk in $\Omega(X)$ by making a product of the transition probabilities set in Table 1. The probability measure on $\Omega(X)$ defined as above is denoted by $P_{X,\eta,\zeta}$.

Proposition 3.1. *Let $X = A, B, \cdots, F$, and let $t = p, q, r, s$, such that bonds of X_t type exist. The probability $P_{X,\eta,\zeta}(\Omega(X,t))$ is equal to the transition probability assigned to X_t given by Table 1 with η and ζ replaced by*

$$(3.1) \qquad Y(\eta,\zeta) = \frac{(ac+a+c)\eta^2 + (ac+a+b)\eta\zeta + (ac+b+c)\eta + ac\zeta}{X(\eta,\zeta)},$$

$$(3.2)\qquad Z(\eta,\zeta) \;=\; \frac{(ab+a+b)\zeta^2+(ab+a+c)\eta\zeta+(ab+b+c)\zeta+ab\eta}{X(\eta,\zeta)},$$

respectively, where

$$X(\eta,\zeta) = bc\eta\zeta + (bc+a+c)\eta + (bc+a+b)\zeta + bc + b + c$$

The proof of this proposition will be sketched later together with that of Proposition 4.2.

Fixing $N \in \mathbf{Z}_+$, we consider sequences $\{\eta_n^{(N)} | 0 \le n \le N\}$ and $\{\zeta_n^{(N)} | 0 \le n \le N\}$ satisfying

$$(3.3)\qquad \eta_{n-1}^{(N)} \;=\; Y(\eta_n^{(N)}, \zeta_n^{(N)}),\ \ n = N, N-1, \cdots, 2, 1,$$

$$(3.4)\qquad \zeta_{n-1}^{(N)} \;=\; Z(\eta_n^{(N)}, \zeta_n^{(N)}),\ \ n = N, N-1, \cdots, 2, 1,$$

and define random walks on H_n with transition probabilities given by Table 1 with $\eta = \eta_n^{(N)}$ and $\zeta = \zeta_n^{(N)}$. The recursion relations (3.3) and (3.4), which we call the renormalization map, are viewed as the consistency condition of probability laws of random walks on H_n and on H_{n-1}, which imply that the random walks on H_n are obtained by neglecting the fine structure of those on $H_m, m > n$.

Fixed points

Let us study fixed points of (3.1) and (3.2):

$$(3.5)\qquad \eta = Y(\eta,\zeta)\ \ ,\ \ \zeta = Z(\eta,\zeta)\ \ ,$$

It is convenient to consider the homogenious equations corresponding to (3.1) and (3.2):

$$(3.6)\qquad X(\xi,\eta,\zeta) \;=\; bc(\xi+\eta)(\xi+\zeta)+\theta\xi,$$

$$(3.7)\qquad Y(\xi,\eta,\zeta) \;=\; ca(\eta+\xi)(\eta+\zeta)+\theta\eta,$$

$$(3.8)\qquad Z(\xi,\eta,\zeta) \;=\; ab(\zeta+\xi)(\zeta+\eta)+\theta\zeta,$$

where

$$\theta \;=\; a(\eta+\zeta)+b(\zeta+\xi)+c(\xi+\eta). \tag{3.9}$$

Note that

$$Y(\eta,\zeta)=\frac{Y(1,\eta,\zeta)}{X(1,\eta,\zeta)}\ ,\ Z(\eta,\zeta)=\frac{Z(1,\eta,\zeta)}{X(1,\eta,\zeta)}\ .$$

Then the fixed point equation (3.5) is written as

$$\xi:\eta:\zeta=X(\xi,\eta,\zeta):Y(\xi,\eta,\zeta):Z(\xi,\eta,\zeta). \tag{3.10}$$

This has trivial solutions

$$\xi:\eta:\zeta=1:0:0\ ,\ 0:1:0\ ,\ 0:0:1\ , \tag{3.11}$$

which give degenerate random walks.

Proposition 3.2. *The necessary and sufficient condition that (3.10) has a nontrivial (positive) solution is*

$$a^{-1}<b^{-1}+c^{-1}, b^{-1}<c^{-1}+a^{-1}, c^{-1}<a^{-1}+b^{-1}. \tag{3.12}$$

Moreover the nontrivial solution is given by

$$\xi:\eta:\zeta=\frac{1}{-a^{-1}+b^{-1}+c^{-1}}:\frac{1}{a^{-1}-b^{-1}+c^{-1}}:\frac{1}{a^{-1}+b^{-1}-c^{-1}}. \tag{3.13}$$

We omit the proof because it is easy and the result is not used later. It is also an easy exercise to show that the above nontrivial solution is a stable fixed point.

