POSITIVITY OF ENTROPY PRODUCTION

IN THE PRESENCE OF A RANDOM THERMOSTAT.

by David Ruelle*

Abstract. We study nonequilibrium statistical mechanics in the presence of a thermostat acting by random forces, and propose a formula for the rate of entropy production $e(\mu)$ in a state μ. When μ is a natural nonequilibrium steady state we show that $e(\mu) \geq 0$, and sometimes we can prove $e(\mu) > 0$.

Key words and phrases: entropy production, nonequilibrium stationary state, nonequilibrium statistical mechanics, random dynamics, SRB state, thermostat.

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0. Physical introduction.

The production of entropy in nonequilibrium statistical mechanics was analyzed in various settings in [19]. Here we continue this analysis by studying the role of a heat bath, and its idealization by random external forces.

As discussed in [19], we maintain a system outside of equilibrium by external forces which, in general, do not keep the energy constant. If no precaution is taken, the time evolution is then represented by an unbounded orbit in the noncompact phase space $\mathcal{S}$ of the system, *i.e.*, the energy grows indefinitely. Physically, this heating up is prevented by coupling the system to a thermostat, or heat bath. The energy changes are diluted in the large heat bath, so that the system keeps a bounded energy.

If Ω is the phase space of the heat bath, we assume thus a deterministic time evolution in $\Omega\times\mathcal{S}$, and we are interested in its projection in $\mathcal{S}$. A priori, however, the projected time evolution in $\mathcal{S}$ has no simple description because the mutual interactions of the system and the heat bath cannot be disentangled. (See however Jakšić and Pillet [7]). One is thus led to considering a simplified model where the heat bath acts on the system, but the system does not act back on the heat bath. We shall return in a moment to the physical meaning of this simplification.

The previous paper [19] and the present one are in the line of a renewed attack on the problems of nonequilibrium mechanics, by using the ideas and methods of differentiable mechanics. This involves in particular the important work of Gallavotti and Cohen [5], Chernov, Eyinck, Lebowitz and Sinai [4], and a number of other papers referred to in [19].

Random dynamics

Our simplified model of a system coupled with a heat bath will be described by a *random dynamical system* (with discrete time for simplicity). The heat bath itself is described by a probability space $(\Omega, \mathbf{P})$, with an invertible measurable $\mathbf{P}$-preserving map τ describing time evolution; $\mathbf{P}$ is assumed to be τ-ergodic. (Further technical assumptions will be made later).

For each $\omega\in\Omega$, a diffeomorphism f_ω of the smooth (noncompact) manifold $\mathcal{S}$ is given, and the time evolution on $\Omega\times\mathcal{S}$ is the skew-product transformation f on $\Omega\times\mathcal{S}$ defined by

$$f(\omega,x)=(\tau\omega,f_\omega x)$$

(Assumptions of smoothness of f_ω and measurability of $\omega\mapsto f_\omega$ are to be discussed later).

Thermostatic action of random forces.

It is easy to understand qualitatively how a heat bath prevents the heating up of a system. Suppose for example that the system is a gas enclosed in a container, and that the thermostatic action of the heat bath takes place when the particles of the gas hit the walls of the container . Shocks with the wall are not elastic . When a particle hits the wall at very high speed, it is released in the average with a lower speed. In this manner the energy of the gas is prevented from increasing indefinitely (even though there are forces acting on the gas, that maintain it outside of equilibrium, and usually transfer energy to it).

The above thermostatic mechanism may be translated into the language of random dynamical systems. Suppose for example that the f_ω are independent, identically distributed, and that there is an energy function $x \mapsto E(x) \geq 0$ and constants $\bar{E} > 0$, $A > 0$, $\alpha > 0$, and $\beta \in (0, 1)$ such that

$$\langle e^{\alpha E(f_\omega x)} \rangle_\omega \leq A \qquad \text{if} \qquad E(x) \leq \bar{E}$$

$$\langle e^{\alpha E(f_\omega x)} \rangle_\omega \leq \beta e^{\alpha E(x)} \qquad \text{if} \qquad E(x) > \bar{E}$$

where $\langle \cdots \rangle_\omega$ is the average over ω. Let ρ_0 be a probability measure on $\mathcal{S}$, and ρ_n the measure obtained from it at time n (*i.e.*, $\rho_{n+1} = \langle f_\omega \rho_n \rangle_\omega$ where $f_\omega \rho$ is the direct image of ρ by f_ω). Write

$$c_n = \int \rho_n(dx) e^{\alpha E(x)}$$

and suppose $c_0 < +\infty$; we have then

$$c_{n+1} = \int \langle (f_\omega \rho_n)(dx) \rangle_\omega e^{\alpha E(x)}$$

$$= \langle \int (f_\omega \rho_n)(dx) e^{\alpha E(x)} \rangle_\omega = \langle \int \rho_n(dx) e^{\alpha E(f_\omega x)} \rangle_\omega = \int \rho_n(dx) \langle e^{\alpha E(f_\omega x)} \rangle_\omega$$

$$\leq A \rho_n \{x : E(x) \leq \bar{E}\} + \beta \int_{E(x) > \bar{E}} \rho_n(dx) e^{\alpha E(x)} \leq A + \beta c_n$$

