Stability of Ginzburg-Landau Vortices*

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Abstract

We consider the question of stability of time-independent vortex solutions of two evolution equations of Ginzburg-Landau type: the nonlinear heat equation, and the nonlinear Schrödinger equation. For the heat equation, we show that n-vortex solutions are stable under real, radially symmetric, but not necessarily small perturbations. For the Schrödinger equation, we set up basic equations for analysis of asymptotic stability.

1 Introduction

In this paper we study the stability of static solutions of two nonlinear evolution equations of Ginzburg-Landau (GL) type: the nonlinear Schrödinger equation (SE),

$$i\dot{\psi}_t = -\triangle\psi_t + (|\psi_t|^2 - 1)\psi_t$$

and the nonlinear heat equation (HE),

$$-\dot{\psi}_t = -\triangle\psi_t + (|\psi_t|^2 - 1)\psi_t,$$

with initial condition

$$\psi_t|_{t=0} = \psi_0.$$

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Here $\psi_t : \mathbf{R^2} \to \mathbf{C}$ for $t \in [0, \infty)$, and $\dot{\psi}_t$ denotes $\partial\psi_t/\partial t$. We impose the boundary condition

$$\lim_{|x|\to\infty} |\psi_t(x)| = 1. \tag{1}$$

Stationary solutions solve the time-independent equation,

$$-\triangle\psi + (|\psi|^2 - 1)\psi = 0. \tag{2}$$

This equation is known to have a unique "n-vortex" solution of the form

$$\psi^{(n)} = f_n(r)e^{in\theta}$$

for each $n \in \mathbf{Z}$, where (r, θ) are polar coordinates on $\mathbf{R^2}$, and $f_n : [0, \infty) \to [0, 1]$ [H, OSI].

The question of vortex dynamics in connection with time-dependent GL equations was brought up in [N], where some of the first important results were obtained. Further results were obtained in [RS, L]. For a reveiw of time-independent GL equations see [JT, BBH, OSI].

The main results of this paper for (HE) concern perturbations of $\psi^{(n)}$ of the form $\psi^{(n)}(r, \theta) + u(r)e^{in\theta}$ with $u(r)$ real, which we will call "real, radially symmetric" perturbations. Theorem 4.1 says that as a solution to (HE), $\psi^{(n)}$ is stable under small such perturbations. In theorem 4.2, we prove that under general such perturbations, $\psi^{(n)}$ is stable in a much weaker sense.

When this paper was essentially completed, we obtained a preprint by M.I. Weinstein and J. Xin, "Dynamic stability of the vortex solutions of the Ginzburg-Landau and nonlinear Schrödinger equations," in which the authors prove similar results. Their methods are more invovled than those used here.

2 The energy functional

The right-hand side of (HE) and (SE) is the variational derivative of the renormalized Ginzburg-Landau energy functional, introduced in [OSI]:

$$E_n(\psi) = \frac{1}{2}\int_{\mathbf{R^2}} \{|\nabla\psi|^2 - \frac{n^2\chi(r)}{r^2} + \frac{1}{2}(|\psi|^2 - 1)^2\} \tag{3}$$

where

$$\chi \in C^\infty([0, \infty); [0, 1]), \quad \chi(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 & \text{if } r \geq 2. \end{cases}$$

That is,

$$\partial_{\bar{\psi}} E(\psi) = -\triangle\psi + (|\psi|^2 - 1)\psi.$$

Functions, ψ, satisfying condition (1), have a (integer) degree, defined as the winding number of the map

$$\left.\frac{\psi}{|\psi|}\right|_{|x|=R} : \mathbf{S^1} \to \mathbf{S^1}$$

for sufficiently large R. We see that $\deg(\psi^{(n)}) = n$. Further, $E_n(\psi^{(n)})$ is finite, and $\psi^{(n)}$ gives [OSI]

$$\begin{cases} \text{a local minimum of } E_n & \text{if } |n| = 1 \\ \text{a saddle point of } E_n & \text{if } |n| \geq 2 \\ \text{an absolute minimum of } E_n & \text{if } n = 0. \end{cases} \tag{4}$$

The last line is, of course, obvious. It is conjectured that $\psi^{(\pm 1)}$ is, in fact, a global minimizer of E_1 among functions of degree ± 1.