On the other hand, the trivial fixed points (3.11) may be stable or unstable. In fact, Proposition 3.3 below implies that the solution $\xi:\eta:\zeta=1:0:0$ to (3.10), i.e. the solution $(\eta,\zeta)=(0,0)$ to (3.5) is unstable if

$$a^{-1}<b^{-1}+c^{-1}. \tag{3.14}$$

In what follows, we assume (3.14). Since at least one of the inequalities of (3.12) must hold, we can assume (3.14) without loss of generality. Put

$$\delta = \frac{(a+1)(b+c)}{bc+b+c}.$$

Then, (3.14) implies $\delta > 1$.

Proposition 3.3. *Choose the initial values*

$$\eta_N^{(N)} = c\delta^{-N}\kappa, \tag{3.15}$$

$$\zeta_N^{(N)} = b\delta^{-N}\kappa, \tag{3.16}$$

for the recursion (3.3) and (3.4), where κ is an arbitrary positive number independent of N. Then, under the assumption (3.14), the limits

$$\eta_n = \lim_{N\to\infty} \eta_n^{(N)} \tag{3.17}$$

$$\zeta_n = \lim_{N\to\infty} \zeta_n^{(N)} \tag{3.18}$$

exist and satisfy

$$\eta_{n-1} = Y(\eta_n, \zeta_n), \ \ n = 1, 2, \cdots, \tag{3.19}$$

$$\zeta_{n-1} = Z(\eta_n, \zeta_n), \ \ n = 1, 2, \cdots, \tag{3.20}$$

$$C_1\delta^{-n} < \eta_n < C_2\delta^{-n}, \ \ n = 1, 2, \cdots, \tag{3.21}$$

$$C_3\delta^{-n} < \zeta_n < C_4\delta^{-n}, \ \ n = 1, 2, \cdots, \tag{3.22}$$

Proof. Since $\delta > 1$, $\eta_N^{(N)}$ and $\zeta_N^{(N)}$ are sufficiently small when N is large. On the other hand, if η and ζ are sufficiently small, (3.1) and (3.2) are written as

$$\begin{pmatrix} Y(\eta,\zeta) \\ Z(\eta,\zeta) \end{pmatrix} = R \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + \begin{pmatrix} \tilde{Y}(\eta,\zeta) \\ \tilde{Z}(\eta,\zeta) \end{pmatrix},$$

where

$$R = \frac{1}{bc+b+c} \begin{pmatrix} ac+b+c & ac \\ ab & ab+b+c \end{pmatrix},$$

and the functions $\tilde{Y}(\eta,\zeta)$ and $\tilde{Z}(\eta,\zeta)$ obey the bounds

$$\begin{aligned} |\tilde{Y}(\eta,\zeta)| &< C_5(\eta^2+\zeta^2), \\ |\tilde{Z}(\eta,\zeta)| &< C_6(\eta^2+\zeta^2), \end{aligned}$$

for some constants independent of η and ζ. Note that the matrix R has eigenvalues $\delta(>1)$ and $\dfrac{b+c}{bc+b+c}(<1)$, with eigenvectors $\begin{pmatrix} c \\ b \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively. Then the remaining part of the proof is just a routine work. □

In the next section, we study the sequence of random walks on $H_n, n = 1,2,3,\cdots$ with transition probabilities given by Table 1 with $\eta = \eta_n$ and $\zeta = \zeta_n$.

4 Hitting times

In our construction of the asymptotically one-dimensional diffusion process, the multi-distributions of numbers of steps of the random walks play the key role as is seen from the arguments for the Sierpiński gasket in [7, Section 2]. To this end, we analyze the generating functions for the numbers of steps of random walks on pre-*abc*-gaskets. A limit theorem related to the discrete-time multi-type non-stationary branching processes [8] is then applied, which, with arguments in [7], proves the existence of the asymptotically one-dimensional diffusions on the *abc*-gaskets. See [7] for details on the actual construction of the process.

