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Wavelet analysis of Transfer Operators acting on n - dimensional Hölder, Zygmund Triebel spaces

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Abstract

In this paper we show how wavelet analysis may be used to analyse spectral properties of the transfer operator when it is acting on function spaces that are well described through wavelet expansions. We prove a fairly explicit formula for the essential spectral radius when the transfer operator is acting on the Hölder-Zygmund Triebel scale of spaces. It involves only the local linearizations of the transfer operator.

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Contents

1 Introduction. **2**

2 Statement of the main results. **3**

3 A formula for ρ_{ess} **5**

4 Wavelet analysis, the basic formulas. **9**
- 4.a The formulas for wavelet analysis and synthesis. 9
- 4.b Relation between $\mathcal{W}$ and $\mathcal{M}$ 10

5 Some function spaces. **11**
- 5.a The analysis of $S_0(\mathbb{R}^n)$ 11
- 5.b Wavelet analysis of $S(\mathbb{R}^n)$. 13
- 5.c Wavelet analysis of $S_0'(\mathbb{R}^n)$. 13
- 5.d Wavelet analysis over $\mathbb{T}^n$ 14
- 5.e More general spaces 15
- 5.f Spaces of periodic functions. 19
- 5.g The scale $F^{p,q}_{\gamma}(\mathbb{T}^n)$. 19

6 Operators in wavelet space. **21**
- 6.a Formal properties. 21
- 6.b Some a priori estimates. 22
- 6.c Compact embedding. 24

7 The transfer operator in wavelet space. **25**
- 7.a The multiplication operator. 25
- 7.b The diffeomorphism in wavelet space 26
- 7.c The locally linear approximation. 28

8 The main theorem **28**
- 8.a The transfer operator. 28
- 8.b A formula for $\rho_{\mathrm{ess}}(\mathcal{L})$ 29
- 8.c A more explicit formula for $\rho_{\mathrm{ess}}(\mathcal{L})$ 31
- 8.d A simpler upper bound. 36

9 The case $\Lambda_{\gamma}(\mathbb{T}^n)$. **37**

10 Extensions. **39**

11 Acknowledgements **40**

1 Introduction.

Let $\mathbb{T}^n$ be the n dimensional torus, which we identify with $\mathbb{R}^n/\mathbb{Z}^n$. For a smooth covering $\phi : \mathbb{T}^n \to \mathbb{T}^n$ and a smooth function $\mu : \mathbb{T} \to \mathbb{C}$ we define the transfer operator

$$\mathcal{L}_{\phi,\mu} : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n), \quad s(x) \mapsto \sum_{y:\phi(y)=x} s(y)\mu(y). \tag{1.1}$$

For the sake of simplicity we assume that the inverse map has only finely many branches so that the sum above is finite. This operator has natural extension acting on periodic functions. Since the there is no danger for confusion we shall not introduce a new symbol for these operators.

The aim of this paper is to study spectral properties of the transfer operators when it acts on Banach spaces of functions or measures of various regularity. This question has been previously considered in [BJLI95] in one dimensions and we shall in particular reproof the theorems of this paper but using a different technique which in turn allows us to extend the results to more general function spaces and higher dimensions.

The main tool of our analysis will be wavelet transform (see e.g. [Dau92], [Mey90], [CM90], [Hol95] for an introduction.).

The main ideas of this paper are easily explained in the one dimensional case. Let $g \in S(\mathbb{R})$, the class of Schwartz of rapidly decaying functions. In addition suppose that g has all moments vanishing

$$\int dx\, x^m g(x) = 0 \tag{1.2}$$

for all $m \in \mathbb{N}$. Then the wavelet transform of $s \in L^p(\mathbb{R})$ with respect to g is defined as the following convolutions

$$\mathcal{W}_g s(b,a) = \mathcal{W}[g,s](b,a) = (\tilde{g}_a * s)(b) = \int dx \frac{1}{a}\overline{g}\left(\frac{x-b}{a}\right) s(x). \tag{1.3}$$

with $a > 0$ and $b \in \mathbb{R}$. Here we have introduced the following notations, that we shall use in the sequel

$$\tilde{g}(t) = \overline{g}(-t),\ g_a = g(\cdot/a)/a,\ g_{b,a} = g_a(\cdot - b). \tag{1.4}$$

The wavelet transform thus maps functions over the real line to functions over the open half-space $\mathbb{H} = \{(b,a) : b \in \mathbb{R}, a > 0\}$. ¿From the definition it is clear that the wavelet transform is a sort of mathematical microscope whose position is fixed by b and whose enlargement is given by $1/a$ or to put it differently $\mathcal{W}[g,s](b,a)$ is obtained by “looking at s at position b and at scale a”. As a general statement one can say that local regularity of s is mirrored in a certain speed of decay of $\mathcal{W}_g s$.

Many function spaces can be characterized in terms of localization properties of wavelet coefficients. So for instance a uniform (in b) decay of $O(a^\infty)$ as $a \to 0$, of the wavelet coefficients equivalent to C^∞ regularity of s. More quantitative information is available. So a uniform decay of $O(a^\alpha)$ with $\alpha \in (0,1)$ is equivalent to $s \in \Lambda^\alpha$, the space of Hölder continuous functions of exponent α (see below).

On the other hand in the case of the transfer operators we have a fairly easy description of the dynamic in wavelet space. At small scale $a \to 0$ the action of $\mathcal{L}$ is essentially a translation and a dilation. So we have for instance (in one dimension!)

$$\mathcal{L}_{\phi,\mu}(g_{b,a}) \approx \mu(\phi(b))\, g_{\phi(b),\phi'(b)a} \tag{1.5}$$

As we shall see this allows control of $\mathcal{L}$ up to smoothing hence compact operators. In higher dimensions the mapping is slightly more complicated since the wavelet will be in general deformed.

Now, combining the fact that wavelet transforms characterize many function spaces with the fact that the dynamic is fairly easy (modulo smoothing) we shall be able to compute bounds for the essential spectral radius for many function spaces.

The rest of the paper is organized as follows. We first state the main result. We then give some formulas for the essential spectral radius of an operator in an abstract setting. We then recall the basic facts about continuous wavelet transforms. In particular we recall how to characterize Banach spaces of various regularity using wavelets. We then analyze the transfer operator in wavelet space. And eventually we combine both results to give formulas for the essential spectral radius.

2 Statement of the main results.

We start by defining the class of transfer operators for which we can compute the essential spectral radius. First some notations. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphisms. In components it reads

$$\phi : (x_j) \mapsto (\phi^k((x_j))). \tag{2.1}$$

Our standing assumption in order to avoid unnecessary difficulties is that this map and its inverse are uniformly smooth, that is all derivatives of all orders of all its components are bounded functions over $\mathbb{R}^n$. This hypothesis may be relaxed. For every $x \in \mathbb{R}^n$ we denote by $\partial\phi_x$ the differential map

$$\partial\phi_x : \mathbb{R}^n \to \mathbb{R}^n, \quad x_k \mapsto \sum_{j=1}^{n} (\partial_j \phi^k(x)) x_j. \tag{2.2}$$

We sometimes also write $\partial[\phi, x]$. As usual the Jacobian is defined as

$$J[\phi](x) = J(x) = |\det \partial\phi_x| \, . \tag{2.3}$$

Consider now the operator associated with ϕ

$$\Phi[\phi] = \Phi : s(x) \mapsto J(\phi^{-1}(x))^{-1} s(\phi^{-1}(x)). \tag{2.4}$$

it maps functions localized around y to functions localized around $\phi(y)$

$$\Phi[\phi]\delta_y = \delta_{\phi(y)}. \tag{2.5}$$

It preserves the L^1-norm. Its adjoint reads

$$\Phi^* : s(x) \mapsto s(\phi(x)), \tag{2.6}$$

and we have for $s \in L^p(\mathbb{R}^n)$, $r \in L^q(\mathbb{R}^n)$, $1/p + 1/q = 1$,

$$\int dx \Phi^* s(x) \, r(x) = \int dx s(x) \Phi s(x) \tag{2.7}$$

For $x \in \mathbb{R}^n$ we denote by $\partial\Phi_x$ the following linear map

$$\partial\Phi[\phi, x] = \partial\Phi_x : s(y) \mapsto (J(x))^{-1} s([\partial\phi_x]^{-1}(y)). \tag{2.8}$$

We have

$$\partial\Phi[\phi, x] = \Phi[\partial\phi_x]. \tag{2.9}$$

It describes the action of the tangent map on functions. Its adjoint reads

$$(\partial\Phi_x)^* : s(y) \mapsto s(\partial\phi_x y). \tag{2.10}$$

For any function $\mu \in L^\infty(\mathbb{R}^n)$ we define the multiplier

$$M[\mu] : s(x) \mapsto \mu(x) \cdot s(x). \tag{2.11}$$

Let ϕ_k, $k = 1, \ldots, K$, be a finite family of diffeomorphisms of $\mathbb{R}^n$. Let further smooth, periodic functions $\mu_k : \mathbb{T}^n \to \mathbb{C}$ be given and let us look at the following operator

$$\mathcal{L} = \sum_{k=1}^{K} \Phi[\phi_k] M[\mu_k] : \quad s(x) \mapsto \sum_{k=1}^{K} \mu_k(\phi_k^{-1}(x)) J[\phi_k](\phi_k^{-1}(x))^{-1} s(\phi_k^{-1}(x)) \tag{2.12}$$

We then suppose further that the ϕ_k and the the μ_k are such that $\mathcal{L}$ defines a mapping of $C^\infty(\mathbb{T}^n)$ into itself in that it maps $[0,1]^n$-periodic functions into periodic functions with the same period. Then for every $m \in \mathbb{N}$ we have that $\mathcal{L}^m$ has a similar form giving rise to a definition of multipliers $\mu_{m,k}$, and diffeomorphisms $\phi_{m,k}$, $k = 1, \ldots, K^m$ in the obvious way:

$$\mathcal{L}^m = \sum_{k=1}^{K^m} \Phi[\phi_{m,k}] M[\mu_{m,k}]. \tag{2.13}$$

Clearly this class contains the operator of the form 1.1, where the ϕ_k are the inverse branches of the lift of ϕ to $\mathbb{R}^n$.

The transfer operators we consider act in many function spaces. Below we shall introduce a family of Banach spaces that are characterized in terms of growth properties of wavelet coefficients. For the moment we shall call these Banach spaces completely regular Banach spaces and we leave the precise definition to later sections. Note however already now that these spaces contain the Hölder - Zygmund scale of spaces $\Lambda^\alpha(\mathbb{T}^n)$ for $\alpha \in \mathbb{R}$ as well as the Sobolev spaces $H^\gamma(\mathbb{T}^n)$, for $\gamma \in \mathbb{R}$. To give more explicit formulas we will consider the Triebel-Lizorkin spaces $F_\gamma^{p,q}(\mathbb{T}^n)$ defined below, that contain the $\Lambda^\alpha(\mathbb{T}^n)$ as well as the $H^\gamma(\mathbb{T}^n)$. Besides the space of periodic functions $B(\mathbb{T}^n)$ we also consider the space of functions $B(\mathbb{R}^n)$ associated with it. Again we postpone the precise definition since it intuitively clear in the case of $\Lambda^\alpha(\mathbb{T}^n)$ or $H^\gamma(\mathbb{T}^n)$.

In order to state the main formula of this paper we need some additional notation. Let $B = B(\mathbb{R}^n)$ be a Banach space of functions over $\mathbb{R}^n$. For any $K \in \mathbb{N}$, consider a sequence $s_k \in B(\mathbb{R}^n)$, $k = 1, \ldots, K$ and look at the following limit

$$F((s_k)) = \lim_{\lambda \to \infty} \sup \left\{ \left\| \sum s_k(\cdot - \beta_k) \right\|_{B(\mathbb{R}^n)} : |\beta_k - \beta_{k'}| > \lambda \ (k \neq k') \right\} \tag{2.14}$$

That is we look at the behavior of the norm as we shift different peaces independently away from each other. For completely regular spaces, the simple limit actually exists. More than that, there are functions $\gamma_K : \mathbb{R}_+^K \to \mathbb{R}_+$, defining a norm on $\mathbb{C}^K$ via $\|(x_k)\| = \gamma_K((|x_k|))$ such that

$$F((s_k)) = \gamma_K((\|s_k\|)). \tag{2.15}$$

We make $B^K = B \times B \ldots \times B$ (K factors) a Banach space with norm

$$\|(s_k)\|_{B^K} = F((s_k)) = \gamma_K((\|s_k\|)). \tag{2.16}$$

Often, the γ_K may be computed explicitly. For instance in the case of $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ we clearly have

$$F((s_k)) = \left(\sum \|s_k\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}, \tag{2.17}$$

with the obvious modification for $p = \infty$.

Now consider a family of linear operators $A_k : B \to B$, $k = 1, \ldots, K$. We then define an operator via

$$\bigoplus_{k=1}^{K} A_k : B^K \to B, \quad (s_k) \mapsto \sum A_k s_k. \tag{2.18}$$

We shall need its norm which explicitly reads

$$\left\| \bigoplus_{k=1}^{K} A_k \right\| = \sup \left\{ \left\| \sum_{k=1}^{K} A_k s_k \right\|_{B(\mathbb{R}^n)} : s_k \in B(\mathbb{R}^n), \gamma_K((\|s_k\|)) = 1 \right\}. \tag{2.19}$$

With these notations we are now ready to state the main theorem.