We lose no generality by considering only non-negative integers, n, as the transformation $\psi \mapsto \bar{\psi}$ takes $\psi^{(-n)}$ to $\psi^{(n)}$, (HE) is invariant under $\psi \mapsto \bar{\psi}$, and (SE) is invariant under $\psi \mapsto \bar{\psi}$, $t \mapsto -t$.

Formally, we have

$$d/dt E_n(\psi_t) = \Re \int_{\mathbf{R^2}} \dot{\psi}_t \partial_\psi E_n(\psi_t) \tag{5}$$

$$= \begin{cases} -\Re \int_{\mathbf{R^2}} |\partial_{\bar{\psi}} E_n(\psi_t)|^2 \leq 0 & \text{if } \psi_t \text{ solves (HE)} \\ -\Re \int_{\mathbf{R^2}} i|\partial_{\bar{\psi}} E_n(\psi_t)|^2 = 0 & \text{if } \psi_t \text{ solves (SE)} \end{cases}$$

In studying the stability of $\psi^{(1)}$ as a solution to (HE), an important role is played by the fact that (HE) is a dissipative equation. The non-dissipative nature of (SE) makes the stability analysis more delicate.

3 Energy lower bounds

Here we state some lower bounds on $E_n(\psi)$ which depend on the location of the zeros of ψ.

Theorem 3.1 *Let $R_0 > 2$ be fixed, and suppose $\deg(\psi) = n$. Write $\psi = |\psi| e^{i\varphi}$.*

(i) If $|\psi| \geq 1/\sqrt{2}$ for $|x| \geq R_0$, then

$$E_n(\psi) \geq -n^2\pi\log(R_0) + \frac{1}{2}\int_{|x|\geq R_0}(|\nabla|\psi||^2 - |\nabla\varphi|^4). \tag{6}$$

(ii) If $\psi \neq 0$ for $|x| \geq R_0$, then

$$E_n(\psi) \geq -n^2\pi\log(R_0) - \frac{n^2\sqrt{\pi}}{2R_0}\left(\int_{|x|\geq R_0}(|\psi|^2-1)^2\right)^{1/2} \tag{7}$$

$$-n\left(\int_{|n|\geq R_0}\frac{|\nabla(\varphi-n\theta)|^2}{r^2}\right)^{1/2}\left(\int_{|x|\geq R_0}(|\psi|^2-1)^2\right)^{1/2}.$$

(iii) If $\psi = ue^{in\theta}$, u real, then

$$E_n(\psi) \geq -\pi n^2(\log(2) + \frac{1}{16}). \tag{8}$$

Proof: Results (i) and (iii) are proved in [OSI]. For (ii), we write $\varphi = n\theta + \lambda$. Denoting,

$$E_{n,\Omega}(\psi) = \frac{1}{2}\int_\Omega\{|\nabla\psi|^2 - \frac{n^2\chi(r)}{r^2} + \frac{1}{2}(|\psi|^2-1)^2\},$$

we have

$$E_n(\psi) = E_{n,\{|x|\leq R_0\}}(\psi) + E_{n,\{|x|\geq R_0\}}(\psi).$$

Call the second integral J. We have

$$J = \frac{1}{2}\int_{|x|>R_0}\{|\nabla|\psi||^2+|\psi|^2(\frac{n^2}{r^2}+2n\nabla\theta\cdot\nabla\lambda+|\nabla\lambda|^2)-\frac{n^2}{r^2}+\frac{1}{2}(|\psi|^2-1)^2\}$$

$$= \frac{1}{2}\int_{|x|>R_0}\{|\nabla|\psi||^2+|\psi|^2|\nabla\lambda|^2+2n\nabla\theta\cdot\nabla\lambda+2n(|\psi|^2-1)\nabla\theta\cdot\nabla\lambda+(|\psi|^2-1)\frac{n^2}{r^2}+\frac{1}{2}(|\psi|^2-1)^2\}.$$