Generating functions

We consider a random walk on H_N with transition probabilities given by Table 1 with $\eta = \eta_N$ and $\zeta = \zeta_N$, where η_N and ζ_N are defined in Proposition 3.3. In order to see the behavior of the walks in the scale of $H_n, n < N$, we generalize the set $\Omega(X,t)$ of walks in Section 3 as follows. Fix a vertex $X \in H_n$ of the type A and denote the "adjacent" vertices of X in H_n by $A''_p, A''_q, A''_r, A''_s$ in an analogous way as in fig. 3. In case (n,N) is $(n-1,n)$, the situation is exactly the same as fig. 3, but in general, the figure has a finer structure. Let $\Omega_{n,N}(X,t),\ t = p,\ q,\ r,\ s$, be the set of all walks starting at the vertex X without reaching to any one of the vertices $A''_p,\ A''_q,\ A''_r$, and A''_s before the end at A''_t. Similarly we define the set $\Omega_{n,N}(X,t)$ of walks for other types B,C,D,E,F of vertices X and for all directions $t = p,q,r,s$ such that bonds of X_t type exist. In particular, $\Omega(X,t) = \Omega_{n-1,n}(X,t)$. Put

$$\Omega_{n,N}(X) = \bigcup_{t=p,q,r,s} \Omega_{n,N}(X,t),\ \ X = A, B, \cdots, F$$

and set the probability $P_{n,N,X}$ on $\Omega_{n,N}(X)$ by assigning to bonds of H_N the transition probabilities given by Table 1 with $\eta = \eta_N$ and $\zeta = \zeta_N$. Then, the probability

$$\pi^{(n)}_{X_t} = P_{n,N,X}(\Omega_{n,N}(X,t))$$

is independent of N and is equal to the transition probability given by Table 1 with $\eta = \eta_n$ and $\zeta = \zeta_n$. We put

$$\Pi_n = \operatorname{diag}(\pi^{(n)}_{A_p}, \pi^{(n)}_{A_q}, \cdots, \pi^{(n)}_{F_r}),$$

where $\operatorname{diag}(\alpha, \beta, \cdots, \gamma)$ stands for the diagonal matrix with diagonal elements $\alpha, \beta, \cdots, \gamma$.

For a walk $\omega \in \Omega_{n,N}(X)$ and for a set of 18 variables $\tilde{u} = (\tilde{u}_{A_p}, \tilde{u}_{A_q}, \cdots, \tilde{u}_{F_r})$, we use the following abbreviation

$$\tilde{u}^\omega = \tilde{u}_{A_p}^{|\omega|_{A_p}} \tilde{u}_{A_q}^{|\omega|_{A_q}} \cdots \tilde{u}_{F_r}^{|\omega|_{F_r}}, \tag{4.1}$$

where $|\omega|_{X_t}$ is the number of passes of the walk ω through bonds of X_t type. Then the generating function of numbers of steps is by definition

$$\Phi_{n,N,X,t}(u) = \Pi_n^{-1} \sum_{\omega \in \Omega_{n,N}(X,t)} (\Pi_N u)^\omega, \quad u = (u_{A_p}, u_{A_q}, \cdots, u_{F_r}) \tag{4.2}$$

and we write

$$\Phi_{n,N}(u) = (\Phi_{n,N,A,p}(u), \Phi_{n,N,A,q}(u), \cdots, \Phi_{n,N,F,r}(u)).$$

Note that (3.19) and (3.20) imply

$$\Phi_{n,N}(\Pi_N \vec{1}) = \vec{1}, \tag{4.3}$$

where $\vec{1}$ denotes the vector with all elements 1.

Renormalization map for generating functions

The generation function has a recursive structure. To see this, we introduce the set of variables $\tilde{u} = (\tilde{u}_{A_p}, \tilde{u}_{A_q}, \cdots, \tilde{u}_{F_r})$ and define the mapping (independent of n)

$$\tilde{U} = \tilde{F}(\tilde{u}) \tag{4.4}$$

by putting

$$\tilde{U}_{X_t} = \sum_{\omega \in \Omega(X,t)} \tilde{u}^{\omega} \tag{4.5}$$

$$\tilde{U} = (\tilde{U}_{A_p}, \tilde{U}_{A_q}, \cdots, \tilde{U}_{F_r}). \tag{4.6}$$

Note that (3.19) and (3.20) imply

$$\Pi_{n-1}\vec{1} = \tilde{F}(\Pi_n \vec{1}). \tag{4.7}$$

Then the set of generating functions is written as

$$\Phi_{n,N}(u) = \Pi_n^{-1} \tilde{F}^{N-n}(\Pi_N u),$$

and hence is decomposed into the product of the mappings

$$U = \Pi_{n-1}^{-1} \tilde{F}(\Pi_n u). \tag{4.8}$$

Reduction of variables

The following lemma is easily shown.