Theorem 2.1 *Let $B(\mathbb{R}^n)$ be a completely regular Banach space and let $B(\mathbb{T}^n)$ be its associated space of periodic functions. Then the essential spectral radius of the transfer operator can be computed as follows in terms of the norms of the linearized maps $\mu_{m,k}(\phi_{m,k}^{-1}(x))\partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)]$, $k = 1, \ldots, K^m$.*

$$\rho_{ess}(\mathcal{L}) = \lim_{m\to\infty} \left(\sup_{x\in\mathbb{T}^n} \left\| \bigoplus_{k=1}^{K^m} \mu_{m,k}(\phi_{m,k}^{-1}(x))\partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)] \right\|_{B^{K^m}(\mathbb{R}^n)\to B(\mathbb{R}^n)} \right)^{1/m} \tag{2.20}$$

This is therefore a sharpening of a result of Ruelle who has given estimates of the essential spectral radius in terms of the local expansion rate in [Rue89]. In the case of the spaces $F_\gamma^{p,q}$ this formula is valid. However we have the following simpler estimate

Theorem 2.2 *We have that $\mathcal{L}$ is acting on $F_\gamma^{p,q}(\mathbb{T}^n)$, $1 \le p \le \infty$, $1 \le q < \infty$ or $p = q = \infty$, and we may estimate*

$$\rho_{ess}(\mathcal{L}) \le \limsup_{m\to\infty} \sup_{x\in\mathbb{T}^n} \left(\sum_{k=1}^{K^m} \left|\mu_{m,k}(\phi_{m,k}^{-1}(x))\right|^v \left\|\partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)]\right\|^v_{F_\gamma^{p,q}(\mathbb{R}^n)\to F_\gamma^{p,q}(\mathbb{R}^n)} \right)^{1/vm} \tag{2.21}$$

with v, the dual exponent of p, $v^{-1} + p^{-1} = 1$.

3 A formula for ρ_{ess}

Let B be an infinite dimensional Banach space and $\mathcal{F}$ an operator acting continuously on B. We denote by B' the dual space consisting of linear functional v, $s \mapsto (v \mid s)$, and by $\mathcal{F}^t$, the transpose operator defined through $(v \mid \mathcal{F}s) = (\mathcal{F}^t v \mid s)$. The following formula for the essential spectral radius of $\mathcal{F}$ has been given in ([Nus70])

$$\rho_{\mathrm{ess}}(\mathcal{F}) = \lim_{m\to\infty} (\inf\{\|\mathcal{F}^m - \mathcal{K}\| : \mathcal{K} \text{ is compact}\})^{1/m}. \tag{3.1}$$

In [Fri95] the formula has been modified to

$$\rho_{\mathrm{ess}}(\mathcal{F}) = \lim_{m\to\infty} (\inf\{\|\mathcal{F}^m - \mathcal{K}\| : \mathcal{K} \text{ is of finite rank}\})^{1/m}. \tag{3.2}$$

Recall that the operators of finite rank can be written as

$$s \mapsto \sum_\alpha v_\alpha(s) u_\alpha \tag{3.3}$$

with a finite collection of $v_\alpha \in B'$ and $u_\alpha \in B$. We now want to show a formula that describes the essential spectral radius in terms of the "infinitesimal (=small scale)" behavior of $\mathcal{F}$. What we precisely mean by that becomes more explicit when we take a concrete realization of the family of operators Π_j below. For the moment we state the formula in an abstract setting.

Theorem 3.1 *Suppose there is a family of bounded operators $\Pi_j : B \to B$, $j = 1, 2, \ldots$, subject to the following conditions*

$$(i) \quad \exists c > 0 : \forall j, \|\Pi_j\| \leq c. \tag{3.4}$$

$$(ii) \quad (\mathbb{1} - \Pi_j) \textit{ is compact} \tag{3.5}$$

Suppose that B satisfies at

$$(iii) \ B_0 = \{r \in B : \lim_{j\to\infty} \Pi_j r = 0\} \textit{ is dense in } B \tag{3.6}$$

Then $B_0 = B$ and we have the following formula

$$\rho_{ess}(\mathcal{F}) = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\Pi_j \mathcal{F}^m\|)^{1/m} = \lim_{m\to\infty} (\liminf_{j\to\infty} \|\Pi_j \mathcal{F}^m\|)^{1/m}. \tag{3.7}$$

If instead we have

$$(iii') \ B'_0 = \{r \in B' : \lim_{j\to\infty} \Pi_j^t r = 0\} \textit{ is dense in } B' \tag{3.8}$$

Then $B'_0 = B'$ and we have the following formula

$$\rho_{ess}(\mathcal{F}) = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\mathcal{F}^m \Pi_j\|)^{1/m} = \lim_{m\to\infty} (\liminf_{j\to\infty} \|\mathcal{F}^m \Pi_j\|)^{1/m}. \tag{3.9}$$

If furthermore the operators Π_j are hierarchical in the sense that they satisfy at

$$(iv) \ \Pi_j \Pi_{j+1} = \Pi_{j+1} \Pi_j = \Pi_{j+1} \tag{3.10}$$

and if the dynamic acts in scales as expressed through

$$(v) \ \exists k \in \mathbb{N} : \lim_{j\to\infty} \|(\mathbb{1} - \Pi_j)\mathcal{F}\Pi_{j+k}\| = \lim_{j\to\infty} \|\Pi_{j+k}\mathcal{F}(\mathbb{1} - \Pi_j)\| = 0. \tag{3.11}$$

Then, if either (iii) or (iii') holds true, any of the above formulas may be used to compute $\rho_{ess}(\mathcal{F})$.

Remark: In the second part of the theorem, if Π_j satisfies that $c \leq 1$, then the interior limit actually exists. Indeed, for any bounded $\mathcal{A}$ by (ii) we have

$$\|\mathcal{A}\Pi_j\| = \|\mathcal{A}\Pi_{j-1}\Pi_j\| \leq c \|\mathcal{A}\Pi_{j-1}\| \leq \|\mathcal{A}\Pi_{j-1}\|. \tag{3.12}$$

Therefore the sequence $\|\mathcal{A}\Pi_j\|$ is monotonically decreasing. The same argument applies to $\|\Pi_j\mathcal{A}\|$.

If one thinks of Π_j as projection operators on the small scales of a multi-resolution analysis on a compact space, then the formula shows, that it is the small scale or infinitesimal behavior of $\mathcal{F}$ that determines the essential spectrum.

Proof. We proceed in eight elementary steps. First suppose that density property (iii') holds. The case (iii) is completely analogue and we leave it to the reader. We also consider only the case lim sup and leave the lim inf part to the reader.

Step 1: We have

$$\inf\{\|\mathcal{F} - \mathcal{K}\| : \mathcal{K} \text{ compact}\} \leq \|\mathcal{F}\Pi_j\|. \tag{3.13}$$

Indeed, we may write

$$\mathcal{F} - \mathcal{F}\Pi_j = \mathcal{F}(\mathbb{1} - \Pi_j) \tag{3.14}$$

The right hand side is a compact operator, since it contains a compact factor by hypothesis (ii) and since $\mathcal{F}$ is bounded.

Step 2: We have $B'_0 = B'$. Indeed, suppose $B'_0 \ni s_k \to s \in B'$. Then for every $\epsilon > 0$, we find K such that $k \geq K$ implies $\|s_k - s\| \leq \epsilon$. It follows that

$$\left\|\Pi_j^t s\right\| \leq \left\|\Pi_j^t s_k\right\| + \left\|\Pi_j^t (s - s_k)\right\| \leq \left\|\Pi_j^t s_k\right\| + c\epsilon \to c\epsilon \quad (j \to \infty). \tag{3.15}$$

Since ϵ was arbitrary, the statement follows.

Let Ξ be the set of finite rank operators

$$s \mapsto \sum_\alpha v_\alpha(s) u_\alpha \tag{3.16}$$

where the sum is finite with $v_\alpha \in B'$ and $u_\alpha \in B$.

Step 3: We have for all $\mathcal{K} \in \Xi$

$$\lim_{j\to\infty} \|\mathcal{K}\Pi_j\| = 0. \tag{3.17}$$

Indeed, we have for all s with $\|s\| = 1$

$$\|\mathcal{K}\Pi_j s\| = \left\|\sum_\alpha (v_\alpha \mid \Pi_j s) u_\alpha\right\| \leq \sum_\alpha \left\|\Pi_j^t v_\alpha\right\|_{B'} \|u_\alpha\|_B\,. \tag{3.18}$$

This tends to 0 as $j \to \infty$, since by step 2 $B'_0 = B'$, and since the sum contains only finitely many terms.

Step 4: We have for all $\mathcal{K} \in \Xi$

$$\|\mathcal{F} - \mathcal{K}\| \geq \frac{1}{c} \limsup_{j\to\infty} \|\mathcal{F}\Pi_j\|\,. \tag{3.19}$$

Here $\infty > c \geq 1$ is given by condition (i). Indeed, we have, since $\|\Pi_j\| \leq c$

$$\|(\mathcal{F} - \mathcal{K})\Pi_j\| \leq c\,\|\mathcal{F} - \mathcal{K}\|\,, \tag{3.20}$$

for each j. Now

$$\|(\mathcal{F} - \mathcal{K})\Pi_j\| \geq \|\mathcal{F}\Pi_j\| - \|\mathcal{K}\Pi_j\|\,. \tag{3.21}$$

We may take the limit superior $j \to \infty$ and obtain the stated estimate.

Step 5: We have, upon replacing $\mathcal{F}$ by $\mathcal{F}^m$, for all m

$$c^{-1/m} \left(\limsup_{j\to\infty} \|\mathcal{F}^m \Pi_j\|\right)^{1/m} \leq \left(\inf\{\|\mathcal{F}^m - \mathcal{K}\| : \mathcal{K} \text{ compact}\}\right)^{1/m} \leq \left(\limsup_{j\to\infty} \|\mathcal{F}^m \Pi_j\|\right)^{1/m} \tag{3.22}$$

We now may go to the limit $m \to \infty$ to conclude for the first set of formulas. The last three steps prove the second part.

Step 6: Note that if $\mathcal{A}$ satisfies at (v) with some k it also satisfies the same with $k+1$ (and hence for all $k' > k$). Indeed, we may write as before

$$\|(\mathbb{1} - \Pi_j)\mathcal{A}\Pi_{j+k+1}\| = \|(\mathbb{1} - \Pi_j)\mathcal{A}\Pi_{j+k}\Pi_{j+k+1}\| \leq c\,\|(\mathbb{1} - \Pi_j)\mathcal{A}\Pi_{j+k}\|\,. \tag{3.23}$$

The last expression tends to 0 by hypothesis on $\mathcal{A}$. The analogue argument applies for the second limit.

Step 7: The set of bounded operators satisfying (v) forms an algebra. Indeed if $\mathcal{G}$ satisfies as $\mathcal{F}$ condition (v) then their sum and scalar multiples obviously satisfy the same and we are left to check the product. By step 6 we may suppose that $\mathcal{F}$ and $\mathcal{G}$ satisfy at (v) with the same k. Then we have thanks to (iv),

$$\|(\mathbb{1}-\Pi_j)\mathcal{F}\mathcal{G}\Pi_{j+2k+1}\| = \|(\mathbb{1}-\Pi_j)\mathcal{F}[\Pi_{j+k}+(\mathbb{1}-\Pi_{j+k})(\mathbb{1}-\Pi_{j+k+1})]\mathcal{G}\Pi_{j+2k+1}\| \,. \tag{3.24}$$

Using the triangular inequality and (i), the last expression may be majorized by

$$\leq c\,\|\mathcal{G}\|\,\|(\mathbb{1}-\Pi_j)\mathcal{F}\Pi_{j+k}\| + (1+c)^2\,\|\mathcal{F}\|\,\|(\mathbb{1}-\Pi_{j+k+1})\mathcal{G}\Pi_{j+2k+1}\| \,. \tag{3.25}$$

By hypothesis, both expressions tend to 0 as $j\to\infty$.