Now,

$$\int_{|x|>R_0}\nabla\theta\cdot\nabla\lambda = \int_{r>R_0}\frac{1}{r}\left[\int_{|x|=r}d\lambda\right]dr$$

but

$$\int_{|x|=r}d\lambda = \int_{|x|=r}d[arg(\psi) - n\theta] = 0$$

(as $2\pi n = 2\pi \deg(\psi) = \int_{|x|=r} d\arg(\psi)$, $r > R_0$). This leaves

$$
\begin{aligned}
J &\geq \frac{1}{2}\int_{|x|>R_0} \left\{ 2n(|\psi|^2 - 1)\nabla\theta \cdot \nabla\lambda + (|\psi|^2 - 1)\frac{n^2}{r^2} \right\} \\
&\geq -n\left(\int_{|x|>R_0} \frac{|\nabla\lambda|^2}{r^2}\right)^{1/2} \left(\int_{|x|>R_0} (|\psi|^2 - 1)^2\right)^{1/2} \\
&\quad -\frac{n^2}{2}\left(\int_{|x|>R_0} \frac{1}{r^4}\right)^{1/2} \left(\int_{|x|>R_0} (|\psi|^2 - 1)^2\right)^{1/2}
\end{aligned}
$$

and the result follows. □

We will need the inequality (8) in what follows. The inequalities (6,7) suggest, in particular, that the renormalized energy should have an absolute lower bound among functions of a given degree with only one zero. Given a degree one function, we can "fill in" pairs of zeros whose local degrees sum to zero to effect an energy decrease. This suggests that the renormalized energy is bounded from below on all degree one functions.

4 Stability results for the heat equation

In this section, we prove that $\psi^{(n)}$ is stable under real, radially symmetric perturbations.

4.1 Small perturbations

We prove that $\psi^{(n)}$ is stable in the H_1 norm under sufficiently small, real, radially symmetric perturbations.

Theorem 4.1 *Suppose the initial conditions for (HE) are of the form*

$$\psi_0 = u_0(r)e^{in\theta}$$

with u_0 real, and $\|\psi_0 - \psi^{(n)}\|_{H_1}$ sufficiently small. Then the solution of (HE) is of the form $\psi_t = u_t(r)e^{in\theta}$ with u_t real, and $\|\psi_t - \psi^{(n)}\|_{H_1}$ remains small.

Proof: The statement that the solution of (HE) is of the form $u_t(r)e^{in\theta}$ with u_t real, relies on existence and uniqueness results for the real, radially symmetric (HE),

$$-\dot{u_t} = -\triangle u_t + \frac{n^2}{r^2}u_t + (|u_t|^2 - 1)u_t. \tag{9}$$

Uniqueness of bounded solutions of the full (HE) (and hence of (9)) is established in [BCPS]. The existence result for (9) will be dealt with in a future article.

The idea of the proof of theorem 4.1 is to use the energy as a Lyapunov function. Our first step is to estimate the Hessian, E'' of E_n at $\psi^{(n)}$ on small, real, radially symmetric functions. For this we require some lemmas. The first is an immediate consequence of a partial convexity result proved in [OSI].

Lemma 4.1 *With* $\xi = w(r)e^{in\theta}$, $w(r) \in \mathbf{R}$,

$$\langle \xi, E''\xi \rangle_{L^2} = \int_{\mathbf{R^2}} f_n^2(|\nabla(f_n w)|^2 + |\xi|^2).$$

This shows that $\langle \xi, E''\xi \rangle$ is roughly like $\|\xi\|_{H_1}$, except it misses the part of ξ near the origin. To deal with a neighbourhood of the origin, we construct cut-off functions, χ_1, $\chi_2 \in C^\infty([0,\infty);[0,1])$ such that $\text{supp}(\chi_1) \subset [0,R]$, $\text{supp}(\chi_2) \subset [2R/3,\infty)$ (for some $R > 0$) and $\chi_1^2 + \chi_2^2 \equiv 1$. Then we apply the IMS localization formula [CFKS] to get

Lemma 4.2

$$\langle \xi, E''\xi \rangle = \langle \chi_1(r)\xi, E''\chi_1(r)\xi \rangle + \langle \chi_2(r)\xi, E''\chi_2(r)\xi \rangle - \langle (|\nabla\chi_1(r)|^2 + |\nabla\chi_2(r)|^2)\xi, \xi \rangle.$$

By lemma 4.1, we can neglect the second term on the r.h.s. to establish the inequality

$$\langle \xi, E''\xi \rangle \geq \langle \chi_1\xi, E''\chi_1\xi \rangle - \langle (|\nabla\chi_1|^2 + |\nabla\chi_2|^2)\xi, \xi \rangle \tag{10}$$

Now we estimate the first term above.