Lemma 4.1. *If the variable $\tilde{u}$ satisfy the 'consistency conditions'*

$$\tilde{u}_{A_s} = \tilde{u}_{A_r}, \quad \tilde{u}_{B_s} = \tilde{u}_{B_p}, \quad \tilde{u}_{C_s} = \tilde{u}_{C_q}, \tag{4.9}$$

$$\tilde{u}_{A_p}\tilde{u}_{B_q} = \tilde{u}_{B_s}\tilde{u}_{A_q}, \quad \tilde{u}_{A_s}\tilde{u}_{B_q} = \tilde{u}_{B_r}\tilde{u}_{A_q}, \quad \tilde{u}_{A_p}\tilde{u}_{C_r} = \tilde{u}_{C_p}\tilde{u}_{A_r}, \tag{4.10}$$

$$\tilde{u}_{A_p}\tilde{u}_{C_s} = \tilde{u}_{C_p}\tilde{u}_{A_q}, \quad \tilde{u}_{D_q}\tilde{u}_{A_p} = \tilde{u}_{D_p}\tilde{u}_{A_q}, \quad \tilde{u}_{C_r}\tilde{u}_{F_p} = \tilde{u}_{C_p}\tilde{u}_{F_r}, \tag{4.11}$$

$$\tilde{u}_{A_p}\tilde{u}_{D_q}\tilde{u}_{E_r} = \tilde{u}_{A_r}\tilde{u}_{E_q}\tilde{u}_{D_p}, \tag{4.12}$$

then, $\tilde{U} = \tilde{F}(\tilde{u})$ also satisfy the same relations:

$$\tilde{U_{A_s}} = \tilde{U_{A_r}}, \quad \tilde{U_{B_s}} = \tilde{U_{B_p}}, \quad \tilde{U_{C_s}} = \tilde{U_{C_q}} \tag{4.13}$$

$$\tilde{U_{A_p}}\tilde{U_{B_q}} = \tilde{U_{B_s}}\tilde{U_{A_q}}, \quad \tilde{U_{A_s}}\tilde{U_{B_q}} = \tilde{U_{B_r}}\tilde{U_{A_q}}, \quad \tilde{U_{A_p}}\tilde{U_{C_r}} = \tilde{U_{C_p}}\tilde{U_{A_r}}, \tag{4.14}$$

$$\tilde{U_{A_p}}\tilde{U_{C_s}} = \tilde{U_{C_p}}\tilde{U_{A_q}}, \quad \tilde{U_{D_q}}\tilde{U_{A_p}} = \tilde{U_{D_p}}\tilde{U_{A_q}}, \quad \tilde{U_{C_r}}\tilde{U_{F_p}} = \tilde{U_{C_p}}\tilde{U_{F_r}}, \tag{4.15}$$

$$\tilde{U_{A_p}}\tilde{U_{D_q}}\tilde{U_{E_r}} = \tilde{U_{A_r}}\tilde{U_{E_q}}\tilde{U_{D_p}}. \tag{4.16}$$

In order to study the total number of steps of random walks, it suffices to analyze the function $\Pi_n^{-1}\tilde{F}^{N-n}(\Pi_N u)$ only for $u_{A_p} = u_{A_q} = \cdots = u_{F_r}$. In this case, $\tilde{u} = \Pi_n u$ satisfies the consistency conditions (4.9)–(4.12), since $\tilde{u} = \Pi_n \vec{1}$ satisfies (4.9)–(4.12). Then, as a result of the Lemma 4.1, we can reduce the 18 kinds of variables to 8 kinds. Our choice is the following:

$$z_1 = u_{A_r},\ z_2 = u_{A_p},\ z_3 = u_{A_q},\ z_4 = u_{B_r},\ z_5 = u_{C_r},$$

$$z_6 = u_{A_p}u_{D_q} = u_{D_p}u_{A_q},\ z_7 = u_{E_r},\ z_8 = u_{F_r}.$$

Namely, using the mapping

$$\Lambda: \; u \mapsto (u_{A_r}, u_{A_p}, u_{A_q}, u_{B_r}, u_{C_r}, u_{A_p}u_{D_q}, u_{E_r}, u_{F_r}),$$

we put

$$z = \Lambda u\,.$$

Note that under the consistency conditions in Lemma 4.1 the mapping Λ can be inverted.