Therefore by induction

$$c_n \leq \beta^n c_0 + \frac{A}{1 - \beta}$$

which shows that

$$\int \rho_n(dx) e^{\alpha E(x)}$$

is bounded independently of n.

In the above example the energy $E(x)$ is unbounded, but states of high energy are rarely visited: the system does not heat up. Notice that we are not here separating the deterministic forces driving the system outside of equilibrium from the random forces associated with the heat bath.

Using a compact phase space.

We have just seen that in the presence of suitable random forces, a system rarely comes close to infinity on $\mathcal{S}$. Physically, this is in agreement with the remarkable metastability of systems that are, strictly speaking, unstable (like a mixture of oxygen and hydrogen at room temperature). In the presence of random forces which prevent heating up of the system we might as well, physically, assume that the phase space $\mathcal{S}$ is compact. In other words we want to modify $\mathcal{S}$ and the f_ω near infinity and replace them by a compact manifold M and diffeomorphisms (again denoted by f_ω) of M. Note that this is a physical approximation, not a mathematical compactification of $\mathcal{S}$.

The reason for this modification of our setup is one of mathematical convenience. Namely that the theorems on random dynamical systems which we shall use have been proved for compact manifolds. Extensions to noncompact manifolds presumably exist, but it does not appear justified to spend much effort in proving them at this time.

Loss of correlations in the heat bath.

Let us briefly return to the difference between a real heat bath and random external forces. When a real heat bath interacts with a system, it is acted upon by the system, which transfers to it energy and information. The transfer of information means that correlations are created between the state of the heat bath and that of the system. Such correlations do not exist in the case of a random dynamical system, where the time evolution in Ω is independent of the factor $\mathcal{S}$. A good heat bath has a short relaxation time: correlations diffuse in it quickly, so that the action on the system appears random (with a short correlation time) independently of the behavior of the system. A random dynamical system is thus an idealization of a dynamical system in contact with a good heat bath. Note that the diffusion and loss of correlations in the heat bath corresponds to an increase of the global entropy. More generally, loss of correlations may be viewed as the basic physical mechanism leading to increase of entropy and irreversibility (this fits with the discussion by Lebowitz [10]). Admittedly, we are lacking a detailed physical understanding of how correlations diffuse and are lost in a heat bath. In the present paper we bypass this fundamental question by using a random dynamical system instead of a realistic heat bath. It is physically reasonable to assume that the $f_{\tau^n\omega}$ are independent and identically distributed (i.i.d.) but since this assumption is unnecessary and tends to confuse the issues, we shall first study the general case.

Scope of the paper.

In Section 1 we review some properties of random differentiable dynamical systems (without i.i.d. assumption). In Section 2 we obtain the formula for the entropy production, study its positivity. In Section 3 we discuss the special i.i.d. case.

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1. Ergodic theory of random dynamical systems.

Assumptions.

Let us fix our mathematical framework*. We consider a random system consisting of a probability space $(\Omega, \mathbf{P})$, a map $\tau : \Omega \to \Omega$ such that $\mathbf{P}$ is τ-ergodic, a compact manifold M, and a family $(f_\omega)_{\omega\in\Omega}$ of diffeomorphisms of M. To be specific, we make the following standing technical assumptions:

- Ω is a Polish space**, $\mathbf{P}$ is a Borel probability measure on Ω, $\tau : \Omega \to \Omega$ is invertible, τ and τ^{-1} are Borel, $\mathbf{P}$ is τ-invariant and ergodic.

- M is a compact C^∞ manifold.

- $\omega \mapsto f_\omega$ is a Borel map $\Omega \to \mathrm{Diff}^r(M)$.

(allowed values of $r \geq 1$ will be specified as needed).

- If J_ω is the Jacobian of f_ω with respect to some Riemann metric, and $\ell(\omega) = \sup_x |\log J_\omega(x)|$, then $\ell \in L^1(\mathbf{P})$

Time entropy.