Lemma 4.3

$$\langle \chi_1\xi, E''\chi_1\xi \rangle \geq (\frac{1}{4R^2} - 2)\|\chi_1\xi\|_{L^2}^2.$$

Proof: We have $E''\xi = -\triangle\xi + (2f_n^2 - 1)\xi + {\psi^{(n)}}^2\bar{\xi}$. By the uncertainty principle, $\langle \chi_1\xi, -\triangle\chi_1\xi\rangle \geq 1/(4R^2)\|\chi_1\xi\|_{L^2}^2$ as $\mathrm{supp}(\chi_1\xi) \subset [0,R]$. Since $f_n \leq 1$,

$$\langle \chi_1\xi, (2f_n^2 - 1)\chi_1\xi + {\psi^{(n)}}^2\chi_1\bar{\xi}\rangle \geq -2\|\chi_1\xi\|_{L^2}^2.$$

The last two inequalities yield the desired estimate. □

Let $D = \sup(|\nabla\chi_1|^2 + |\nabla\chi_2|^2)$. We choose $R = 1/4$ so that the estimate from lemma 4.2 becomes

$$\langle \chi_1\xi, E''\chi_1\xi\rangle \geq 2\|\chi_1\xi\|_{L^2}^2.$$

Combining the estimates from lemma 4.1 and lemma 4.3 (with weight γ, $\gamma < f_n^2(1/6)/D$), we have

$$(1+\gamma)\langle \xi, E''\xi\rangle \geq \int_{\mathbf{R^2}} (f_n^2 + 2\gamma\chi_1^2 - f_n^2(1/6)\tilde{\chi})|\xi|^2$$

where $\tilde{\chi}$ is the characteristic function of $[1/6, 1/4]$. We know that f_n is a positive, increasing function ([OSI]), so we can bound the quantity in the parentheses from below by a constant, F. Then

$$\frac{1+\gamma}{F}\langle \xi, E''\xi\rangle \geq \|\xi\|_{L^2}^2. \tag{11}$$

Now

$$\begin{aligned}\langle \xi, E''\xi\rangle &= \int_{\mathbf{R^2}} (|\nabla\xi|^2 + (2f_n^2 - 1)|\xi|^2 + {\psi^{(n)}}^2\bar{\xi}^2) \\ &\geq \|\nabla\xi\|_{L^2}^2 - 2\|\xi\|_{L^2}^2\end{aligned}$$

so that

$$\|\nabla\xi\|_{L^2}^2 \leq (1 + \frac{2+2\gamma}{F})\langle \xi, E''\xi\rangle. \tag{12}$$

Combining (11) and (12) we get

$$\alpha\|\xi\|_{H_1}^2 \leq \langle \xi, E''\xi\rangle$$

(α a constant). Now we expand $E_n(\psi^{(n)} + \xi)$ in a Taylor series to get

$$E_n(\psi^{(n)} + \xi) - E_n(\psi^{(n)}) = (1/2)\langle \xi, E''\xi\rangle + R(\xi)$$

where

$$|R(\xi)| \leq (3/2)\|\xi\|_{L^3}^3 \leq (3/2)(\pi/2)^{1/2}\|\xi\|_{H_1}^3$$

by a Sobolev inequality. Hence, there are positive constants a, b such that

$$E_n(\psi^{(n)} + \xi) - E_n(\psi^{(n)}) \geq a\|\xi\|_{H_1}^2 - b\|\xi\|_{H_1}^3. \tag{13}$$

Now, by (5), the energy decreases during the evolution. By (13), taking $\|\psi_0 - \psi^{(n)}\|_{H_1}$ sufficiently small ensures that $\|\psi_t - \psi^{(n)}\|_{H_1}$ remains small. □

4.2 More general perturbations

In what follows, we apply a standard technique for dissipative equations to show that finite-energy, real, radially symmetric solutions of (HE) converge in a weak sense to $\psi^{(n)}$. We require the energy of the solution of (HE) to be bounded from below. The bound (8) ensures this is so for real, radially symetric solutions. In general, the bounds we have (6, 7) require the zeros of the solution to be contained. If $\deg(\psi_t) \geq 2$, the zeros of ψ_t will not remain contained in general, so this result does not extend to general solutions. Of course by (4), $\psi^{(n)}$ is not stable under arbitrary perturbations if $n \geq 2$.