Define the matrix

$$\begin{aligned} D_n = \mathrm{diag} \quad (& \frac{1}{2+\eta_n+\zeta_n}, \frac{\eta_n}{2+\eta_n+\zeta_n}, \frac{\zeta_n}{2+\eta_n+\zeta_n}, \frac{1}{1+2\eta_n+\zeta_n}, \\ & \frac{1}{1+\eta_n+2\zeta_n}, \frac{\eta_n\zeta_n}{(2+\eta_n+\zeta_n)(\eta_n+\zeta_n)}, \frac{1}{1+\zeta_n}, \frac{1}{1+\eta_n}). \end{aligned}$$

Since

$$D_n\Lambda = \Lambda\Pi_n, \tag{4.17}$$

the relation (4.8) is written as

$$Z = D_{n-1}^{-1}F(D_n z)\,,$$

where

$$\begin{aligned} F &= \Lambda \circ \tilde{F} \circ \Lambda^{-1}\,, && (4.18) \\ Z &= \Lambda U\,. && (4.19) \end{aligned}$$

As a result, we have

$$\Phi_{n,N}(u) = \Lambda^{-1}D_n^{-1}F^{N-n}(D_N\Lambda u)\,. \tag{4.20}$$

Analysis of the renormalization map

The analysis of the generating function $\Phi_{n,N}$ reduces to that of the *renormalization map* F.

The following proposition is the technical core of our work.

Proposition 4.2. *The function $F = \Lambda \circ \tilde{F} \circ \Lambda^{-1}$ has the expression*

$$F(D_n\vec{1} + t) = D_{n-1}\vec{1} + A_n t + R_n(t)\,, \quad n \in \mathbf{Z}_+,\ \|t\| < C_7\delta^{-n}, \tag{4.21}$$

where A_n is an 8×8 matrix independent of t and R_n is a $\mathbf{C}^8$-valued function with the following properties:

(1) The function R_n is analytic on $\{t|\ \|t\| < C_8\delta^{-n}\}$ with the bound

$$\|R_n(t)\| \le C_9\delta^n\|t\|^2, \quad \|t\| < C_{10}\delta^{-n},\ n \in \mathbf{Z}_+,$$

for some positive constants C_9 and C_{10} independent of N and t;

(2) There exists a matrix A such that

$$\|A - A_n\| < C_{11}\delta^{-n},$$

where C_{11} is a constant independent of n;

(3) The matrix A has eigenvalues

$$(a+1)(a+b+c),\ \frac{(a+1)(2bc+b+c)}{bc+b+c},$$
$$a+1,\ a+1,\ a+1,\ a+1,\ \frac{(a+1)(b+c)}{bc+b+c},\ \frac{b+c}{bc+b+c}.$$

The eight eigenvalues of A satisfy

$$(a+1)(a+b+c) > \frac{(a+1)(2bc+b+c)}{bc+b+c}$$
$$> a+1 = a+1 = a+1 = a+1 > \frac{(a+1)(b+c)}{bc+b+c} > \frac{b+c}{bc+b+c}.$$

In particular, the largest eigenvalue $\ell = (a+1)(a+b+c)$ of A is simple. The eigenvalue $\delta = \dfrac{(a+1)(b+c)}{bc+b+c}$ appeared in Proposition 3.3. The eigenvalues ℓ and δ are denoted by τ and β^{-1} in [4], respectively.

Sketch of proofs of Proposition 3.1 and Proposition 4.2

Proposition 3.1 is the result of the explicit calculation of the right hand side of (4.7). The proof of Proposition 4.2 needs the first order Taylor expansion of the left hand side of (4.21) with the remainder estimate. In fact the complete proof of Proposition 4.2 is very long. As mentioned in the Introduction, the algebraic part of the proof is computer-aided. The output of computer amounts to about 9×10^5 Bytes. Then, it would not be worthwhile to describe the detail of the calculations but it is reasonable to clarify the logical structure of our procedure so that a reader in principle can reproduce our calculations in a straightforward (long) way.