A Borel map $f : \Omega \times M \to \Omega \times M$ is defined by $f(\omega, x) = (\tau\omega, f_\omega x)$ and we denote by $\pi : \Omega \times M \to \Omega$ the canonical projection. We assume that μ is an f-invariant probability measure on $\Omega \times M$ with projection $\pi\mu = \mathbf{P}$. A disintegration $(\mu_\omega)_{\omega\in\Omega}$ of μ then exists ($\mathbf{P}$-a.e. unique) such that the μ_ω are probability measures on M and

$$\mu(d\omega dx) = \mathbf{P}(d\omega)\mu_\omega(dx)$$

It is convenient to write $f^k(\omega, x) = (f^k)_\omega(x)$ (so that for instance $(f^{-1})_\omega = (f_{\tau^{-1}\omega})^{-1}$). Let then β be a finite Borel partition of M, and

$$\beta_\omega^{(n)} = \beta \vee f_\omega^{-1}\beta \vee \cdots \vee ((f^{n-1})_\omega)^{-1}\beta$$

If we write

$$H(\mu_\omega, \beta_\omega^{(n)}) = -\sum_{B\in\beta_\omega^{(n)}} \mu_\omega(B)\log\mu_\omega(B)$$

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* We follow here the presentation by Liu [14] which can be consulted for some more details and references. The special i.i.d. case, which was studied earlier (see Kiefer [8], Ledrappier and Young [13]), will be discussed in Section 3. For further background material, see Kiefer [9],Liu and Qian [15].

** A topological space Ω is called a Polish space if (a) Ω is separable,*i.e.*, it contains a countable dense set, (b) there is a metric on Ω which defines the topology, and for which Ω is complete. Part of the results below would hold without supposing Ω Polish, but this assumption (made by Liu [14]) is quite acceptable for our purposes (we could in fact make the stronger assumption that Ω is metrizable compact).

the limit

$$h(\mu,\beta)=\lim_{n\to\infty}\frac{1}{n}H(\mu_\omega,\beta_\omega^{(n)})$$

exists and is constant for $\mathbf{P}$-almost all ω. The *fiber entropy* is defined by

$$h(\mu)=\sup_\beta h(\mu,\beta)$$

and turns out to coincide with the relative (or conditional) entropy of (μ,f) with respect to $\pi:\Omega\times M\to\Omega$. (For these results see Bogenschütz [3], and for background the book of Kifer [8]). Note that h is a time entropy, different from the statistical mechanical entropy S to be discussed later.

Lyapunov exponents and unstable manifolds.

Choose a Riemann metric on the tangent bundle TM (with associated distance d on M) and assume that

$$\int[\log^+\parallel T_xf_\omega\parallel+\log^+\parallel T_xf_\omega^{-1}\parallel]\mu(d\omega dx)<\infty$$

One may then write

$$-\infty<\lambda^{(1)}(\omega,x)<\lambda^{(2)}(\omega,x)<\cdots<\lambda^{(r(\omega,x))}(\omega,x)<\infty$$

and

$$T_xM=E^{(1)}(\omega,x)\oplus\cdots\oplus E^{(r(\omega,x))}(\omega,x)$$

so that r, the $\lambda^{(i)}$ and $E^{(i)}$ are Borel, and for μ-almost all (ω,x)

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\parallel T_x(f^n)_\omega\xi\parallel=\lambda^{(i)}(\omega,x)$$

if $0\neq\xi\in E^{(i)}(\omega,x)$, $1\leq i\leq r(\omega,x)$. This is a form of the multiplicative ergodic theorem of Oseledec; the $\lambda^{(i)}$ are called *Lyapunov exponents*, and the $m^{(i)}=\dim E^{(i)}$ are their multiplicities. We define $E^u(\omega,x)$ to be the sum $\oplus E^{(i)}(\omega,x)$ extended over those i for which $\lambda^{(i)}(\omega,x)\geq 0$ (unstable space).

Suppose now that the f_ω are C^2 (*i.e.*, $r\geq 2$),and that

$$\int[\log^+\parallel f_\omega\parallel_{C^2}+\log^+\parallel f_\omega^{-1}\parallel_{C^2}]\mathbf{P}(d\omega)<\infty$$

Given $\omega\in\Omega$, we partition M into *unstable manifolds* W_ω^u such that the W_ω^u containing x is

$$W_\omega^u(x)=\{y\in M:\limsup_{n\to+\infty}\frac{1}{n}\log d((f^{-n})_\omega x,(f^{-n})_\omega y)<0\}$$

One can construct an f-invariant Borel set $\Delta \subset \Omega \times M$, with $\mu(\Delta) = 1$ such that, if $(\omega, x) \in \Delta$, $W^u_\omega(x)$ is the image of $E^u(\omega, x)$ by a $C^{1,1}$ injective immersion*. The proof of the above results can be obtained by the methods of Ruelle [17] as noted by Liu [14].