Theorem 4.2 *Suppose ψ_t solves (HE) with $\psi_0 = u_0(r)e^{in\theta}$, u_0 real, $|E_n(\psi_0)| < \infty$ and $\int_{\mathbf{R^2}} |\partial_\psi E_n(\psi_0)|^2 < \infty$. Then there is a set, $L \subset [0, \infty)$, whose complement has arbitrarily small measure, such that for any sequence, $\{t_m\} \subset L$ with $t_m \to \infty$ as $m \to \infty$, there is a subsequence, $\{t_{m_k}\}$ such that $\psi_{t_{m_k}} \to \psi^{(n)}$ a.e. in $\mathbf{R^2}$ and weakly in H_1 on bounded sets.*

Proof: As in the proof of theorem 4.1, ψ_t is of the form $u_t(r)e^{in\theta}$, u_t real. By (5), and the hypotheses on ψ_0, we have

$$-E_n(\psi_T) + E_n(\psi_0) = \int_0^T \int_{\mathbf{R^2}} |\partial_\psi E_n(\psi_t)|^2$$

for any $T > 0$. The bound (8) ensures that the energy is bounded during the evolution. Letting $T \to \infty$ gives

$$\int_0^\infty \int_{\mathbf{R^2}} |\partial_\psi E_n(\psi_t)|^2 < \infty. \tag{14}$$

It follows that there is a set $L \subset [0, \infty)$ whose complement has arbitrarily small measure, such that any sequence $\{t_r\} \subset L$ such that $t_r \to \infty$ satisfies

$$\int_{\mathbf{R^2}} |\partial_\psi E_n(\psi_{t_r})|^2 \to 0 \tag{15}$$

as $r \to \infty$. Let $\{t_r\}$ be such a sequence. Fix $k \in \mathbf{Z}^+$. Let B_k be the disk of radius k. The proof of the bound (8) ensures the energy is bounded on B_k, and hence that $\|\psi_{t_r}\|_{H_1(B_k)} \leq C$ for all $t > 0$ (C a constant). By the Banach-Alaoglu theorem, and Rellich's theorem, there is a subsequence $\{t_{r_k}\}$ such that $\psi_{t_{r_k}} \to \psi_k$ weakly in $H_1(B_k)$, strongly in $L^2(B_k)$, and a.e., for some $\psi_k \in H_1(B_k)$. We now let $k = 1, 2, 3, \cdots$, and construct a diagonal subsequence, denoted $\{t_m\}$, which converges weakly in $H_1(B_k)$, strongly in $L^2(B_k)$, and a.e. to ψ_k for all k. Thus ψ_k is an almost everywhere extension of ψ_l for $l < k$, and we can define ψ a.e. in $\mathbf{R^2}$ such that $\psi|_{B_k} = \psi_k$ a.e. and $\psi_k \to \psi$ a.e. in $\mathbf{R^2}$. It remains to show that $\psi = \psi^{(n)}$. Fix B_k and let $\xi \in C_0^\infty(B_k)$ $(\xi|_{\partial B_k} = 0)$. Then

$$|\int_{B_k} (\nabla\psi \cdot \nabla\xi + (|\psi|^2 - 1)\psi\xi)|$$

$$\leq \int_{B_k} |(\nabla\psi - \nabla\psi_{t_m}) \cdot \nabla\xi + (|\psi|^2 - 1)(\psi - \psi_{t_m})\xi|$$

$$+ \int_{B_k} | - \triangle\psi_{t_m} + (|\psi|^2 - 1)\psi_{t_m}||\xi|.$$

The first integral $\to 0$ as $m \to \infty$ because $\psi_{t_m} \to \psi$ weakly in $H_1(B_k)$. The second integral, I, satisfies

$$I \leq \int_{B_k} | - \triangle\psi_{t_m} + (|\psi_{t_m}|^2 - 1)\psi_{t_m}||\xi| + \int_{B_k} ||\psi|^2 - |\psi_{t_m}|^2||\psi_{t_m}\xi|.$$