Step 8: Conclusion. For any $\mathcal{A}$, bounded in B satisfying (v) and thus in particular for $\mathcal{A}=\mathcal{F}^m$ we have

$$\|\mathcal{A}\Pi_{j+k}\| \leq \|\Pi_j\mathcal{A}\Pi_{j+k}\| + \|(\mathbb{1}-\Pi_j)\mathcal{A}\Pi_{j+k}\| \tag{3.26}$$

In the limit $j\to\infty$ we obtain

$$\limsup_{j\to\infty}\|\mathcal{A}\Pi_{j+k}\| \leq \limsup_{j\to\infty}\|\Pi_j\mathcal{A}\|\,\|\Pi_{j+k}\| \leq c\,\limsup_{j\to\infty}\|\Pi_j\mathcal{A}\| \,. \tag{3.27}$$

In the same way way obtain

$$\limsup_{j\to\infty}\|\Pi_{j+k}\mathcal{A}\| \leq c\limsup_{j\to\infty}\|\mathcal{A}\Pi_j\| \tag{3.28}$$

The constant c does not depend on $\mathcal{A}$ and thus the stated equality of all the limits follows. □

As the proof shows we have the following corollary

Corollary 3.2 *Suppose Π_j satisfies (i), (ii) and (iv) and suppose that $\mathcal{F}$ satisfies at (v) then the following four limits are equal*

$$\limsup_{m\to\infty}(\inf)\,(\limsup_{j\to\infty}(inf)\,\|\mathcal{F}^m\Pi_j\|)^{1/m}. \tag{3.29}$$

We shall need a second theorem for the essential spectral radius. This is because the as it will turn out, the space $\Lambda^\alpha(\mathbb{R}^n)$ does not satisfy the density properties (iii) nor (iii'). However we will be able to use the following theorem that is similar to a theorem of Keller proved in [Kel84].

Theorem 3.3 *Suppose that Π_j satisfies conditions (i) and (ii) of the previous theorem. The following condition is supposed to hold for $\mathcal{F}$ and all its iterates $\mathcal{F}^m$. We suppose that there is a double sequence of infinite dimensional closed spaces $V_j^m\subset\Pi_jB$, j, $m=1,\,2,\,\ldots$ and some constant $c'>0$ such that for each fixed m the following holds: for all sequences $s_j\in V_j^m$ with $\|s_j\|=1$ we have*

$$\limsup_{j\to\infty}\|\mathcal{F}^m s_j\| \geq c'\,\limsup_{j\to\infty}\|\mathcal{F}^m\Pi_j\| \,, \tag{3.30}$$

then again we have

$$\rho_{ess}(\mathcal{F}) = \lim_{m\to\infty}(\limsup_{j\to\infty}\|\mathcal{F}^m\Pi_j\|)^{1/m}. \tag{3.31}$$

Proof. We proceed in 4 elementary steps. We argue for fixed, but arbitrary m.

Step 1: As before we have

$$\inf\{\|\mathcal{F}^m-\mathcal{K}\| : \mathcal{K}\text{ compact}\} \leq \limsup_{j\to\infty}\|\mathcal{F}^m\Pi_j\| \tag{3.32}$$

Step 2: Since $\mathcal{K}$ is compact, there is a sequence $s_j \in V_j^m$ with $\|s_j\| = 1$ but

$$\lim_{j\to\infty} \|\mathcal{K}s_j\| = 0. \tag{3.33}$$

Indeed, the inverse of $\mathcal{K}$ is necessarily unbounded (the dimension of B is infinite), and thus for each j there is a sequence $s_j^k \in V_j$ with $\left\|s_j^k\right\| = 1$ and $\mathcal{K}s_j^k \to 0$ as $k \to \infty$. A diagonal sequence $s_j = s_j^{p(j)}$ with some function $p(j)$ tending to ∞ fast enough will do.

Step 3: We therefore have

$$\|\mathcal{F}^m - \mathcal{K}\| \geq \|(\mathcal{F}^m - \mathcal{K})s_j\| \geq \|\mathcal{F}^m s_j\| - \|\mathcal{K}s_j\|. \tag{3.34}$$

Taking limits, the first terms is $\geq \limsup_{j\to\infty} \|\mathcal{F}\Pi_j\| / c'$ whereas the last term tends to 0.

Step 4: This shows that again

$$c'^{-1/m} \limsup_{j\to\infty} \|\mathcal{F}^m\Pi_j\|^{1/m} \leq \inf\{\|\mathcal{F}^m - \mathcal{K}\| : \mathcal{K} \text{ compact}\}^{1/m} \leq \limsup_{j\to\infty} \|\mathcal{F}^m\Pi_j\|^{1/m} \tag{3.35}$$

We now may go to the limit $m \to \infty$ to conclude. □

4 Wavelet analysis, the basic formulas.

For the convenience of the reader we shall list here the basic formulas of wavelet analysis. We limit ourselves to formal expressions. They actually have a precise meaning when we consider the wavelet analysis in $S_0(\mathbb{R}^n)$ or $S_0'(\mathbb{R}^n)$ (see below).

4.a The formulas for wavelet analysis and synthesis.

Let f be a complex valued function over $\mathbb{R}^n$. Let g be another such function. The wavelet transform of f with respect to the analyzing wavelet g is defined through (we write dx for n-dimensional Lebesgue measure)

$$\mathcal{W}_g f(b,a) = \int_{\mathbb{R}^n} dx\, \frac{1}{a^n}\overline{g}\left(\frac{x-b}{a}\right) f(x), \quad b \in \mathbb{R}^n, a > 0. \tag{4.1}$$

We also write $\mathcal{W}[g,f](b,a)$ instead of $\mathcal{W}_g f(b,a)$. Here $b \in \mathbb{R}^n$ is a position parameter and $a \in \mathbb{R}_+$ is a scale parameter. The wavelet transform of a function over $\mathbb{R}^n$ is thus a function over the position scale half-space $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}_+$.

If we introduce the dilation (D_a) and translation operators (T_b)

$$T_b s(x) = s(x-b), \quad D_a s(x) = s(x/a)/a^n, \tag{4.2}$$

then we may also write the wavelet transform as a family of scalar products

$$\mathcal{W}_g f(b,a) = \langle g_{b,a} \mid f\rangle, \quad g_{b,a} = T_b D_a g, \tag{4.3}$$

or as a family of convolutions indexed by a scale parameter

$$\mathcal{W}_g f(b,a) = (\tilde{g}_a * f)(b), \quad \tilde{g}_a = D_a\tilde{g}, \quad \tilde{g}(x) = \overline{g}(-x). \tag{4.4}$$

The convolution product is defined as usual

$$(s * r)(x) = \int_{\mathbb{R}^n} dy s(x-y)r(y) = (r * s)(x) \tag{4.5}$$

If we introduce the Fourier transform

$$\widehat{s}(k) = \int_{\mathbb{R}^n} dx e^{-ikx} s(x), \quad s(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} dk e^{ikx} \widehat{s}(k) \tag{4.6}$$

then the wavelet transform may also be written as

$$\mathcal{W}_g f(b,a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} dk \, \overline{\widehat{g}(ak)} \, e^{ibk} \, \widehat{f}(k). \tag{4.7}$$

The wavelet synthesis $\mathcal{M}$ maps functions over the position - scale half-space to functions over $\mathbb{R}^n$. Let $r = r(b,a)$ be a complex valued function over $\mathbb{H}^n$ and h a function over $\mathbb{R}^n$. Then the wavelet synthesis of r with respect to the synthesizing wavelet h is defined as

$$\mathcal{M}_h r(x) = \int_{\mathbb{H}^n} \frac{dbda}{a} \, r(b,a) \, \frac{1}{a^n} \, h\left(\frac{x-b}{a}\right). \tag{4.8}$$

4.b Relation between $\mathcal{W}$ and $\mathcal{M}$

We now list some relations between $\mathcal{W}$ and $\mathcal{M}$. The wavelet synthesis is the adjoint of the wavelet transform—both with respect to the same wavelet—$\mathcal{W}_g^* = \mathcal{M}_g$

$$\int_{\mathbb{H}^n} \frac{dbda}{a} \overline{\mathcal{W}_g s(b,a)} r(b,a) = \int_{\mathbb{R}^n} dx \, \overline{s(x)} \, \mathcal{M}_g r(x). \tag{4.9}$$

The combination $\mathcal{M}_h \mathcal{W}_g$ reads in Fourier space

$$\mathcal{M}_h \mathcal{W}_g : \widehat{s}(k) \mapsto m_{g,h}(k) \widehat{s}(k), \quad m_{g,h}(k) = \int_0^\infty \frac{da}{a} \overline{\widehat{g}(ak)} \, \widehat{h}(ak) \tag{4.10}$$

Note that the Fourier multiplier $m_{g,h}$ depends only on the direction of k, $m_{g,h} = m_{g,h}(k/|k|)$. This is because the measure da/a is scaling invariant. In case that g and h are such that

$$m_{g,h}(k) = \int_0^\infty \frac{da}{a} \overline{\widehat{g}(ak)} \, \widehat{h}(ak) = c_{g,h} \tag{4.11}$$

with $0 < |c_{g,h}| < \infty$, we say that g, h are an *analysis reconstruction pair*, or that h is a *reconstruction wavelet* for g. In this case we have

$$\mathcal{M}_h \mathcal{W}_g = c_{g,h} \mathbb{1}. \tag{4.12}$$

We say that g is *strictly admissible* if g is its own reconstruction wavelet, or (what is the same) if

$$\forall k \in \mathbb{R}^n \backslash \{0\} : \quad \int_0^\infty \frac{da}{a} \, |\widehat{g}(ak)|^2 = c_g \tag{4.13}$$

A sufficient condition for g to have a reconstruction wavelet r is that for some $c > 1$ we have

$$c^{-1} \le \int_0^\infty \frac{da}{a} \, |\widehat{g}(ak)|^2 \le c. \tag{4.14}$$

If this condition holds, we call g *admissible*. In this case the following function r will be a reconstruction wavelet for g

$$\widehat{r}(k) = \widehat{g}(k) \Big/ \int_0^\infty \frac{da}{a} \, |\widehat{g}(ak)|^2. \tag{4.15}$$

If g and h are an analysis reconstruction pair, then the following formula holds

$$\int_{\mathbb{H}^n} \frac{dbda}{a} \overline{\mathcal{W}_g s(b,a)} \mathcal{W}_h r(b,a) = c_{g,h} \int_{\mathbb{R}^n} dx\, \overline{s}(x)\, r(x). \tag{4.16}$$

In particular if g is strictly admissible, then we have conservation of energy ($c_g = c_{g,g}$)

$$\int_{\mathbb{H}^n} \frac{dbda}{a} \left|\mathcal{W}_g s(b,a)\right|^2 = c_g \int_{\mathbb{R}^n} dx\, \left|s(x)\right|^2. \tag{4.17}$$

Let P and r be functions over the half-space $\mathbb{H}^n$. We define a (non-commutative) convolution for functions over the half-space via

$$(P * r)(b,a) = \int_{\mathbb{H}^n} \frac{db' da'}{a'} \frac{1}{a'^n} P\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) r(b', a'). \tag{4.18}$$

Formally we can write for the combination $\mathcal{W}_g \mathcal{M}_h$

$$(\mathcal{W}_g \mathcal{M}_h s)(b,a) = P_{g,h} * s(b,a), \quad P_{g,h} = \mathcal{W}_g h. \tag{4.19}$$

In the case where g, and h are an analysis reconstruction pair with $c_{g,h} = 1$ we clearly have $(\mathcal{W}_g \mathcal{M}_h)^2 = \mathcal{W}_g \mathcal{M}_h$ and hence

$$P_{g,h} * P_{g,h} = P_{g,h}. \tag{4.20}$$

The mapping $\Pi_{g,h} : r \mapsto P_{g,h} * r$ is a projector into the range of the wavelet transform $\mathcal{W}_g$

$$P_{g,h} * \mathcal{W}_g = \mathcal{W}_g. \tag{4.21}$$

Because of this property the function $P_{g,h}$ is called the a reproducing kernel. In case that g is strictly admissible with $c_{g,g} = 1$, we have that convolution with $P_{g,g}$ is an orthogonal projector onto the range of $\mathcal{W}_g$.

The relation between $\mathcal{W}_g$ and $\mathcal{W}_h$ is given by the following. Consider a function s over $\mathbb{R}^n$. Suppose that g is admissible. It thus has a reconstruction wavelet r. An explicit formula has been given in 4.15. Then, since $\mathcal{W}_h = (\mathcal{W}_h \mathcal{M}_r)\mathcal{W}_g$, the wavelet transform of s with respect to g and the one with respect to h are related via the so called cross kernel equation.

$$\mathcal{W}_h s = P_{g\to h} * \mathcal{W}_g s, \quad P_{g\to h} = \mathcal{W}_h r. \tag{4.22}$$

5 Some function spaces.

All formulas that we have given so far have a well defined meaning if the wavelets are taken in some subset of the class of Schwartz as we will recall now.

5.a The analysis of $S_0(\mathbb{R}^n)$

Let $S(\mathbb{R}^n)$ denote the class of Schwartz consisting of those functions that together with their derivatives decay faster than any polynomial such that the following norms are all finite for all multi-indices α and β

$$\|s\|_{\alpha,\beta} = \sup_{t\in\mathbb{R}^n} \left| t^\alpha \partial^\beta s(t) \right| < \infty. \tag{5.1}$$

They generate a locally convex topology which makes $S(\mathbb{R}^n)$ a Fréchet space. We denote by $S_0(\mathbb{R}^n)$ the closed set of functions in $S(\mathbb{R}^n)$ for which all moments vanish

$$\forall \alpha \in \mathbb{N}^n, \int dx\, s(x)\, x^\alpha = 0 \iff \forall m > 0,\ \widehat{s}(k) = o(k^m)\ (|k| \to 0). \tag{5.2}$$

The Schwartz space of functions over the half-plane $\mathbb{H}^n$ shall be denoted by $S(\mathbb{H})$. It consists of those functions r for which the following norms are all finite

$$\|r\|_{k,l} = \sup_{(b,a)\in\mathbb{H}^n} \left|(a+1/a)^k (1+|b|)^l\, r(b,a)\right| < \infty. \tag{5.3}$$

Note that this means that r decays rapidly for large b and for large or small a. It can be shown that

$$\mathcal{W} : S_0(\mathbb{R}^n) \times S_0(\mathbb{R}^n) \to S(\mathbb{H}^n), \quad (g,s) \mapsto \mathcal{W}_g s \tag{5.4}$$

is continuous. The localization in b follows easily from the definition of the wavelet transform in terms of a family of convolutions, whereas the localization in a follows from the formula (4.7) in Fourier space and the fact that for functions in $S_0(\mathbb{R}^n)$ the Fourier transform is "flat" around 0 and ∞.