The SRB condition for the measure μ is, roughly speaking, that the conditional measures on the unstable manifolds $(\omega, W^u_\omega) \subset \Omega \times M$ be absolutely continuous with respect to the Riemann measure. Technically, however, one cannot directly define conditional measures** on the (ω, W^u_ω). This is because they usually do not form a measurable partition of $\Omega \times M$ (each W^u_ω may be folded over upon itself so that its closure is M). One defines a *local unstable manifold* $W^u_\omega(x, \text{local})$ to be the graph of a smooth map from an open neighborhood of x in $E^u(\omega, x)$ to $E^u(\omega, x)^\perp$. It is then possible to define measurable partitions of $\Omega \times M$ into sets (ω, S) where S is on open subset of a local unstable manifold (for its induced topology). If the conditional measures of μ on the (ω, S) are absolutely continuous with respect to the Riemann measure of the unstable manifolds, then μ is said to satisfy the SRB condition.

We can now state two results of the ergodic theory of random differentiable dynamical systems, which we shall use later in the discussion of entropy production. (For the proofs we refer to the original papers).

1.1 Theorem. If $r \geq 1$ and

$$\int \mathbf{P}(d\omega) \log^+ \| f_\omega \|_{C^1} \leq +\infty$$

then the fiber entropy (=conditional entropy of (μ, f) with respect to the projection $\pi : \Omega \times M \to \Omega$) satisfies

$$h(\mu) \leq \int \mu(d\omega dx) \sum_{i:\lambda^{(i)}>0} \lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)$$

This was first proved for a single C^1 map (*i.e.*, Ω reduced to one point), see Ruelle [18]. For the extension to the present situation see Bahnmüller and Bogenschütz [2]*** (their paper gives a history of related results).

1.2 Theorem. If $r \geq 2$ and

$$\int \mathbf{P}(d\omega) [\log^+ \| f_\omega \|_{C^2} + \log^+ \| f_\omega^{-1} \|_{C^2}] \leq +\infty$$

* The class $C^{1,1}$ consists of functions with Lipschitz continuous first order derivatives.

** For the theory of conditional measures associated with measurable partitions see Rohlin [16]

*** A gap in [2] is fixed in [1]. Note that, for Theorem 1.1, Ω need not be Polish, and the f_ω are not required to be diffeomorphisms (it suffices, as done in [18], to assume that they are C^1 maps).

and if μ satisfies the SRB condition, then

$$h(\mu) = \int \mu(d\omega dx) \sum_{i:\lambda^{(i)}>0} \lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)$$

In the case of a single map, the above equality is known as Pesin's formula, and it was shown to follow from the SRB condition by Ledrappier and Strelcyn [11]. (In fact Pesin's formula is equivalent to the SRB condition, as proved by Ledrappier and Young [12]). The generalization to random dynamical systems of the result of Ledrappier and Strelcyn is due to Liu [14]. Liu's assumption that Ω is Polish allows him to apply Lusin's Theorem* to connect arguments of abstract measure theory (measurable partitions of Lebesgue spaces) and the topology of the problem.

Time reversal.

The measure μ is invariant under $f^{-1} : (\omega, x) \mapsto (\tau^{-1}\omega, f_{\tau^{-1}}^{-1} x)$. The fiber entropy computed with respect to f^{-1} is again $h(\mu)$. (To see this one can for instance use the fact that the relative entropies of (μ, f) and (μ, f^{-1}) with respect to the projection $\Omega \times M \to \Omega$ are the same). It follows from the definition that the Lyapunov exponents associated with f^{-1} are the $-\lambda^{(i)}(\omega, x)$, with multiplicity $m^{(i)}(\omega, x)$. The unstable manifolds associated with f^{-1} are the stable manifolds associated with f (we shall not use them).

* To the effect that a measurable function is in fact continuous outside of a set of small measure. More precisely, let E be a Polish space, F a topological space with countable base, μ a (Borel, bounded) positive measure on E, and $f : E \to F$ a Borel map. Then there is $\tilde{f}$ equal μ-a.e. to f, and for each $\epsilon > 0$ there is a compact set $K \subset E$ such that $\mu(E \backslash K) \leq \epsilon$ and $\tilde{f}|K$ is continuous.