The first term here vanishes as $m \to \infty$ by (15), and by Hölder's inequality. The second term $\to 0$ because $\psi_{t_m} \to \psi$ in $L^2(B_k)$. Thus,

$$\int_{B_k} (\nabla\psi \cdot \nabla\xi + (|\psi|^2 - 1)\psi\xi) = 0$$

for all k, and so ψ is a weak solution of equation (2). By elliptic regularity, ψ is a classical solution. The only radially symmetric solution with finite energy is $\psi^{(n)}$. Hence $\psi = \psi^{(n)}$. $\square$

5 Nonlinear Schrödinger equation

In this section we discuss the linearized asymptotic stability of $\psi^{(1)}$ along the lines of [SW, BP].

We say the vortex $\psi^{(1)}$ is "asymptotically stable" if given initial conditions, ψ_0, with $\psi_0 - \psi^{(1)}$ sufficiently small, the solution, ψ_t, of (SE), satisfies

$$\inf_{g \in G_{SE}} \|g\psi^{(1)} - \psi_t\| \to 0$$

as $t \to \infty$, where G_{SE} is the symmetry group of (SE), and $\|\cdot\|$ is some norm.

Our goal is to split the original (SE) into a dispersive equation, coupled to a dynamical system.

We present the solution ψ_t in the form

$$\psi_t = g(t)\psi^{(1)} + \eta_t \tag{16}$$

where $g(t) \in G_{SE}$. The goal of the asymptotic stability analysis is to show that $\eta_t \to 0$ in some norm which may be local in nature (eg L^∞ or weighted L^2), and to show that g(t) converges to some $g \in G_{SE}$.

Substitution of (16) into (SE) results in the equation

$$i\dot{g}(t)\psi^{(1)} + i\dot{\eta}_t = E''(g(t)\psi^{(1)})\eta_t + \tilde{N}(\eta_t)$$

or

$$ig^{-1}(t)\dot{g}(t)\psi^{(1)} + i\dot{\xi}_t = L\xi_t + N(\xi_t) \tag{17}$$

where $\xi_t = g^{-1}(t)\eta_t$, $L = E''(\psi^{(1)})$ is the linearized operator of E_1 around $\psi^{(1)}$, and

$$N(\xi_t) = \xi_t(|\xi_t|^2 + 2\psi^{(1)}\bar{\xi}_t + \bar{\psi^{(1)}}\xi_t) - g^{-1}(t)\dot{g(t)}\xi_t$$

contains the terms nonlinear in ξ_t.

The linearized operator,

$$L\xi = -\triangle\xi + (2|\psi^{(1)}|^2 - 1)\xi + {\psi^{(1)}}^2\bar{\xi}$$

has continuous spectrum equal to $[0, \infty)$ ([OSI]). In addition, it has several zero modes (zero-eigenvectors) corresponding to symmetries of the time-independent equation (2) broken by $\psi^{(1)}$, and root vectors corresponding to symmetries of the time-dependent equation (SE). The continuous symmetry group of (2) consists of $U(1)$ symmetry, and rigid motions of $\mathbf{R^2}$. These result in the zero modes

$$\zeta_1 = i\psi^{(1)} \quad \zeta_{2,3} = \partial_{1,2}\psi^{(1)}$$

(for $\psi^{(1)}$, $U(1)$ and rotational symmetries coincide). In considering the time-dependent equation, we add time-translational symmetry, and a Galilean symmetry given by

$$\psi_t(x) \mapsto e^{i(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t)}\psi_t(x-vt)$$

for $v \in \mathbf{R^2}$. The latter results in two root vectors,

$$\zeta_{4,5} = -\frac{i}{2}x_{1,2}\psi^{(1)}.$$

To see this, we differentiate $e^{i(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t)}\psi^{(1)}(x-vt)$ with respect to v_j at $v=0$. The result, $-t\partial_j\psi^{(1)} + (i/2)x_j\psi^{(1)}$ is a solution of the linearized (SE), $i\psi_t = L\psi_t$. This leads to the following relations:

$$L\zeta_1 = L\zeta_2 = L\zeta_3 = 0$$

$$L\zeta_4 = i\zeta_2, \quad L\zeta_5 = i\zeta_3.$$

The 1-vortex is a member of the energy space

$$X = \{\psi : \mathbf{R^2} \to \mathbf{C} \mid |E_1(\psi)| < \infty\}$$

and the finite-energy perturbations of $\psi^{(1)}$ which we consider belong to

$$T_{\psi^{(1)}}X = \{\xi : \mathbf{R^2} \to \mathbf{C} \mid |E_1(\psi^{(1)} + \xi)| < \infty\}.$$