The analogue holds for the wavelet synthesis defined through

$$\mathcal{M}_h r(x) = \int_{\mathbb{H}^n} \frac{dbda}{a}\, r(b,a)\, \frac{1}{a^n} h\left(\frac{x-b}{a}\right), \tag{5.5}$$

and we have that

$$\mathcal{M} : S_0(\mathbb{R}^n) \times S(\mathbb{H}^n) \to S_0(\mathbb{R}^n), \quad (h,r) \mapsto \mathcal{M}_h r \tag{5.6}$$

is continuous too.

This shows that $S_0(\mathbb{R}^n)$ can be characterized through the localization of its wavelet coefficients. More precisely, formula (5.4) and (5.6) show that for g admissible we have $s \in S_0(\mathbb{R}^n)$ if and only if $\mathcal{W}_g s \in S(\mathbb{H}^n)$. In particular we have the following lemma that we state here for later reference.

Lemma 5.1 *Let $\Omega \subset S_0(\mathbb{R}^n)$ and $\Gamma \subset S_0(\mathbb{R}^n)$ both be bounded sets. Then there is $F \in S_0(\mathbb{H}^n)$ such that*

$$(g,s) \in \Gamma \times \Omega \Rightarrow |\mathcal{W}_g s(b,a)| \le F(b,a). \tag{5.7}$$

Remark. It can be shown that the image of the wavelet transform consists actually of smooth function. More precisely we have that for $r = \mathcal{W}_g s$

$$\|r\|_{k,l,m,n} = \sup_{(b,a)\in\mathbb{H}^n} \left|(a+1/a)^k (1+|b|)^l\, \partial_b^m \partial_a^n r(b,a)\right| < \infty. \tag{5.8}$$

Moreover when restricted to the image of the wavelet transform these norms are equivalent with the previous ones. Note however that throughout this paper $S(\mathbb{H}^n)$ is understood with its topology generated through (5.3) only. Thus in particular multiplication with bounded functions is a continuous operator.

5.b Wavelet analysis of $S(\mathbb{R}^n)$.

It is not possible to characterize in an easy way $S(\mathbb{R}^n)$ in wavelet space. This is due to the fact that functions, that are not of 0-mean can only poorly be approximated by functions that have all moments vanishing. However we still have the following result. Consider the set of functions over $\mathbb{H}^n$ for which the following semi-norms are finite

$$\|r\|_{k,l} = \sup_{(b,a)\in\mathbb{H}^n} \left|(1+1/a)^k(1+a)^n(1+|b|/[1+a])^l\, r(b,a)\right| < \infty. \tag{5.9}$$

This means that for some function $f \in S(\mathbb{R}^n)$ we have

$$|r(b,a)| \leq \rho(a)\frac{1}{(1+a)^n} f\left(\frac{b}{1+a}\right) \tag{5.10}$$

where $\rho(a) \geq 0$ satisfies at

$$\forall K > 0\ \rho(a) \leq O(a^K),\ (a \to 0), \quad \rho(a) \leq O(1), (a \to \infty). \tag{5.11}$$

Denote this space of functions by $\Sigma(\mathbb{H}^n)$ and equip it with the topology generated by the above semi-norms. It can be shown, that the mapping

$$\mathcal{W} : S_0(\mathbb{R}^n) \times S(\mathbb{R}^n) \to \Sigma(\mathbb{H}^n), \quad (g,s) \mapsto \mathcal{W}_g s \tag{5.12}$$

is continuous. In particular we have the following result that we state as a lemma for reference

Lemma 5.2 *Let $\Omega \subset S(\mathbb{R}^n)$ and $\Gamma \subset S_0(\mathbb{R}^n)$ both be bounded sets. Then there is $f \in S(\mathbb{R}^n)$ and a function $\rho(a)$ satisfying at (5.11) such that*

$$(g,s) \in \Gamma \times \Omega \Rightarrow |\mathcal{W}_g s(b,a)| \leq \rho(a)\frac{1}{(1+a)^n} f\left(\frac{b}{1+a}\right) \tag{5.13}$$

Note that there is no analogue for the wavelet synthesis. It is not true in general that a localization of the above type for r implies that $\mathcal{M}_h r$ is localized.

5.c Wavelet analysis of $S_0'(\mathbb{R}^n)$.

We denote the space of linear continuous functionals $\eta : S_0(\mathbb{R}^n) \to \mathbb{C}$ by $S_0'(\mathbb{R}^n)$. We consider it together with its natural weak-$*$ topology. The space $S_0'(\mathbb{R}^n)$ can canonically be identified with $S'(\mathbb{R})/P(\mathbb{R}^n)$, where $P(\mathbb{R}^n)$ is the space of polynomials in n variables. The wavelet transform of $\eta \in S_0'(\mathbb{R}^n)$ can now be defined pointwise as

$$\mathcal{W}_g\eta(b,a) = \eta(\overline{g}_{b,a}). \tag{5.14}$$

This is a smooth function (e.g. [Hol95]) that satisfies at

$$|\mathcal{W}_g\eta(b,a)| \leq c(1+|b|)^m(a+a^{-1})^m \tag{5.15}$$

for some $m > 0$. By duality we have that the mapping (this time for fixed wavelet $g \in S_0(\mathbb{R}^n)$)

$$\mathcal{W}_g : S_0'(\mathbb{R}^n) \to S'(\mathbb{H}^n), \quad \eta \mapsto \mathcal{W}_g\eta \tag{5.16}$$

is continuous. Here $S'(\mathbb{H}^n)$ is the dual of $S(\mathbb{H}^n)$ together with the weak-$*$ topology. Vice versa let $r \in S'(\mathbb{H}^n)$. Then we may set for $s \in S_0(\mathbb{R}^n)$

$$(\mathcal{M}_h r)(s) = r(\mathcal{W}_{\overline{h}} s) \tag{5.17}$$

Clearly again ($h \in S_0(\mathbb{R}^n)$)

$$\mathcal{M}_h : S'(\mathbb{H}^n) \to S_0'(\mathbb{R}^n), \quad r \mapsto \mathcal{M}_h r \tag{5.18}$$

is continuous. Thus as before in the case of $S_0(\mathbb{R}^n)$, we may characterize $S_0'(\mathbb{R}^n)$ through the localization of its wavelet coefficients (5.15).

5.d Wavelet analysis over $\mathbb{T}^n$

All results we have shown so far carry over in a natural way if we consider spaces of periodic functions. More precisely consider the n-dimensional torus $\mathbb{T}^n$ that we identify with $\mathbb{R}^n/\mathbb{Z}^n$. Accordingly we identify functions over $\mathbb{T}^n$ with periodic functions. From the translation covariance of the wavelet transform it follows that the wavelet transform of a periodic function is a function that is periodic in the first variable

$$\mathcal{W}_g s(b+k,a) = \mathcal{W}_g s(b,a), \quad k \in \mathbb{Z}^n. \tag{5.19}$$

Therefore, if we introduce the notation $\mathbb{M}^n = \mathbb{T}^n \times \mathbb{R}_+$, then we may identify wavelet transforms of functions over $\mathbb{T}^n$ with functions over the n-dimensional half-cylinder $\mathbb{M}^n$.

Accordingly we write $S_0(\mathbb{T}^n)$ for the space of smooth functions together with the norms ($m \in \mathbb{N}^n$)

$$\|s\|_m = \|\partial^m s\|_{L^\infty(\mathbb{T}^n)}, \tag{5.20}$$

and for which we have

$$\int_{\mathbb{T}^n} s = 0. \tag{5.21}$$

Its dual, the space of periodic distributions shall be denoted by $S_0'(\mathbb{T}^n)$. On $\mathbb{M}^n$ let $S(\mathbb{M}^n)$ denote the space of functions for which

$$\|s\|_k = \sup_{(b,a)\in\mathbb{M}^n} \left|(a+1/a)^k s(b,a)\right| < \infty. \tag{5.22}$$

Then it can be shown that as before

$$\mathcal{W} : S_0(\mathbb{R}^n) \times S_0(\mathbb{T}^n) \to S(\mathbb{M}^n), \quad (g,s) \mapsto \mathcal{W}_g s \tag{5.23}$$

is continuous.

The analogue holds for the wavelet synthesis defined as before upon identifying functions over $\mathbb{M}^n$ with periodic functions over $\mathbb{H}^n$. Again we have that

$$\mathcal{M} : S_0(\mathbb{R}^n) \times S(\mathbb{M}^n) \to S_0(\mathbb{T}^n), \quad (h,r) \mapsto \mathcal{M}_h r \tag{5.24}$$

is continuous.

The analogue statements hold for $S_0'(\mathbb{T}^n)$ and $S'(\mathbb{M}^n)$.

Note that the wavelet analysis and synthesis can also be formulated with the help of periodized wavelets. For this let us for $g \in S(\mathbb{R}^n)$ denote by g° its periodization

$$g^\circ(x) = \sum_{k\in\mathbb{Z}^n} g(x+k). \tag{5.25}$$

Thus in particular $g^\circ \in S_0(\mathbb{T}^n)$ for $g \in S_0(\mathbb{R}^n)$. Then the periodized wavelet $g^\circ_{b,a} = (g_{b,a})^\circ$ may be used to write ($s \in S_0(\mathbb{T}^n)$, $r \in S(\mathbb{M}^n)$)

$$\mathcal{W}_g s(b,a) = \int_{\mathbb{T}^n} dx\, \overline{g^\circ_{b,a}}(x)\, s(x), \quad \mathcal{M}_h r(x) = \int_{\mathbb{M}^n} \frac{db da}{a}\, r(b,a)\, h^\circ_{b,a}(x). \tag{5.26}$$

Since there is a natural maximal length scale (=the period) it is no surprise that the wavelet coefficients decay rapidly at large scale regardless of the distribution we analyzed

$$|\mathcal{W}_g s(b,a)| \le O(a^{-K}), \quad (a \to \infty), \forall K > 0. \tag{5.27}$$

To see this note that

$$\mathcal{W}_g s(b,a) = \sum_{k\in\mathbb{Z}^n} \overline{\widehat{g}}(2\pi ak)\, e^{2i\pi kb}\, \widehat{s}_k, \tag{5.28}$$

where the Fourier coefficients of s read

$$\widehat{s}_k = \int_{\mathbb{T}^n} s(x) e^{-2i\pi kx}. \tag{5.29}$$

Since $\widehat{g}(0)=0$, it follows that the sum is rapidly decaying as $a\to\infty$ thanks to the rapid decay of $\widehat{g}$. In particular we may choose g in such a way that for all $s\in S'(\mathbb{T}^n)$ we have

$$a\geq 1 \Rightarrow \mathcal{W}_g s(b,a) = 0. \tag{5.30}$$

Indeed take $\widehat{g}$ compactly supported with support contained in $[-2\pi,2\pi]^n$. This remark will play a role in some demonstrations.

5.e More general spaces

Many other function spaces can be characterized in terms of wavelet coefficients. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular. We have already encountered this, since we have seen that

$$S_0(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset S_0'(\mathbb{R}^n) \tag{5.31}$$

are characterized by the fact that the wavelet coefficients are in the following scale of spaces

$$S_0(\mathbb{H}^n) \subset L^2(\mathbb{H}^n) \subset S_0'(\mathbb{H}^n) \tag{5.32}$$

We generalize this situation and we recall the general principals of characterization of functions spaces through wavelet coefficients (see e.g. [FG89a],[FG89b] for a more abstract setting) For this consider a Banach space $B(\mathbb{H}^n)$ locally integrable functions. We always require that it satisfies the following properties.

(A1) We have

$$S(\mathbb{H}^n) \subset B(\mathbb{H}^n) \subset S'(\mathbb{H}^n). \tag{5.33}$$

and the inclusions hold in the topological sense.

(A2) We have a Banach lattice

$$\|\,|s|\,\|_{B(\mathbb{H}^n)} = \|s\|_{B(\mathbb{H}^n)}\,. \tag{5.34}$$

(A3) It is tempered in the following sense. We suppose that there is $K>0$ and $c>0$ such that for all $s\in B(\mathbb{H}^n)$ we can estimate† for $s_{\beta,\alpha}(b,a) = s(\beta+b,\alpha a)$

$$\|s_{\beta,\alpha}\|_{B(\mathbb{H}^n)} \leq c\,\|s\|_{B(\mathbb{H}^n)}\,(\alpha+1/\alpha)^K. \tag{5.35}$$

†More generally we would allow estimates of the form $\leq c\,\|s\|_{B(\mathbb{H}^n)}\,(\alpha+1/\alpha)^K(1+|\beta|)^K$, but since we are essentially interested in function spaces of periodic functions, we suppose translation independence of the norm.