We consider the 'interior' of a block of H_n and number the six vertices adjacent to 'boundary points' of the block as fig. 4. Let $\Omega_{ij},\ i,j = 1,2,\cdots,6$, be the set of all walks in the interior of the block starting at i and ending at j without reaching any one of the six vertices except for the start and the end. We define the matrix $T = T(\tilde{u}) = (T_{ij})$ by

$$T_{ij} = \sum_{\omega \in \Omega_{ij}} \tilde{u}^{\omega},\ i,j = 1,2,\cdots,6.$$

In particular, we have

$$T_{23} = \tilde{u}_{A_q},\ T_{32} = \tilde{u}_{B_q},\ \text{etc.}$$

It is also an easy exercise to show the following:

$$T_{11} = -\frac{\lambda^a - \mu^a}{\lambda^{a-1} - \mu^{a-1}} + 1 - \tilde{u}_{A_p}\tilde{u}_{D_p}\,, \tag{4.22}$$

$$T_{22} = -\frac{\lambda^a - \mu^a}{\lambda^{a-1} - \mu^{a-1}} + 1 - \tilde{u}_{A_q}\tilde{u}_{D_q}\,, \tag{4.23}$$

$$T_{23} = T_{32} = -\frac{\lambda - \mu}{\lambda^{a-1} - \mu^{a-1}}(\tilde{u}_{A_r} + \tilde{u}_{A_q}\tilde{u}_{D_p})^{a-1}, \tag{4.24}$$

where λ and μ are the roots of the quadratic equation

$$x^2 - (1 - \tilde{u}_{A_p}\tilde{u}_{D_p} - \tilde{u}_{A_q}\tilde{u}_{D_q})x + (\tilde{u}_{A_r} + \tilde{u}_{A_q}\tilde{u}_{D_p})^2 = 0\,. \tag{4.25}$$

If $\lambda = \mu \neq 0$, the fractions should be evaluated after reductions. The other elements of the matrix T are given similarly. Put

$$W = (I - T)^{-1}.$$

Then W_{ij} gives the sum of $\tilde{u}^{\omega}$ over the set of all walks starting at i and ending at j, if the sum converges.

Let us describe the program to calculate $\tilde{F}$ defined by (4.4). As an example, we consider $\tilde{U}_{A_p}$ in the right hand side of (4.6). Connecting two blocks as fig. 3, we obtain

$$\tilde{U}_{A_p} = \frac{1}{\Theta_A}(\tilde{u}_{A_p}(W_{34}\tilde{u}_{B_p} + W_{35}\tilde{u}_{C_q}) + \tilde{u}_{A_r}(W_{24}\tilde{u}_{B_p} + W_{25}\tilde{u}_{C_q})),$$

where

$$\begin{aligned} \Theta_A = 1 \quad &- \quad \tilde{u}_{A_p}(W_{33}\tilde{u}_{B_s} + W_{32}\tilde{u}_{A_s}) - \tilde{u}_{A_q}(W_{66}\tilde{u}_{C_q} + W_{61}\tilde{u}_{A_r}) \\ &- \quad \tilde{u}_{A_r}(W_{23}\tilde{u}_{B_s} + W_{22}\tilde{u}_{A_s}) - \tilde{u}_{A_s}(W_{16}\tilde{u}_{C_q} + W_{11}\tilde{u}_{A_r}). \end{aligned}$$

The other components of $\tilde{U} = \tilde{F}(\tilde{u})$ are given similarly.