We can characterize $T_{\psi^{(1)}}X$ as follows ([OSI])

$$T_{\psi^{(1)}}X = \{\xi \in L^4(\mathbf{R^2};\mathbf{C}) \mid \int |\Re(e^{-i\theta}\xi)|^2 < \infty, \int |\nabla\xi|^2 < \infty\}.$$

Because L is not complex linear, we replace ξ by the vector

$$\tilde{\xi} = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$$

and consider L acting on $TX = T_{\psi^{(1)}}X \oplus T_{\bar{\psi^{(1)}}}X$ via

$$\tilde{L\xi} = \begin{pmatrix} L\xi \\ \bar{L\xi} \end{pmatrix} = \tilde{L}\tilde{\xi}$$

where

$$\tilde{L} = \begin{pmatrix} -\Delta + 2|\psi^{(1)}|^2 - 1 & {\psi^{(1)}}^2 \\ {\bar{\psi^{(1)}}}^2 & -\Delta + 2|\psi^{(1)}|^2 - 1 \end{pmatrix}.$$

We will, however, continue to use "L" and "ξ" in place of "$\tilde{L}$" and "$\tilde{\xi}$".

Of the symmetry zero-modes ζ_i, only $\zeta_{2,3}$ lie in TX. We make the assumption that $\{\zeta_2, \zeta_3\}$ are the only 0-eigenfunctions in TX, and therefore form a basis of the nullspace of L on TX. Let $\langle \cdot, \cdot \rangle$ be a pairing between $(TX)^*$ and TX, and let $\{\zeta_2^*, \zeta_3^*\}$ be a basis of null(L^*) on $(TX)^*$ such that the matrix $A = (\langle \zeta_i^*, \sigma\zeta_j \rangle)$ is non-singular, where

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Returning now to equation (17), we decompose ξ and $N(\xi)$ as follows

$$\xi_t = c(t) \cdot \zeta + h_t$$

where

$$< \zeta^*, \sigma h_t >= 0 \tag{18}$$

and

$$N(\xi_t) = ib(t) \cdot \sigma\zeta + N_t$$

where

$$< \zeta^*, N_t >= 0 \tag{19}$$

(here $b, c \in \mathbf{C^2}$, $\zeta = (\zeta_2, \zeta_3)$, $\zeta^* = (\zeta_2^*, \zeta_3^*)$). The non-singularity of A ensures the validity of this decomposition. Substituting these decompositions into equation 17 and applying ζ^* to the result, gives

$$i < \zeta^*, \sigma g^{-1}(t)\dot{g}(t)\psi^{(1)} > + i\dot{c}(t)\cdot < \zeta^*, \sigma\zeta >= ib(t)\cdot < \zeta^*, \sigma\zeta > .$$

Now $L(g^{-1}(t)\dot{g}(t)\psi^{(1)}) = 0$ (symmetry zero-modes) so $g^{-1}(t)\dot{g(t)}\psi^{(1)} = d(t) \cdot \zeta$ giving

$$A(d(t) + \dot{c}(t) - b(t)) = 0$$

or

$$d(t) + \dot{c}(t) - b(t) = 0. \tag{20}$$

Left over in equation 17 is

$$i\sigma\dot{h_t} = Lh_t + N_t. \tag{21}$$

We have acheived a splitting of (SE) into a dynamical system (20) coupled to a dispersive equation (21) from which the zero-modes (non-decaying solutions of the linearized equation) have been removed (in the sense of (18,19)).

Observe, finally, that though the symmetry modes $\zeta_{1,4,5}$ do not lie in the space TX, they may affect the decay rate of the time evolution.

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