(A4) Its multiplier algebra contains the bounded functions

$$\mu \in L^\infty(\mathbb{H}^n), s \in B(\mathbb{H}^n) \Rightarrow \mu \cdot s \in B(\mathbb{H}^n), \tag{5.36}$$

and we have, if we denote by $M[\mu]$ the operator $s \mapsto \mu \cdot s$

$$\|M[\mu]\|_{B(\mathbb{H}^n)\to B(\mathbb{H}^n)} = \|\mu\|_{L^\infty(\mathbb{H}^n)} \tag{5.37}$$

These are minimal requirements and we say that $B(\mathbb{H}^n)$ is an *admissible* Banach space.

¿From the first two conditions it follows easily that $B(\mathbb{H}^n)$ is invariant under convolutions with highly localized kernels.

$$r \in S(\mathbb{H}^n),\ s \in B(\mathbb{H}^n) \quad \Rightarrow \quad r * s,\ s * r \in B(\mathbb{H}^n). \tag{5.38}$$

The convolution of two functions over $\mathbb{H}^n$ is defined through 4.18. More than that, we have the following lemma.

Lemma 5.3 *For fixed $P \in S(\mathbb{H}^n)$ we have*

$$A[P] : B(\mathbb{H}^n) \to B(\mathbb{H}^n), \quad r \mapsto P * r \tag{5.39}$$

is continuous.

Proof. By a simple change of variables, we may write

$$P * s = \int_{\mathbb{H}^n} \frac{d\beta d\alpha}{\alpha} P(-\beta/\alpha, 1/\alpha) s(\cdot + \beta, \cdot \alpha) \tag{5.40}$$

Upon passing to the absolute value and then to the norm on both sides we have using hypothesis (A3)

$$\|P * s\|_{B(\mathbb{H}^n)} \leq \|s\|_{B(\mathbb{H}^n)}\ c \int_{\mathbb{H}^n} \frac{d\beta d\alpha}{\alpha}\ |P(-\beta/\alpha, 1/\alpha)|\ (\alpha + 1/\alpha)^K. \tag{5.41}$$

The last integral is a finite constant. □

It therefore then makes sense to pull back $B(\mathbb{H}^n)$ to a vector space of distributions over $\mathbb{R}^n$. We shall denote this space of distributions by $B(\mathbb{R}^n)$. More precisely we define for g admissible

$$B(\mathbb{R}^n) = \{\eta \in S_0'(\mathbb{R}^n) : \mathcal{W}_g\eta \in B(\mathbb{H}^n)\}. \tag{5.42}$$

It is well defined defined because the passage from $\mathcal{W}_g\eta$ to $\mathcal{W}_h\eta$ is given by the highly localized cross kernel (4.22). And as we have seen in lemma (5.3), this operation leaves invariant $B(\mathbb{H}^n)$. We can make $B(\mathbb{R}^n)$ a Banach space space by setting

$$\|s\|_{B(\mathbb{R}^n)} = \|\mathcal{W}_g s\|_{B(\mathbb{H}^n)}. \tag{5.43}$$

This is well defined, since for different admissible wavelets we obtain equivalent norms since convolutions with highly localized kernels is continuous.

We have the following easy convergence result that we state it here for later reference.

Lemma 5.4 *For $B(\mathbb{H}^n)$ satisfying (A1) the following holds. Suppose that the family $s_m \in B(\mathbb{H}^n)$ is uniformly continuous compact sets. Suppose further that $s_m(b,a) \to s(b,a)$ pointwise. Suppose further that $|s_m(b,a)| \leq r(b,a)$ for some $r \in S(\mathbb{H}^n)$. Then*

$$\lim_{m\to\infty} \|s_m - s\|_{B(\mathbb{H}^n)} = 0. \tag{5.44}$$

Proof. Consider the compact sets $P_w = \{(a+1/a) + |b| \leq w\}$ and denote the multiplier with its characteristic function with the same symbol. We have uniformly in m, that $(\mathbb{1} - P_w)(s_m - s) \to 0$ in the topology of $S(\mathbb{H}^n)$ by localization. Therefore a forteriori by (A1)

$$\lim_{w\to\infty} \|P_w s_m\|_{B(\mathbb{H}^n)} = 0 \quad \text{uniformly in } m. \tag{5.45}$$

On the other hand, for fixed w we have $P_w s_m(b,a) \to P_w s(b,a)$ uniformly on the compact set P_w beause the s_m are uniformly continuous. Therefore, for fixed w we have $P_w s_m \to P_w s$ in $S(\mathbb{H}^n)$ and a forteriori in $B(\mathbb{H}^n)$. □

We now list some additional conditions, that we shall need in order to prove the subsequent theorems. In definition 5.7 below we shall summarize the different concepts we need.

We sometimes have to replace the third condition by the slightly stronger condition.

(ST) It is *strongly tempered* in the following sense. We suppose that there is ϵ, K and $c > 0$ such that for all $s \in B(\mathbb{H}^n)$ we can estimate with $s_{\beta,\alpha} = s(\cdot + \beta, \alpha\cdot)$

$$\|s_{\beta,\alpha}\|_{B(\mathbb{H}^n)} \leq c\,\|s\|_{B(\mathbb{H}^n)}\, \alpha^{\epsilon}(1+\alpha)^K. \tag{5.46}$$

A homogeneity property is also needed.

(H) We say $B(\mathbb{H}^n)$ is homogeneous (of degree $\gamma \in \mathbb{R}$) if

$$\|s(\alpha\cdot+\beta,\cdot\alpha)\|_{B(\mathbb{H}^n)} = \alpha^{\gamma}\;\|s\|_{B(\mathbb{H}^n)} \iff \|T_\beta D_\alpha r\|_{B(\mathbb{R}^n)} = \alpha^{-n-\gamma}\,\|r\|_{B(\mathbb{R}^n)} \tag{5.47}$$

for some $\gamma \in \mathbb{R}$

We sometimes need the following density requirement.

(D) The embedding

$$i : S(\mathbb{H}^n) \to B(\mathbb{H}^n) \tag{5.48}$$

has dense range.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that all its derivatives are bounded functions. We then may require

(R) The space is *deformable* in the following sense. The mapping

$$\Psi[\phi] : s(b,a) \mapsto s(\phi^{-1}(b),a)/J(\phi^{-1}(b)) = (\Phi[\phi]s(\cdot,a))(b). \tag{5.49}$$

defines a continuous operator acting in $B(\mathbb{H}^n)$ for any such diffeomorphism ϕ.

We need a local approximation property for this operator. For this we consider the space of diffeomorphisms of $\mathbb{R}^n$ equipped with the topology of uniform convergence $\phi_m \to \phi$ and $\partial\phi_m \to \partial\phi$ on compact sets. For $\epsilon > 0$ let $\Pi_{x,\epsilon}$ be the multiplication operator with the characteristic function of $\{|b-x| \leq \epsilon, 0 < a \leq \epsilon\}$. We then may require.

(CR1) The *local trivialization* property holds if

$$\phi \mapsto \|\Psi[\phi]\Pi_{0,1}\|_{B(\mathbb{H}^n)\to B(\mathbb{H}^n)} \tag{5.50}$$

is continuous.

In particular this implies the following. Suppose $\phi(x) = y$. Now let $\delta\phi_x$ denote the linearized map at x, $\delta\phi_x : z \mapsto y + \partial\phi_x(z - x)$. We may leave the proof of the next lemma to the the reader.

Lemma 5.5 *Let $B(\mathbb{H}^n)$ satisfy (A4), be homogeneous (H) and satisfy (CR1). Let ϕ be a fixed, uniformly smooth diffeomorphism. Then for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $\rho \leq \delta$ and all x we have*

$$\left| \left\| \Psi[(\delta\phi_x)^{-1}\phi]\Pi_{x,\rho} \right\|_{B(\mathbb{H}^n)\to B(\mathbb{H}^n)} - 1 \right| \leq \epsilon. \tag{5.51}$$

We also need an estimation of the norm of projectors onto disjoint regions. More precisely, for $x \in \mathbb{R}^n$ we call *influence cone of x* the set of points in $\mathbb{H}^n$ defined through $|b - x| \leq a$. Note that the wavelet transform of δ_x with respect to a wavelet supported by the unit ball around 0, is supported by the influence cone of x. Now, for any set $\Gamma \subset \mathbb{H}^n$ we call its *base* the set of $x \in \mathbb{R}^n$ such that the influence cone of x does not meet Γ.

(CR2) The norm is *asymptotically free* in the following sense. Suppose that there are functions $\gamma_K : \mathbb{R}_+^K \to \mathbb{R}_+$ such that for any family of functions $s_k \in B(\mathbb{H}^n)$, $k = 1, 2, \ldots K$ with supports whose bases intersect only on a set of measure zero we have

$$\left\| \sum s_k \right\|_{B(\mathbb{H}^n)} = \gamma_K((\|s_k\|)) \tag{5.52}$$

The functions γ_K define a norm on $\mathbb{C}^K$ via $\|x\| = \gamma_K((|x_k|))$. From property (A4) it follows that

$$\gamma_K((\lambda_k x_k)) \leq \max_k |\lambda_k| \; \gamma_K((x_k)) \tag{5.53}$$

Let $s_k \in S(\mathbb{H}^n)$, $k = 1, 2, \ldots K$. Then by a simple approximation argument we have that the following limit

$$F((s_k)) = \lim_{|\beta_k - \beta_{k'}|\to\infty} \left\| \sum s_k(\cdot - \beta_k, \cdot) \right\|_{B(\mathbb{R}^n)} \tag{5.54}$$

exists and we have

$$F((s_k)) = \gamma_K((\|s_k\|)) \tag{5.55}$$

In case that (D) holds, this equation holds true for all functions $s_k \in B(\mathbb{H}^n)$.

We will need the following property of theses spaces. For any set Γ we denote by the same symbol the multiplier with its characteristic functions. Thus $s \mapsto \Gamma s$ is a projector onto the space of functions with support in the set Γ.

Lemma 5.6 *Let $B(\mathbb{H}^n)$ satisfy property (A4) and (CR2). Let $\Gamma_k \subset \mathbb{H}^n$ be a finite family of sets whose bases intersect only on sets of measure zero. Suppose Ω_k is a second such family of equal cardinality. Let A be a bounded, linear operator acting in $B(\mathbb{H}^n)$. Then we have*

$$\left\| \sum \Omega_k A \Gamma_k \right\| \leq \max_k \|\Omega_k A \Gamma_k\| \tag{5.56}$$

Proof. We have

$$\left\| \sum \Omega_k A \Gamma_k \right\| = \sup_{\|s\|=1} \left\| \sum \Omega_k A \Gamma_k s \right\| \leq \sup_{\gamma_K((\|s_k\|))=1} \left\| \sum \Omega_k A \Gamma_k s_k \right\| \tag{5.57}$$

where the s_k satisfy $\Gamma_k s_k = s_k$. The last inequality follows from the fact that $\|\sum_k \Gamma_k s\| \leq \|s\|$ because of (A4). Now

$$\sup_{\gamma_K(\|s_k\|)=1} \left\| \sum \Omega_k A \Gamma_k s_k \right\| = \sup_{\gamma_K((\|s_k\|))=1} \gamma_K((\|\Omega_k A \Gamma_k s_k\|)) \tag{5.58}$$

We may conclude by noting that

$$\gamma_K((\|\Omega_k A \Gamma_k s_k\|)) \leq \gamma_K((\|\Omega_k A \Gamma_k\| \, \|s_k\|)) \leq (\max_k \|\Omega_k A \Gamma_k\|) \gamma_K((\|s_k\|)), \tag{5.59}$$

thanks to (5.53). □

To summarize we have the following definition for the different types of spaces we consider.

Definition 5.7 *We say a space $B(\mathbb{H}^n)$ is admissible if it satisfies conditions (A1) - (A4). It is strongly tempered if it satisfies (ST). It is deformable if it satisfies (D). An admissible, deformable space for which (R) holds true is called* regular. *A regular space which is homogeneous (H) and for which in addition conditions (CR1) and (CR2) hold is called* completely regular.

5.f Spaces of periodic functions.