2. Entropy production.

Keeping the assumptions of Sections 1, we let J_ω denote the absolute value of the Jacobian of f_ω with respect to some Riemann metric. If ρ is any (Borel) probability measure on $\Omega \times M$ we may define the entropy production $e_f(\rho)$ by

$$e_f(\rho) = -\int \rho(d\omega dx) \log J_\omega(x) \tag{2.1}$$

We shall use this definition only when $\pi\rho = \mathbf{P}$ (where π is the projection $\Omega \times M \to \Omega$). We have then

$$|e_f(\rho)| \leq \int \mathbf{P}(d\omega) \sup_x |\log J_\omega(x)| \leq \infty$$

2.1 Lemma. The entropy production $e_f(\mu)$ for the f-invariant probability measure μ is independent of the choice of Riemann metric on M, and

$$e_f(\mu) = -\int \mu(d\omega dx) \sum_i \lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)$$

This expression of $e_f(\mu)$ in terms of the Lyapunov exponents follows from the multiplicative ergodic theorem, and implies independence of the choice of metric. $\square$

2.2 Theorem. If the f-invariant probability measure μ satisfies the SRB condition, then $e_f(\mu) \geq 0$.

By Lemma 2.1, we have

$$e_f(\mu) = -\int \mu(d\omega dx) \sum_i \lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)$$

$$= h(\mu) - \int \mu(d\omega dx) \sum_{i:\lambda^{(i)}>0} \lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)$$

$$-[h(\mu) + \int \mu(d\omega dx) \sum_{i:\lambda^{(i)}<0} \lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)]$$

so that, by Theorem 1.2,

$$e_f(\mu) = -[h(\mu) - \int \mu(d\omega dx) \sum_{i:\lambda^{(i)}<0} |\lambda^{(i)}(\omega, x) m^{(i)}(\omega, x)|]$$

If $\bar{h}$, $\bar{\lambda}$ denote the fiber entropy and Lyapunov exponents for the time reversed system (*i.e.*, when f is replaced by f^{-1}, see end of Section 1) we have thus

$$e_f(\mu) = -[\bar{h}(\mu) - \int \mu(d\omega dx) \sum_{i:\bar{\lambda}^{(i)}>0} \bar{\lambda}^{(i)}(\omega, x) m^{(i)}(\omega, x)]$$

Finally, Theorem 1.1 applied to the time reversed system gives $e_f(\mu) \geq 0$. $\square$

Comment.

If the probability measure ρ is absolutely continuous with respect to the Riemann volume on M, one expects a limit μ of $f^k\rho$ when $k \to +\infty$ to be smooth along unstable directions, *i.e.*, to be an SRB measure. If we accept that (2.1) represents the physical entropy production (see below) then Theorem 2.2 means that the physical entropy production is positive.

Relation of $e_f(\rho)$ with statistical mecanical entropy.

For any (Borel) probability measure ρ on $\Omega \times M$, such that $\pi\rho = \mathbf{P}$, we have the ($\mathbf{P}$-a.e. unique) disintegration $(\rho_\omega)_{\omega\in\Omega}$ where the ρ_ω are probability measures on M, and

$$\rho(d\omega dx) = \mathbf{P}(d\omega)\rho_\omega(dx)$$

We shall use in a moment the fact that*

$$(f\rho)(d\omega dx) = \mathbf{P}(d\omega) \times (f_{\tau^{-1}\omega}\rho_{\tau^{-1}\omega})(dx) \tag{2.2}$$

Let us assume that the ρ_ω are absolutely continuous with respect to the Riemann volume on M. We write then $\rho_\omega(dx) = \underline{\rho}_\omega(x)dx$, where $\underline{\rho}_\omega$ is the density of ρ_ω with respect to the Riemann volume element dx. Interpreting dx as phase space volume element we define the entropy

$$S(\underline{\rho}_\omega) = -\int dx\,\underline{\rho}_\omega(x) \log \underline{\rho}_\omega(x)$$

(which is $\leq$log vol$M$). The entropy corresponding to ρ is then the average

$$S_\rho = \int \mathbf{P}(d\omega) S(\underline{\rho}_\omega) = -\int \rho(d\omega dx) \log \underline{\rho}_\omega(x)$$

and $-\infty \leq S_\rho \leq$log vol$M$.

Note that

$$(f_\omega\rho_\omega)(dx) = \frac{\underline{\rho}_\omega(f_\omega^{-1}x)}{J_\omega(f_\omega^{-1}x)}dx$$

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* We have indeed

$$\int (f\rho)(d\omega dx)\Phi(\omega,x) = \int \rho(d\omega dx)\Phi(\tau\omega, f_\omega x) = \int \mathbf{P}(d\omega)\rho_\omega(dx)\Phi(\tau\omega, f_\omega x)$$

$$= \int \mathbf{P}(d\omega)(f_\omega\rho_\omega)(dx)\Phi(\tau\omega, x) = \int \mathbf{P}(d\omega)(f_{\tau^{-1}\omega}\rho_{\tau^{-1}\omega})(dx)\Phi(\omega,x)$$