Proof of Proposition 3.1. For $\tilde{u} = \Pi_n\vec{1}$, the equation (4.25) has a double root $\lambda = \mu = \tilde{u}_{A_r} + \tilde{u}_{A_q}\tilde{u}_{D_p}$. Noting this fact, we explicitly calculate $\tilde{F}(\Pi_n\vec{1})$ and obtain (4.7). □

Proof of Proposition 4.2. The first term of the right hand side of (4.21) is obtained from (4.18),(4.17) and (4.7) as follows:

$$F(D_n\vec{1}) = \Lambda \circ \tilde{F} \circ \Lambda^{-1}(D_n\vec{1}) = \Lambda \circ \tilde{F}(\Pi_n\vec{1}) = \Lambda(\Pi_{n-1}\vec{1}) = D_{n-1}\vec{1}.$$

In order to obtain the second term, i.e. the derivative of $\tilde{F}$ at $\tilde{u} = \Pi_n\vec{1}$, we used REDUCE programme on computer. In view of the explicit form of A_n produced by REDUCE, we can show the statements on the matrices A_n and A. The bound on the third term is obtained by looking into the remainder term produced at each step on the way from $\tilde{u}$ to $\tilde{U}$ described above. $\square$

Convergence to the diffusion process

In order to show the convergence of the sequence of random walks on H_N considered above in the limit of $N \to \infty$, we study the asymptotic behavior of the generating function $\Phi_{n,N}$ or equivalently that of F^{N-n} (see (4.20)) as $N \to \infty$ with a fixed n. To this end, we put

$$f^{(n,N)}(s) = F^{N-n}(D_N \exp(-\ell^{-N} D_N^{-1} s)) .$$

Theorem 4.3. *There exist positive constants ϵ and C_{12} such that, for every $n \in \mathbf{Z}_+$:*

(1) The function $f^{(n,N)}(s)$, $N \geq n$, is analytic on $\|s\| < \epsilon\, \ell^n \delta^{-n}$ and converges uniformly to an analytic function $f^{(n)}(s)$ on $\|s\| < \epsilon\, \ell^n \delta^{-n}$ as $N \to \infty$.

(2) The function $f^{(n)}(s)$ has the expression

$$f^{(n)}(s) = D_n \vec{1} - \ell^{-n} B_n\, s + r^{(n)}(s)\, , \ \ \|s\| < \epsilon\, \ell^n \delta^{-n} \, ,$$

with the bound

$$\|r^{(n)}(s)\| \leq C_{12} \delta^n \ell^{-2n} \|s\|^2 \, , \ \ \|s\| < \epsilon\, \ell^n \delta^{-n} \, ,$$

where

$$B_n = \lim_{N \to \infty} \ell^{-N+n} A_{n+1} A_{n+2} \cdots A_N .$$

Proof. Since the largest eigenvalue ℓ of A in Proposition 4.2 is simple, the function $F = \Lambda \circ \tilde{F} \circ \Lambda^{-1}$ is $(\{D_n, \ell\})$-regular. [For the terminology, see [8].] Then we can apply Theorem 2 in [8] to our F and obtain the theorem. $\square$

The above theorem claims that the time unit of random walks on H_N should be scaled as l^{-N}. Under this scaling, the standard tightness argument applies. The argument goes exactly parallel with that for the Sierpiński gasket. [See [7] for detail. Theorem 4.3 above corresponds to Proposition 2.4 in [7].] Thus we arrive at the conclusion:

Theorem 4.4. *The sequence of random walks on the pre-abc-gasket $H_N, N = 1, 2, 3, \cdots$ with transition probabilities given by Table 1 with $\eta = \eta_N$ and $\zeta = \zeta_N$ weakly converges to a diffusion process on the abc-gasket, where η_N and ζ_N are defined in Proposition 3.3.*

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Appendix

Choice of random walk sequence for the multi-parameter case

An important point in the choice of parameter sequences for the *abc*-gaskets in Proposition 3.3 is that we are dealing with the multi-parameter case. Namely, we are considering cases where random walks on a (finitely ramified) pre-fractal is specified by a set of more than one non-negative parameters. In the case of the *abc*-gaskets, two parameters η and ζ are introduced. Another example of the multi-parameter case is the snowflake fractal [12]. As regards the asymptotically one-dimensional (lower dimensional) diffusions, such cases have not been considered for other fractals than the *abc*-gaskets. We here propose a general strategy suggested by the choice in Proposition 3.3, which may be applicable to other fractals. We leave it as an open problem to find general conditions with which the following idea may be applicable to construct asymptotically lower dimensional diffusions on fractals.