For any admissible Banach space $B(\mathbb{H}^n)$ we can associate a second admissible Banach space $B_{\rm loc}(\mathbb{H}^n)$ in the following way. Let χ denote the characteristic function of $\mathbb{M}^n = [0,1)^n \times \mathbb{R}_+$. Then we may define a norm via

$$\sup_k \|\chi \cdot s(\cdot - k, \cdot)\|_{B(\mathbb{H}^n)} \,. \tag{5.60}$$

It is plain to see that this new Banach space is again admissible. Consider the subspace, denoted by $B(\mathbb{M}^n) \subset B_{\rm loc}(\mathbb{H}^n)$ of periodic functions. On $B(\mathbb{M}^n)$, we have the following norm for $s \in B(\mathbb{M}^n) \subset B_{\rm loc}(\mathbb{H}^n)$ which is equivalent to the induced norm

$$\|s\|_{B(\mathbb{M}^n)} =: \|s\|_{B_{\rm loc}(\mathbb{H}^n)} = \|\chi \cdot s\|_{B(\mathbb{H}^n)} \,. \tag{5.61}$$

This shows that $B(\mathbb{M}^n)$ is closed. Indeed, if $s_m \in B(\mathbb{M}^n)$ and $s_m \to s$ in $B_{\rm loc}(\mathbb{H}^n)$ then $\chi \cdot s_n \to \chi s$ in $B(\mathbb{H}^n)$. In addition $s(b,a)$ is periodic in b since the lattice translation operator $s(\cdot,\cdot) \to s(\cdot - k, \cdot)$, $k \in \mathbb{Z}^n$ is continuous in $B_{\rm loc}(\mathbb{H}^n)$. This space may be pulled back to a space of periodic functions over $\mathbb{T}^n$ via a wavelet synthesis. We denote this space by $B(\mathbb{T}^n)$.

For later reference we state the following density lemma.

Lemma 5.8 *If $S(\mathbb{H}^n)$ is dense in $B(\mathbb{H}^n)$, then $S(\mathbb{M}^n)$ is dense in $B(\mathbb{M}^n)$.*

Proof. Let $s \in B(\mathbb{M}^n)$. Then $\chi \cdot s \in B(\mathbb{H}^n)$. We therefore can find an approximating sequence $r_m \in S(\mathbb{H}^n)$ that converges towards $\chi \cdot s$. But then $\chi \cdot r_m \to \chi \cdot s$ too since $M[\chi]$ is a continuous projector. If we set s_m the unique periodic function that satisfies at $\chi s_m = \chi r_m$ we have found an approximating sequence. □

5.g The scale $F^{p,q}_\gamma(\mathbb{T}^n)$.

Many functions spaces of day to day functional analysis can be characterized with the easy concept we have presented in section 5.f. Most of them are contained in the following two scales of spaces. For $\gamma \in \mathbb{R}$ the following expressions define norms for functions over the half-space $1 \leq p,q \leq \infty$

$$\|r\|_{B^{p,q}_\gamma(\mathbb{H}^n)} = \left\{ \int_0^\infty \frac{da}{a} a^{-\gamma p} \left(\int_{-\infty}^{+\infty} db \;\, |r(b,a)|^q \right)^{p/q} \right\}^{1/p} \tag{5.62}$$

or for $1 \leq p \leq \infty$, $1 \leq q < \infty$ or $p = q = \infty$

$$\|r\|_{F^{p,q}_\gamma(\mathbb{H}^n)} = \left\{ \int_{-\infty}^{+\infty} db \left(\int_0^\infty \frac{da}{a} a^{-\gamma q} \, |r(b,a)|^q \right)^{p/q} \right\}^{1/p}. \tag{5.63}$$

The associated Banach spaces are all admissible and thus they may be pulled back to $\mathbb{R}^n$ giving rise to two scales of spaces. The first scale of spaces contains the Besov spaces, whereas the second scale are the Triebel Lizorkin spaces, which contains in e.g. the L^p-potential spaces and Sobolev spaces (see e.g. [Mey90], [CM90], [Hol95], [Tri84] for more details).

For $1 \leq p, q < \infty$, both scales of spaces are regular (definition 5.7). The Triebel scale $F^{p,q}_\gamma(\mathbb{H}^n)$ is even completely regular but the Besov scale $B^{p,q}_\gamma(\mathbb{H}^n)$ is not since the norm is not asymptotically free. For either p or $q = \infty$, $S(\mathbb{H}^n)$ is not dense any more and the spaces are not regular any more. However, the closures $f^{p,q}_\gamma(\mathbb{H}^n)$ respectively $b^{p,q}_\gamma(\mathbb{H}^n)$ of $S(\mathbb{H}^n)$ in $F^{p,q}_\gamma(\mathbb{H}^n)$ respectively $B^{p,q}_\gamma(\mathbb{H}^n)$ are regular, and in the case of $f^{p,q}_\gamma(\mathbb{H}^n)$ completely regular.

We now shall focus on the periodic case and we shall consider the spaces which are of Triebel Lizorkin type. We consider two special cases for illustration. $\Lambda_\gamma(\mathbb{H}^n)$ of functions over $\mathbb{H}^n$ for following norms are finite

$$\|r\|_\gamma = \operatorname{ess\,sup}\, a^{-\gamma} \, |r(b,a)| < \infty. \tag{5.64}$$

Clearly $\Lambda_\gamma(\mathbb{H}^n) = F^{\infty,\infty}_\gamma(\mathbb{H}^n)$. These spaces may be interpreted in terms of local regularity as we recall now. For proofs see e.g. [Jaf89] and [HT91]. If $\gamma \in (0,1)$ then $\|s\|_\gamma$ is equivalent to the homogeneous Hölder (semi)-norm

$$\sup \frac{|s(x) - s(y)|}{|x-y|^\gamma}. \tag{5.65}$$

Actually this is a norm if we quotient by the constant functions for which the above expression is 0. If $\gamma = 1$ then the above norm is equivalent to the Zygmund (semi) norm

$$\sup \frac{|s(x+y) + s(x-y) - 2s(x)|}{|y|}. \tag{5.66}$$

This is actually a norm if restrict ourselves to continuous functions and if in addition we quotient by the linear functions. Note however that $\Lambda_1(\mathbb{T}^n)$ is always defined modulo the constant functions since they are the only periodic polynomials.

The other values of $\gamma > 0$ and $\alpha \leq 0$ are obtained upon passing to derivatives or primitives thanks to the following formula

$$\mathcal{W}_g \sqrt{-\Delta} s = a^{-1} \mathcal{W}_{\sqrt{-\Delta} g} s. \tag{5.67}$$

Note that if g is admissible, then the same is true for $\sqrt{-\Delta} g$. Therefore $s \in \Lambda_\gamma(\mathbb{R}^n)$ iff $(-\Delta)^{\alpha/2} s \in \Lambda_{\gamma-\alpha}(\mathbb{R}^n)$.

This scale of spaces does not satisfy the density property (5.48), that is the smooth functions are not dense in $\Lambda_\gamma(\mathbb{T}^n)$. The closure of $S(\mathbb{H}^n)$ in $\Lambda_\gamma(\mathbb{H}^n)$ will be denoted by $\lambda_\gamma(\mathbb{H}^n)$. We then obtain by the general construction spaces $\lambda_\gamma(\mathbb{T}^n)$. The functions in this space satisfy instead the stronger regularity $(0 < \gamma < 1)$.

$$\lim_{\epsilon \to 0} \sup_{|x-y| \leq \epsilon} \frac{|s(x) - s(y)|}{|x-y|^\gamma} = 0. \tag{5.68}$$

A second well known scale of spaces of Triebel Lizorkin type are the Sobolev spaces with weight a^γ. Here we set

$$\|r\|_{H_\gamma(\mathbb{H}^n)} = \sqrt{\int_{\mathbb{H}^n} \frac{dbda}{a} \, |a^{-\gamma} r(b,a)|^2} \tag{5.69}$$

This is an admissible Banach space and we denote it by $H_\gamma(\mathbb{H}^n)$. Clearly we have $H_\gamma(\mathbb{H}^n) = F^{2,2}_\gamma(\mathbb{H}^n)$. It can easily be shown that the norm in $H_\gamma(\mathbb{T}^n)$ is equivalent to the following (semi-norm)

$$\|r\|'_{H_\gamma(\mathbb{T}^n)} = \sqrt{\sum_{n\neq 0} |n^\gamma \widehat{r}_n|^2} \tag{5.70}$$

where $\widehat{r}_n$ are the Fourier coefficients of r. For $\gamma < 0$ it is a norm and for $\gamma \geq 0$ is a norm modulo the constant functions.

6 Operators in wavelet space.

6.a Formal properties.

For any function $E : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$ we define the following operator (formally!), denoted by $Op[E]$ and acting on functions over $\mathbb{H}^n$

$$Op[E] : r(b,a) \mapsto \int_{\mathbb{H}^n} \frac{db'da'}{a'} E(b,a;b',a') \, r(b,a) \tag{6.1}$$

On the other hand for any operator $\mathcal{E}$ acting on functions over $\mathbb{R}^n$ we associate the following function called its symbol with respect to g and h.

$$\sigma_{g,h}[\mathcal{E}](b,a;b',a') = \langle g_{b,a} \mid \mathcal{E} h_{b',a'} \rangle = \mathcal{W}_g \mathcal{E} h_{b',a'}(b,a). \tag{6.2}$$

with $g_{b,a} = a^{-n} g([x-b]/a)$. We sometimes write simply $\sigma[\mathcal{E}]$ if no confusion is possible. For a fixed, analysis reconstruction pair g, h it is clear that $\mathcal{E}$ is determined by its matrix elements E. Indeed, if we introduce the following notation

$$\mathcal{E}^\sharp = Op[\sigma[\mathcal{E}]], \tag{6.3}$$

we then have

$$\mathcal{E} = \mathcal{M}_h \mathcal{E}^\sharp \mathcal{W}_g. \tag{6.4}$$

Less obviously we also have

$$\mathcal{W}_g \mathcal{E} = \mathcal{E}^\sharp \mathcal{W}_g \tag{6.5}$$

But this holds true, since for fixed (b',a'), $\sigma[\mathcal{E}](\cdot,\cdot;b',a')$ is in the image of $\mathcal{W}_g$. Therefore we have $\mathcal{W}_g \mathcal{M}_h \sigma[\mathcal{E}](\cdot,\cdot;b',a') = \sigma[\mathcal{E}](\cdot,\cdot;b',a')$, and thus the second equation follows from the first.

The following equations are then easily verified

$$(\mathcal{E}\mathcal{F})^\sharp = \mathcal{E}^\sharp \mathcal{F}^\sharp. \tag{6.6}$$

We also introduce the notation $\mathcal{F}^\flat$ for the operator acting on functions on $\mathbb{R}^n$ obtained via the pull back of an operator $\mathcal{F}$ acting on functions over $\mathbb{H}^n$.

$$\mathcal{F}^\flat = \mathcal{M}_h \mathcal{F} \mathcal{W}_g. \tag{6.7}$$

We therefore have for any $\mathcal{E}$ acting on functions on $\mathbb{R}^n$

$$\mathcal{E} = (\mathcal{E}^\sharp)^\flat, \tag{6.8}$$

but for general $\mathcal{F}$ acting on functions on $\mathbb{H}^n$ we rather have

$$(\mathcal{F}^\flat)^\sharp = \Pi\mathcal{F}\Pi = A[P]\mathcal{F}A[P], \tag{6.9}$$

with P the reproducing kernel associated with g and h and Π the corresponding projector onto the image of $\mathcal{W}_g$. Here we have introduced the following notation for the convolution operator over $\mathbb{H}^n$ (compare 4.18)

$$A[P] : s \mapsto P * s. \tag{6.10}$$

By direct computation we can easily verify the following formulas for dilation and translation

$$\sigma[T_\beta D_\alpha \mathcal{E}](b,a;b',a') = \alpha^{-n}\sigma[\mathcal{E}]([b-\beta]/\alpha, a/\alpha; b', a'), \tag{6.11}$$

$\sigma[\mathcal{E}T_\beta D_\alpha](b,a;b',a') = \sigma[\mathcal{E}](b,a;\alpha b'+\beta, \alpha a')$(6.12)

6.b Some a priori estimates.

We now show how estimates of localization properties of E help to understand continuity properties of $\mathcal{E} = Op[E]$. When acting in a Banach lattice over $\mathbb{H}^n$, with norm $\|\cdot\|$, we have the following fact

$$\|Op[E]\| \leq \|Op[|E|]\|. \tag{6.13}$$

In addition, suppose $|E(b,a;b',a')| \leq F(b,a;b',a')$ then

$$\|Op[E]\| \leq \|Op[F]\|. \tag{6.14}$$

Therefore to prove continuity of an operator $\mathcal{E}$ acting in some admissible Banach space $B(\mathbb{R}^n)$ is enough prove continuity for $Op[F]$ for some $F \geq |\sigma[\mathcal{E}]|$.

Lemma 6.1 *Suppose that $E(b,a;b',a')$ satisfies the following estimate*

$$|E(b,a;b',a')| \leq \frac{1}{a'^n} F\left(\frac{b-b'}{a'}, \frac{a}{a'}\right), \tag{6.15}$$

with some $F \in S(\mathbb{H}^n)$. Then $Op[E]$ acts continuous in every admissible Banach space $B(\mathbb{H}^n)$.

Proof. A majorization by a convolution with the F together with lemma 5.3 will do. □

Remark. Operators for which the matrix elements satisfy the estimates of this lemma are called highly regular Calderón Zygmund Operators. They form an algebra.

In case that the kernel F is well localized in b and small a but only poorly for a large is treated by the next lemma. We then have to require the stronger condition 5.46 to hold for $B(\mathbb{H}^n)$.