Using (2.2) thus yields

$$(f\rho)(d\omega dx) = \mathbf{P}(d\omega) \times \underline{\rho}_{\tau^{-1}\omega}((f^{-1})_\omega x)\bar{J}_{\tau^{-1}\omega}((f^{-1})_\omega x)dx$$

where we have written $\bar{J} = 1/J$. Therefore

$$S_{f\rho} = -\int (f\rho)(d\omega dx)[\log \underline{\rho}_{\tau^{-1}\omega}(f_\omega^{-1}x) + \log \bar{J}_{\tau^{-1}\omega}(f_\omega^{-1}x)]$$

$$= -\int \rho(d\omega dx)[\log \underline{\rho}_\omega((f^{-1})_{\tau\omega}f_\omega x) + \log J_\omega((f^{-1})_{\tau\omega}f_\omega x)]$$

$$= -\int \rho(d\omega dx)[\log \underline{\rho}_\omega(x) + \log \bar{J}_\omega(x)]$$

so that

$$-[S_{f\rho} - S_\rho] = \int \rho(d\omega dx) \log \bar{J}_\omega(x) \tag{2.3}$$

This expression is minus the entropy put into the system in one time step, which is equal to the entropy pumped out of the system, or produced by the system.

The expression (2.3) agrees with the expression (2.1) postulated for the entropy production. This justifies our definition in a special case, and in more general cases obtained by suitable limits. We are thus led to investigating the topology of probability measures μ such that $\pi\mu = \mathbf{P}$.

P-*vague topology.*

Let us define

$$E = \{\mu : \mu \text{ is a Borel probability measure on } \Omega \times M \text{ and } \pi\mu = \mathbf{P}\}$$

and

$$\Psi_{A\phi}(\mu) = \int_{A\times M} \mu(d\omega dx)\phi(x)$$

where A is a $\mathbf{P}$-measurable subset of Ω, and ϕ a continuous function $M \to \mathbf{C}$. We call $\mathbf{P}$-vague topology the coarsest topology on E for which all the functions $\Psi_{A\phi}$ are continuous*.

2.3 Proposition. With respect to the **P**-*vague topology, E is metrizable compact.*

In order to prove this we shall first replace Ω by a metrizable compact space. This is possible because on a Polish space Ω there is a metrizable compact topology which gives the same Borel sets [6]. More simply we can use the fact that a Polish space is homeomorphic to a countable intersection of open sets in a metrizable compact space.

* Given a finite number of pairs $(A_1, \phi_1), \ldots, (A_n, \phi_n)$, and $\epsilon > 0$, let $N = \{\mu \in E : |\Psi_{A_i\phi_i}(\mu) - \Psi_{A_i\phi_i}(\mu_0)| < \epsilon$ for $i = 1, \ldots, n\}$. Such sets form a basis of neighborhoods of $\mu_0 \in E$ for the $\mathbf{P}$-vague topology.

Now that $\Omega \times M$ is metrizable compact, the Borel measures coincide with the Radon measures, and the set of all probability measures on $\Omega \times M$, with the vague topology* is metrizable compact. The subset E of probability measures such that $\pi\mu = \mathbf{P}$ is vaguely closed, and E is thus metrizable compact for the vague topology. We are going to see that this vague topology on E coincides with the $\mathbf{P}$-vague topology defined above.

Let us show that the $\mathbf{P}$-vague topology on E is coarser than the vague topology. It suffices to prove that

$$N = \{\mu \in E : |\Psi_{A\phi}(\mu) - \Psi_{A\phi}(\mu_0)| < \epsilon\}$$

contains a vague neighborhood of μ_0 in E. We may assume $\phi \neq 0$ and choose a continuous function $\psi : \Omega \to \mathbf{R}$ so that it is close to the characteristic function of A in the $L^1(\mathbf{P})$ norm:

$$\| \psi - \chi_A \|_{L^1} \leq \frac{\epsilon}{3 \| \phi \|}$$

where $\| \phi \|$ is the uniform (=sup) norm of ϕ.Then

$$|\mu(\psi \otimes \phi) - \Psi_{A\phi}(\mu)| = |\int \mu(d\omega dx)[\psi(\omega) - \chi_A(\omega)]\phi(x)]|$$

$$\leq \| \phi \| \cdot \| \psi - \chi_A \|_{L^1} \leq \epsilon/3$$

The vague neighborhood of μ_0 defined by

$$\{\mu \in E : |\mu(\psi \otimes \phi) - \mu_0(\psi \otimes \phi)| \leq \epsilon/3\}$$

is then contained in N as announced.