Let $r \geq 1$ be the number of parameters. Let $G_0 \subset G_1 \subset G_2 \subset \cdots$ be a sequence of vertices of pre-fractals, and assume that for each $n \in \mathbf{Z}_+$ a random walk Z_n on G_n is specified by the non-negative parameters $p^{(n)} = {}^t(p_1^{(n)}, p_2^{(n)}, \cdots, p_r^{(n)})$, chosen in such a way that for $m < n$, the m-decimated random walk of Z_n is Z_m. As in (3.1) and (3.2), there exists an $\mathbf{R}^r$–valued rational function $\Gamma = {}^t(\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ in r variables such that

$$p^{(n)} = \Gamma(p^{(n+1)}), \quad n \in \mathbf{Z}_+ .$$

Let $p^* = {}^t(p_1^*, \cdots, p_r^*)$ be a fixed point of Γ with non-negative elements:

$$\Gamma(p^*) = p^* , \quad p_i^* \geq 0, \ i = 1, 2, \cdots, r.$$

Denote by $J = J(p)$ an r-dimensional matrix whose (i,j) element $J_{i,j}$ is given by $J_{ij}(p) = \dfrac{\partial \Gamma_i}{\partial p_j}(p)$. Let δ be that eigenvalue of $J(p^*)$ which has the largest absolute value. Assume that δ uniquely exists and is real and positive. To construct an asymptotically lower dimensional diffusion around p^*, it is necessary that p^* is an unstable fixed point of Γ, which implies $\delta > 1$. Let q^* be an eigenvector of $J(p^*)$ corresponding to the eigenvalue δ.

An intuitive choice is $p^{(n)} \approx p^* + \delta^{-n} q^*$, but from technical point of view, it is nicer to proceed in the following way. For each $N \in \mathbf{Z}_+$, put $p^{(N,N)} = p^* + \delta^{-N} q^*$, and define $p^{(N,n)}$, $n = 0, 1, 2, \cdots, N$, inductively by

$$p^{(N,n-1)} = \Gamma(p^{(N,n)}), \quad n = N, N-1, \cdots, 1.$$

Assume that

$$\tilde{\Gamma}(p) \stackrel{\mathrm{def}}{=} \Gamma(p) - p^* - J(p^*)(p - p^*)$$

satisfies

$$|\tilde{\Gamma}_i(p)| \le C \sum_{j=1}^{r} (p_j - p_j^*)^2, \quad i = 1, 2, \cdots, r$$

for some positive constant C independent of p. Let $p^{(n)} = \lim_{N\to\infty} p^{(N,n)}$, $n \in \mathbf{Z}_+$. Then

$$p^{(n-1)} = \Gamma(p^{(n)}), \quad C_1 \delta^{-n} q_i^* \le p_i^{(n)} - p_i^* < C_2 \delta^{-n} q_i^*, \quad i = 1, 2, \cdots, r, \ n \in \mathbf{Z}_+,$$

for some positive constants C_1 and C_2 independent of p. This provides a generalized form of the Proposition 3.3.

General conditions on the fractals, for which this idea works, is left as an open problem.

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	p	q	r	s
A	$\eta/(2+\eta+\zeta)$	$\zeta/(2+\eta+\zeta)$	$1/(2+\eta+\zeta)$	$1/(2+\eta+\zeta)$
B	$\eta/(1+2\eta+\zeta)$	$\zeta/(1+2\eta+\zeta)$	$1/(1+2\eta+\zeta)$	$\eta/(1+2\eta+\zeta)$
C	$\eta/(1+\eta+2\zeta)$	$\zeta/(1+\eta+2\zeta)$	$1/(1+\eta+2\zeta)$	$\zeta/(1+\eta+2\zeta)$
D	$\eta/(\eta+\zeta)$	$\zeta/(\eta+\zeta)$		
E		$\zeta/(1+\zeta)$	$1/(1+\zeta)$	
F	$\eta/(1+\eta)$		$1/(1+\eta)$	

Table 1: Transition probabilities

Figure captions.

Figure 1: The block of pre-abc-gasket with $a = 2, b = 4, c = 3$.

Figure 2: We classify vertices of a pre-abc gasket into 6 groups A, B, C, D, E, and F according to the possible directions of transitions from each vertex. We refer to the possible directions by the symbols p, q, r, and s.

Figure 3: The vertex $A \in H_{n-1}$ belongs to two blocks of H_n. We denote the vertices of H_{n-1} contained in the two blocks by A, A'_p, A'_q, A'_r, and A'_s.

Figure 4: The 'interior' of a block of H_n. We number the six vertices adjacent to 'boundary points' of the block as above.