Lemma 6.2 *Suppose that $E(b,a;b',a')$ satisfies the following estimate*

$$|E(b,a;b',a')| \leq \frac{1}{a'^n} F\left(\frac{b-b'}{a'}, \frac{a}{a'}\right), \quad F(b,a) \leq \frac{\rho(a)}{(1+a)^n} f\left(\frac{b}{1+a}\right) \tag{6.16}$$

with some $f \in S(\mathbb{R}^n)$ and some function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying for all $m > 0$ at

$$\rho(a) \leq O(a^m(1+a)^{-m}). \tag{6.17}$$

Then $Op[E]$ acts continuously in every admissible Banach space $B(\mathbb{H}^n)$ subject in addition condition (5.46).

Proof. We have for $s = Op[E]r$

$$s(b,a) = \int_{\mathbb{H}^n} \frac{db'da'}{a'} E(b,a;b',a')r(b',a'). \tag{6.18}$$

A simple change of variables leads to

$$s(b,a) = \int_{\mathbb{H}^n} \frac{db'da'}{a'} E(b,a;b'+b,aa')r(b'+b,a'a). \tag{6.19}$$

Using the various estimates on E and r we have

$$\|s\| \leq O(1)\,\|r\| \int_{\mathbb{H}^n} \frac{db'da'}{a'} \rho(1/a') \frac{1}{(1+a')^n} f\left(\frac{-b'}{1+a'}\right) a'^{\epsilon}(1+a')^M. \tag{6.20}$$

Changing b' to $(1+a')b'$ we may estimate the integral by

$$\int_{\mathbb{R}^n} db' f(b') \int_0^\infty \frac{da'}{a'} \rho(1/a') a'^{\epsilon}(1+a')^M < \infty \tag{6.21}$$

This shows the lemma. □

Let $\gamma \in \mathbb{R}$. Then denote by $a^\gamma B(\mathbb{H}^n)$ the space of functions $a^\gamma r(b,a)$ with $r \in B(\mathbb{H}^n)$. This new space is a Banach space by choosing the final topology for the map $r \mapsto a^\gamma r$. It is plain to see that if $B(\mathbb{H}^n)$ is admissible, (regular, completely regular) , then the same holds true for $a^\gamma B(\mathbb{H}^n)$. In case $\gamma \geq 0$, if $B(\mathbb{H}^n)$ is strongly tempered then the same holds true for $a^\gamma B(\mathbb{H}^n)$.

Remark. For all admissible $B(\mathbb{H}^n)$, there is $\gamma \in \mathbb{R}$ such that $a^\gamma B(\mathbb{H}^n)$ is strongly tempered. This will play a role in subsequent demonstrations.

As partial integration shows that we have the following formula

$$\mathcal{W}_g(-\Delta)^{\gamma/2} s(b,a) = a^{-\gamma}\mathcal{W}_h s(b,a), \quad h = (-\Delta)^{\gamma/2} g. \tag{6.22}$$

Therefore multiplication with powers of a essentially acts like a smoothing operator. More generally, let $\rho : \mathbb{R}_+ r \to \mathbb{R}_+$ satisfy at

$$\rho(\alpha a) \leq O((\alpha + 1/\alpha)^m)\rho(a) \tag{6.23}$$

for some $m > 0$. Then with the obvious notations we define $\rho \cdot B(\mathbb{H}^n)$ and norm it through

$$\|s\|_{\rho\cdot B(\mathbb{H}^n)} = \|s/\rho\|_{B(\mathbb{H}^n)} \tag{6.24}$$

Then $\rho \cdot B(\mathbb{H}^n)$ is admissible if $B(\mathbb{H}^n)$ is. With these notations we have the following corollaries.

Corollary 6.3 *Keep the notations of lemma 6.1 respectively of lemma 6.2. Let E satisfy instead*

$$|E(b,a;b',a')| \leq \rho(a') \frac{1}{a'^n} F\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) \tag{6.25}$$

or

$$|E(b,a;b',a')| \leq \rho(a) \frac{1}{a'^n} F\left(\frac{b-b'}{a'}, \frac{a}{a'}\right). \tag{6.26}$$

If F satisfies the the localization conditions of lemma 6.1, then $Op[E]$ maps $B(\mathbb{H}^n)$ continuously into $\rho \cdot B(\mathbb{H}^n)$ for every admissible Banach space. Suppose now that F only satisfies the conditions of lemma 6.2. Then if E satisfies at 6.25 and if $\rho \cdot B(\mathbb{H}^n)$ is strongly tempered then again $Op[E]$ maps $B(\mathbb{H}^n)$ into $\rho \cdot B(\mathbb{H}^n)$ continuously. The same holds true if E satisfies at 6.26 and if $B(\mathbb{H}^n)$ itself is strongly tempered.

Proof. $Op[E]$ is the composition of two operators. The first (respectively second) maps $r(b,a) \to \rho(a)r(b,a)$. The second (first) has a Schwartz kernel that satisfies the estimates of the previous lemma. The first (second) operator is trivially continuous, and for the second lemma 6.1 respectively lemma 6.2 applies. □

6.c Compact embedding.

We proof a fairly general theorem.

Theorem 6.4 *Let $\rho(a)$, $a > 0$ be a tempered weight function that is $o(1)$ as $a \to 0$. Then $\rho B(\mathbb{T}^n) \subset B(\mathbb{T}^n)$ and the embedding is compact for any admissible Banach space $B(\mathbb{T}^n)$.*

We will need the following lemma

Lemma 6.5 *For $B(\mathbb{H}^n)$ admissible, we have for $P \in S(\mathbb{H}^n)$ and $r \in B(\mathbb{H}^n)$ that*

$$|P * r(b,a)| \leq c \, \|r\|_{B(\mathbb{H}^n)} (a + 1/a)^M \tag{6.27}$$

where $c > 0$ depends only on P and $B(\mathbb{H}^n)$ and $M > 0$ depends only on $B(\mathbb{H}^n)$.

Proof. By a simple change of variables we may write as before

$$P * r(b,a) = \int_{\mathbb{H}^n} \frac{dbda}{a} P(-b'/a', 1/a') \, r(b+b', aa'). \tag{6.28}$$

Now we have that $\tilde{P}(b,a) = \overline{P}(-b/a, 1/a)$ is in $S(\mathbb{H}^n)$ and hence, by the first hypothesis on $B(\mathbb{H}^n)$, it is in the dual, $B'(\mathbb{H}^n)$ of $B(\mathbb{H}^n)$. We therefore may estimate using the temperdness of the norm in $B(\mathbb{H}^n)$, that

$$|P * r(b,a)| \leq c \, \|r\|_{B(\mathbb{H}^n)} \left\|\tilde{P}\right\|_{B'(\mathbb{H}^n)} (a + 1/a)^K \leq c' \, \|r\|_{B(\mathbb{H}^n)} (a+1/a)^K. \tag{6.29}$$

This is the desired estimate. □

Corollary 6.6 *Suppose $\partial_b^\alpha \partial_a^m P \in S(\mathbb{R}^n)$ for every mulit-index α and every $m \in \mathbb{N}$. Then we have that the growth behavior of the derivatives of $P * r$ may be estimated in terms of the norms of r as follows*

$$|\partial_b^\alpha P * r(b,a)| \leq c \, \|r\|_{B(\mathbb{H}^n)} (a + 1/a)^{M+|\alpha|} \tag{6.30}$$

and

$$|\partial_a P * r(b,a)| \leq c \, \|r\|_{B(\mathbb{H}^n)} (a + 1/a)^{M+1} \tag{6.31}$$

with some c depending on α and m only.

Proof. (of theorem 6.4.) We have

$$\partial_b^\alpha \frac{1}{a'^n} P\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) = a'^{-|\alpha|} \frac{1}{a'^n} (\partial_b^\alpha P) \left(\frac{b-b'}{a'}, \frac{a}{a'}\right) \tag{6.32}$$

and

$$\partial_a \frac{1}{a'^n} P\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) = a'^{-1} \frac{1}{a'^n} (a\partial_a P) \left(\frac{b-b'}{a'}, \frac{a}{a'}\right) \tag{6.33}$$

□

In particular this shows thanks to the reproducing kernel equation $\mathcal{W}_g s = P_{g,h} * \mathcal{W}_g s$, (compare 4.21) and the fact 5.8 that $\mathcal{W}_g s(b,a)$ all its derivatives have an at most tempered growth for uniformly for s ins some bounded set of $B(\mathbb{R}^n)$.

We now can proof the theorem.

Proof. We pick as we may a wavelet g such that the wavelet transform of periodic distributions is supported by $\{a \le 1\}$. We now have to show that given a sequence $s_n(b,a) = \mathcal{W}_g r_n(b,a)$, with $r(x)$ periodic in x such that $\|s_n\|_{\rho B(\mathbb{M}^n)} \le 1$ we can extract a subsequence $\tilde{s}_m = s_{m(n)}$, that converges in $B(\mathbb{M}^n)$:

$$\lim_{k,m\to\infty} \|\tilde{s}_k - \tilde{s}_m\|_{B(\mathbb{M}^n)} = 0. \tag{6.34}$$

We first observe that thanks to lemma 6.5 and the periodicity of s_n in b we have that that for $|\alpha| = 1$

$$|s_n(b,a)| + |\partial_b^\alpha s_n(b,a)| + |\partial_a s_n(b,a)| \le c a^{-M} \tag{6.35}$$

for some $c > 0$ and $M > 0$. Now for every $k \in \mathbb{N}$ we may select a sub-sequence of the s_n that converges uniformly in $\{a \ge 1/k\}$. Therefore using a diagonal sequence, we have that $\tilde{s}_m$ converges uniformly in $\{a \ge 1/k\}$ for all $k \in \mathbb{N}$ to some continuous function $\tilde{s}(b,a)$. Since $B(\mathbb{H}^n)$ is tempered, we have that

$$\forall k \in \mathbb{N}: \quad \left\| \chi_{\{a\ge 1/k\}}(\tilde{s}_l - \tilde{s}_m) \right\|_{B(\mathbb{H}^n)} \to 0 \quad (l,m\to\infty). \tag{6.36}$$

Here χ_I is the characteristic function of I. On the other hand

$$\begin{aligned}
\left\| \chi_{\{a<1/k\}} \tilde{s}_m \right\|_{B(\mathbb{H}^n)} &= \left\| \rho(a) \chi_{\{a<1/k\}} \tilde{s}_m / \rho(a) \right\|_{B(\mathbb{H}^n)} && (6.37)\\
&\le \left\| \rho(a) \chi_{\{a<1/k\}} \right\|_{L^\infty(\mathbb{H}^n)} \|\tilde{s}_m\|_{\rho B(\mathbb{H}^n)} && (6.38)\\
&\le c_1 \rho(1/k). && (6.39)
\end{aligned}$$

In the last inequality we have used the boundedness of s_n in $\rho B(\mathbb{H}^n)$. Therefore we end up with

$$\begin{aligned}
&\lim_{l,m\to\infty} \|\tilde{s}_l - \tilde{s}_m\|_{B(\mathbb{H}^n)} && (6.40)\\
&\le \lim_{l,m\to\infty} \left\| \chi_{\{a\ge 1/k\}}(\tilde{s}_l - \tilde{s}_m) \right\|_{B(\mathbb{H}^n)} + \lim_{l,m\to\infty} \left\| \chi_{\{a<1/k\}}(\tilde{s}_l - \tilde{s}_m) \right\|_{B(\mathbb{H}^n)} && (6.41)\\
&\le O(1)\,(\rho(1/k)) && (6.42)
\end{aligned}$$

Now as k gets large this is arbitrary small by hypothesis on ρ and the theorem follows. □

7 The transfer operator in wavelet space.

We now shall analyse the transfer operator in wavelet space. It is clear that it is a combination of multiplication operators and coordinate change operators. We shall analyze both of them separately.

To simplify the expressions we work with a fixed strictly admissible wavelet g. Recall that this means g, g is an analysis reconstruction pair.

7.a The multiplication operator.

For any smooth function $\mu : \mathbb{R}^n \to \mathbb{C}$ we associate the multiplication operator

$$M = M[\mu]: \quad s(x) \mapsto \mu(x) \cdot s(x). \tag{7.1}$$

$$\begin{aligned}
\sigma_{g,g}[M[\mu]](b,a;b',a') &= \mu(b') \frac{1}{a'^n} P\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) && (7.2)\\
&+ \sum_{k=1}^{K-1} a'^k F_k(b,a;b',a') + a'^K R_K(b,a;b',a') && (7.3)
\end{aligned}$$

where $P = \mathcal{W}_g g$ is the reproducing kernel and R_K satisfies the estimates of lemma 6.2.

The first term acts like the identity in the image of the wavelet transform thanks to the reproducing kernel equation. Otherwise $A[P] : s \mapsto P * s$, acts as a projection into the image of $\mathcal{W}_g$. Therefore if for for any function $r(b,a)$ over $\mathbb{H}^n$ we associate the multiplier (denoted again by $M[r]$),

$$M[r] : \quad s(b,a) \mapsto r(b,a) \cdot s(b,a), \tag{7.4}$$

we have up to smoothing operators that the multiplication operator reads in wavelet space look like ($\mu'(b,a) = \mu(b)$)

$$M[\mu]^\sharp = M[\mu'] + \mathcal{R} \tag{7.5}$$

Note that upon choosing K large enough, the reminder $\mathcal{R}$ becomes smoothing from $B(\mathbb{H}^n)$ to $aB(\mathbb{H}^n)$ for any tempered Banach space. This is because for K large enough, $a^K B(\mathbb{H}^n)$ is strongly tempered.