The $\mathbf{P}$-vague topology separates points of E and is coarser than the vague topology, for which E is compact. Therefore the $\mathbf{P}$-vague and the vague topology coincide on E, and the proposition follows. $\square$

2.4 Proposition. If $\rho^{(m)}$ tends to μ for the $\mathbf{P}$-vague topology, then

$$\lim_{m\to\infty} \int \rho^{(m)}(d\omega dx) \log \bar{J}_\omega(x) = \int \mu(d\omega dx) \log \bar{J}_\omega(x)$$

As in the previous proof we may take Ω to be compact metrizable. We apply Lusin's theorem to the measure $(1 + \ell(\omega))\mathbf{P}(d\omega)$ where $\ell(\omega) = \sup_x |\log J_\omega(x)|$ and to the map $\omega \to \log \bar{J}_\omega(\cdot)$ of Ω to $\mathcal{C}(M)$ (space of continuous functions on M, with the sup-norm). We obtain thus a compact set $K \subset \Omega$ with $\mathbf{P}(K) \geq 1 - \epsilon$, $\int_{\Omega\setminus K} \mathbf{P}(d\omega)\ell(\omega) \leq \epsilon$, and $(\omega, x) \mapsto \log \bar{J}_\omega(x)$ continuous on $K \times M$. The restriction of $\rho^{(m)}$ to $K \times M$ tends vaguely

* The vague topology on measures on the compact set $\Omega \times M$ is the topology of pointwise convergence on the space $\mathcal{C}(\Omega \times M)$, of continuous functions. In other words the vague topology is the w^*-topology of the dual $\mathcal{C}(\Omega \times M)^*$.

to the restriction of μ to $K \times M$ (this results from the proof of Proposition 2.3 with Ω replaced by K), therefore

$$\int_{K\times M} \rho^{(m)}(d\omega dx) \log \bar{J}_\omega(x) \to \int_{K\times M} \mu(d\omega dx) \log \bar{J}_\omega(x)$$

and

$$|\int_{(\Omega\backslash K)\times M} \rho^{(m)}(d\omega dx) \log \bar{J}_\omega(x)| \le \int_{\Omega\backslash K} \mathbf{P}(d\omega)\ell(\omega) < \epsilon$$

$$|\int_{(\Omega\backslash K)\times M} \mu(d\omega dx) \log \bar{J}_\omega(x)| \le \int_{\Omega\backslash K} \mathbf{P}(d\omega)\ell(\omega) < \epsilon$$

Since ϵ is arbitrarily small, the proposition follows. $\square$

2.5 Theorem. Let $\rho(d\omega dx) = \mathbf{P}(d\omega)\rho_\omega(dx)$. Assume that the ρ_ω are absolutely continuous with respect to the Riemann volume, and that $S_\rho > +\infty$. If μ is any $\mathbf{P}$-vague limit of the measures $\rho^{(m)} = (1/m)\sum_{k=0}^{m-1} f^k\rho$, then $e_f(\mu) \ge 0$.

By Proposition 2.4 and equation (2.3) we have

$$e_f(\mu) = \lim_{m\to\infty} e_f(\rho^{(m)}) = \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1}[-S_{f^{k+1}\rho} + S_{f^k\rho}] = \lim_{m\to\infty} \frac{1}{m}[-S_{f^m\rho} + S_\rho]$$

Since S_ρ is finite, and $S_{f^m}\rho$ bounded above by log vol$M$, the limit in the right hand side is ≥ 0 as announced. $\square$

3. The i.i.d. case.*

To describe the case where the $f_{\tau^k\omega}$ are independent identically distributed, we write $\omega = (\alpha_i)_{i\in\mathbf{Z}}$ with $\alpha_i \in A$, so that $\Omega = A^{\mathbf{Z}}$. We also write $\mathbf{P}(d\omega) = \prod_{i\in\mathbf{Z}} p(d\alpha_i)$ where p is a probability measure on A. We take f_ω to depend only on α_0 and write $f_\omega = f_{(\alpha_0)}$. Our standing assumptions of Section 1 are thus replaced by the following.

- A is a Polish space, p a Borel probability measure on A. (We denote by τ the shift on $A^{\mathbf{Z}}$).
- M is a compact C^∞ manifold.
- $\alpha \mapsto f_{(\alpha)}$ is a Borel map $A \to \mathrm{Diff}^r(M)$, $r \geq 2$.
- If $J_{(\alpha)}$ is the Jacobian of $f_{(\alpha)}$ with respect to some Riemann metric, and $\ell(\alpha) = \sup_x |\log J_{(\alpha)}(x)|$, then $\ell \in L^1(p)$.

If the probability measure μ on $\Omega \times M$ has projection $\pi\mu = \mathbf{P}$ on Ω, we have the disintegration $\mu(d\omega dx) = \mathbf{P}(d\omega)\mu_\omega(dx)$. We denote by $E^{\leq}$ the set of those μ such that μ_ω does not depend on the *future*, i.e., μ_ω depends only on α_i for $i \leq 0$. Define also the maps $s : \Omega \times M \to A \times M$, $t : A \times M \to M$, and $\theta = ts : \Omega \times M \to M$ such that

$$(\omega, x) \xrightarrow{s} (\alpha_0, x) \xrightarrow{t} x \qquad , \qquad (\omega, x) \xrightarrow{\theta} x$$

and let

$$m = \theta\mu \qquad , \qquad m_1 = \theta f\mu$$

It is readily seen that if $\mu \in E^{\leq}$, then the image of $f\mu$ by s is of the form

$$(sf\mu)(d\alpha_1 dx) = p(d\alpha_1)m_1(dx)$$

and

$$m_1 = \int p(d\alpha) f_{(\alpha)} m \tag{3.1}$$

Also $fE^{\leq} \subset E^{\leq}$ and the entropy production for $\mu \in fE^{\leq}$ has the expression

$$e_f(\mu) = \int p(d\alpha)m(dx) \log \bar{J}_{(\alpha)}(x) \tag{3.2}$$

3.1 Proposition. Let $\mu \in fE^{\leq}$, and assume that $m = \theta\mu$ has density $\underline{m}$ with respect to the Riemann volume element dx, i.e., $m(dx) = \underline{m}(x)dx$.

(a) The density $\underline{m}_{(\alpha)}$ of $f_{(\alpha)}m$ and the density $\underline{m}_1$ of $\theta f m$ are given by

$$\underline{m}_{(\alpha)}(x) = \underline{m}(f_{(\alpha)}^{-1}x)\bar{J}(f_{(\alpha)}^{-1}x)$$

---

* See Ledrappier and Young [13] and, for background, Kifer[8].

$$\underline{m}_1(x) = \int p(d\alpha)\underline{m}_{(\alpha)}(x)$$

(b) If $S(\underline{m}) = -\int dx\, \underline{m}(x) \log \underline{m}(x) > -\infty$*, then*

$$e_f(\mu) = -\int p(d\alpha)S(\underline{m}_{(\alpha)}) + S(\underline{m})$$

(c) Let $\delta(\alpha, x) = \underline{m}_{(\alpha)}/\underline{m}_1(x)$ *, then*

$$\int p(d\alpha)S(\underline{m}_{(\alpha)}) - S(\underline{m}_1) = \int m_1(dx)[-\int p(d\alpha)\delta(\alpha, x)\log\delta(\alpha, x)] \leq 0$$

(d) In particular if $\underline{m} = \underline{m}_1$ *(i.e. if* μ *is* f*-invariant) we have*

$$e_f \geq 0$$

and $e_f > 0$ *unless* $\underline{m}_{(\alpha)}(x) = \underline{m}(x)$ *a.e. with respect to* $p(d\alpha)\underline{m}(x)dx$*.**

(a) follows from the definitions and (3.1). We have

$$-\int p(d\alpha)S(\underline{m}_{(\alpha)}) + S(\underline{m}) = \int p(d\alpha)[-S(\underline{m}_{(\alpha)}) + S(\underline{m})]$$

$$= \int p(d\alpha) \int dx\, \underline{m}_{(\alpha)}(x) \log \bar{J}_{(\alpha)}(x)$$

and (b) follows from (3.2). Note that we have $\int p(d\alpha)\delta(\alpha, x) = 1$; therefore

$$\int p(d\alpha)S(\underline{m}_{(\alpha)}) - S(\underline{m}_1) = \int p(d\alpha)S(\delta(\alpha, \cdot)\underline{m}_1(\cdot)) - S(\underline{m}_1)$$

$$= \int m_1(dx)[-\int p(d\alpha)\delta(\alpha, x)\log\delta(\alpha, x)]$$

but $\int p(d\alpha)\delta(\alpha, x) = 1$ also implies

$$\int p(d\alpha)\delta(\alpha, x)\log\frac{1}{\delta(\alpha, x)} \leq \int p(d\alpha)\log 1 \leq 0$$

proving (c). From (b) and (c) we obtain $e_f \geq 0$ when $\underline{m} = \underline{m}_1$, and in fact $e_f > 0$ unless $\delta(\alpha, x)$ (equal to $\underline{m}_{(\alpha)}(x)/\underline{m}(x)$) is 1 almost everywhere. This proves (d). $\square$

3.2 Remarks.

(a) While in the general case (Section 2) f and f^{-1} play symmetric roles, this symmetry is broken in the present Section because $(f^{-1})_{(\alpha_0)} \neq (f_{(\alpha_0)})^{-1}$.

(b) From Proposition 3.1(b),(c) it is clear that, in the stready state, the entropy produced by the system is equal to minus the entropy that it extracts from the heat bath.

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* This has been noted by Kifer [8] and Ledrappier and Young [13] Proposition (2.4.2).

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