Proof. As we may we choose g such that $\mathcal{W}_g\eta(b,a) = 0$ for $a > 1$ and for η any function on $\mathbb{T}^n$. A change of variables $x \to a'x + b'$ shows that we have

$$\sigma_{g,g}[M[\mu]](b,a;b',a') = \int dx \, \frac{1}{a^n} \, \overline{g} \left(\frac{x - [b-b']/a'}{a/a'} \right) \mu(a'x + b') g(x). \tag{7.6}$$

Now Taylors theorem reads

$$\mu(a'x + b') = \mu(b') + \sum_{|\alpha|<K} (\partial^\alpha \mu)(b')(\alpha!)^{-1} {a'}^{|\alpha|} x^\alpha + B_K(b', a'x) \tag{7.7}$$

Inserting this we obtain the first term. The localization properties of the F_k follows from the fact that $x^\alpha g \in S(\mathbb{R}^n)$ if $g \in S_0(\mathbb{R}^n)$. To estimate the reminder, observe that Taylor's theorem shows that for $(b'a') \in \mathbb{T}^n \times (0,1]$ we have

$${a'}^{-K} B_K(b', a'x) g(x) \in \Sigma \tag{7.8}$$

for some bounded set $\Sigma \subset S(\mathbb{R}^n)$. It follows from the localization result 5.2 that the estimates of lemma 6.2 hold for the reminder. □

7.b The diffeomorphism in wavelet space

We use the notations and hypothesis of section 2 concerning the diffeomorphism ϕ and the operators associated with it. In analogy we define the following operator acting on functions over $\mathbb{H}^n$

$$\Psi = \Psi[\phi] : \; s(b,a) \mapsto J(\phi^{-1}(x))^{-1} \, s(\phi^{-1}(b), a) = (\Phi s(\cdot, a))(b). \tag{7.9}$$

For any map $P : \mathbb{R}^n \to S(\mathbb{H}^n)$, $x \mapsto P(x) = P(x;b,a)$, we define the following position dependent convolution

$$\Omega[P] : s(b,a) \mapsto \int_{\mathbb{H}^n} \frac{db'da'}{a'} \frac{1}{{a'}^n} P\left(b'; \frac{b-b'}{a'}, \frac{a}{a'}\right) \, s(b',a'). \tag{7.10}$$

We shall prove that with a smoothing operator $\mathcal{R}$ we have

$$\Phi[\phi]^\sharp = \Omega[E] \, \Psi[\phi] \; + \mathcal{R} \tag{7.11}$$

with $E(x;b,a) = \mathcal{W}[g, \partial\Phi[g, \phi^{-1}(x)]](b,a)$. This formula is easily verified for linear diffeomorphisms. And it is therefore no surprise that it becomes a good approximation at small scale.

Lemma 7.1 *There is a sequence of symbols $F_k(b,a;b',a')$, $k=1,2,\dots$ satisfying the estimates of lemma 6.1 such that for all $K\in\mathbb{N}$ we may write for $a,\ a'\le 1$*

$$\sigma_{g,g}[\Phi[\phi]](b,a;b',a') = \frac{1}{a'^n}P\left(b';\frac{b-\phi(b')}{a'},\frac{a}{a'}\right) \tag{7.12}$$

$$+ \sum_{k=1}^{K-1} a'^k F_k(b,a;\phi(b'),a') + a'^K R_K(b,a;\phi(b'),a') \tag{7.13}$$

where $P(b';b,a)=\mathcal{W}[g,\partial\Phi_{b'}g](b,a)$. The remainder R_K are satisfying the estimates of lemma 6.2.

Remark. If g is such that its Fourier transform is supported by a corona $\{c^{-1}\le|\xi|\le c\}$, it follows that the principal term vanishes whenever a/a' is too large or too small. Indeed, all wavelets $\partial\Phi_{b'}g$ are supported by a corona and there is one that contains them all. Hence all the functions $P(b';\cdot,\cdot)$ are supported by some strip $a+1/a\le c$. Therefore the principal part of $\Phi[\phi]$ leaves invariant the spaces of functions whose wavelet transforms with respect to g are supported by a strip. This remark will play a role in some demonstrations.

Proof. First observe that $\partial\Phi_b$ is the small scale behavior of Φ at $b\in\mathbb{R}^n$ in the following precise sense. Let as usual $D_a: s(x)\mapsto s(x/a)/a^n$ denote the dilation operator and let T_b denote the translation operator $s(x)\mapsto s(x-b)$. We then have

$$D_a^{-1}T_{\phi(b)}^{-1}\Phi[\phi]T_bD_ag(x) = \frac{1}{J(\phi^{-1}(\phi(b)+ax))}\,g\left(\frac{\phi^{-1}(ax+\phi(b))-b}{a}\right) \tag{7.14}$$

and thus it follows that

$$\Phi_{b,a}[\phi] = D_a^{-1}T_{\phi(b)}^{-1}\Phi[\phi]T_bD_a \to \partial\Phi_b, \quad (a\to 0). \tag{7.15}$$

More precisely, let us introduce, for each b the following operators $A_k(b)$ via ($s\in S(\mathbb{R}^n)$)

$$(A_k(b)s)(x) = \lim_{a\to 0}\frac{d^k}{da^k}\Phi_{b,a}[\phi]s(x) \tag{7.16}$$

Repeatedly application of the chain rule shows that the $A_k(b)$ are in the algebra of partial differential operators generated by x^α and ∂_x^β

$$A_k(b) = \sum_{|\alpha|+|\beta|\le M_k} c_{k,\alpha,\beta}(b)x^\alpha\partial_x^\beta \tag{7.17}$$

Moreover, since the diffeomorphism is uniformly smooth we have that all the $c_{k,\alpha,\beta}(b)$ are bounded functions. It follows that $\{A_k(b)g : b\in\mathbb{R}^n\}$ is a bounded subset of $S_0(\mathbb{R}^n)$. Now Taylor's theorem takes the form

$$\Phi_{b,a}[\phi]g = \partial\Phi_b g + \sum_{k=1}^{K-1} a^k A_k(b)g + R(b,a), \tag{7.18}$$

and the remainder $R=R_K$ satisfy at

$$a^{-K}R(b,a)\in\Sigma, \tag{7.19}$$

for some bounded set $\Sigma \subset S(\mathbb{R}^n)$. We may write using the fact that $(T_\beta)^* = T_\beta^{-1}$ and $(D_\alpha)^* = \alpha^{-n} D_\alpha^{-1}$

$$\begin{aligned}
\sigma_{g,g}[\Phi[\phi]](b,a;b',a') &= \langle g_{b,a} \mid \Phi g_{b',a'} \rangle & (7.20)\\
&= \langle g_{b,a} \mid \Phi T_{b'} D_{a'} g \rangle & (7.21)\\
&= a'^{-n} \left\langle D_{a'}^{-1} T_{\phi(b')}^{-1} g_{b,a} \mid \Phi_{b',a'} g \right\rangle & (7.22)\\
&= a'^{-n} \left\langle g_{[b-\phi(b')]/a', a/a'} \mid \Phi_{b',a'} g \right\rangle . & (7.23)
\end{aligned}$$

We now may insert expression 7.18. The principal term is the one stated in the theorem. The terms F_k are localized since they are contained in the set of wavelet transforms of a bounded set of functions in $S_0(\mathbb{R}^n)$. Finally, by lemma 5.2 the reminder satisfies the estimate of lemma 6.2. □

In exactly the same we can show the next lemma, the proof of which is left to the reader.

Lemma 7.2 *There is a sequence of symbols $F_k(b,a;b',a')$, $k = 1,2,\dots$ satisfying the estimates of lemma 6.1 such that for all $K \in \mathbb{N}$ we may write for a, $a' \leq 1$*

$$\begin{aligned}
\sigma_{g,g}[\Phi[\phi]M[\mu]](b,a;b',a') &= \mu(b') \frac{1}{a'^n} P\left(b'; \frac{b-\phi(b')}{a'}, \frac{a}{a'}\right) & (7.24)\\
&+ \sum_{k=1}^{K-1} a'^k F_k(b,a;\phi(b'),a') + a'^K R_K(b,a;\phi(b'),a') & (7.25)
\end{aligned}$$

where $P(b';b,a) = \mathcal{W}[g, \partial\Phi_{b'} g](b,a)$. The remainder R_K are satisfying the estimates of lemma 6.2.

7.c The locally linear approximation.

We even may go one step further and consider the diffeomorphism at small scale and locally.

Lemma 7.3 *For a, a', $|b'-\beta| \leq 1$, $\beta \in \mathbb{R}^n$, we have*

$$\frac{1}{a'^n} P\left(b'; \frac{b-\phi(b')}{a'}, \frac{a}{a'}\right) = \frac{1}{a'^n} P\left(\beta; \frac{b-\phi(b')}{a'}, \frac{a}{a'}\right) + |b'-\beta|\, R(b,a;\phi(b'),a') \tag{7.26}$$

with $P(b';b,a) = \mathcal{W}[g, \partial\Phi_{b'} g](b,a)$. The remainder R satisfies the estimates of lemma 6.1.

Note however that in general, we may not replace $\phi(b)$ in the kernel by some linear approximation.

Proof. For $g \in S_0(\mathbb{R}^n)$, there is a bounded set $\Sigma \subset S_0(\mathbb{R}^n)$, such that for $|b'-\beta| \leq 1$ we have

$$|b'-\beta|^{-1} (\partial\Phi_{b'} - \partial\Phi_\beta) g \subset \Sigma. \tag{7.27}$$

Therefore, upon taking a wavelet transform, the lemma follows. □

8 The main theorem

8.a The transfer operator.

We now can analyse the transfer operators introduced in section 2

$$\mathcal{L} = \sum_{k=1}^{K} \Phi[\phi_k] M[\mu_k] : \quad s(x) \mapsto \sum_{k=1}^{K} \mu_k(\phi_k^{-1}(x)) J[\phi_k](\phi_k^{-1}(x)) s(\phi_k^{-1}(x)) \tag{8.1}$$

We fix a strictly admissible wavelet g. Later we will impose some more conditions on it. Let $\sigma[\mathcal{L}]$ be the symbol of the transfer operator, with respect to g, and let $\mathcal{L}^\sharp$ be the associated operator acting on functions over $\mathbb{H}^n$

$$\mathcal{L}^\sharp = Op[\sigma[\mathcal{L}]]. \tag{8.2}$$

As we have seen in lemma 7.2, the main term of $\mathcal{L}^\sharp$ at small scale is the form $\mathcal{L}_\partial = Op[L]$ with

$$L(b,a;b',a') = \sum_{k=1}^{K} L_k(b,a;b',a') = \sum_{k=1}^{K} \mu_k(b') \frac{1}{a'^n} P_k\left(b'; \frac{b-\phi_k(b')}{a'}, \frac{a}{a'}\right), \tag{8.3}$$

with $P_k(b';b,a) = \mathcal{W}[g, \partial\Phi[\phi_k, b']g](b,a)$.

Proposition 8.1 *Let $B(\mathbb{H}^n)$ be an admissible, deformable Banach space. Then the operator $\mathcal{L}$ is acting continuously on $B(\mathbb{T}^n)$.*

Proof. We write

$$\mathcal{L}^\sharp = \mathcal{L}_\partial + \mathcal{R}, \tag{8.4}$$

Then, upon choosing K large enough such that $a^K B(\mathbb{H}^n)$ is strongly tempered it follows from lemma 7.2 that $\mathcal{R}$ maps $B(\mathbb{H}^n)$ continuously to $a^K B(\mathbb{H}^n)$ since $a^K B(\mathbb{H}^n)$ satisfies again condition 5.49. The principal term $\mathcal{L}_\partial$ is continuous in $B(\mathbb{H}^n)$ and the proof is finished. □

Note that the principal part $\mathcal{L}_\partial$ is acting on $B(\mathbb{M}^n)$ and on $\Pi B(\mathbb{M}^n)$, where Π is the extension to $B(\mathbb{H}^n)$ of the orthogonal projection onto the image of $L^2(\mathbb{R}^n)$ under $\mathcal{W}_g$; that is, $\Pi s = P_g * s$, $P_g = \mathcal{W}_g g$. This follows from the fact that

$$\Pi \mathcal{L}_\partial = \Pi Op[L] = \mathcal{L}_\partial \tag{8.5}$$

since the for fixed b', a' the kernel $L(\cdot, \cdot; b', a')$ is in the imag of the wavelet transform. In the same way we may consider the iterates $\mathcal{L}^m$ to define $(\mathcal{L}^m)_\partial$. Indeed, all powers of $\mathcal{L}$ can be written as

$$\mathcal{L}^m = \sum_{k=1}^{K^m} \Phi[\phi_{m,k}] M[\mu_{m,k}]. \tag{8.6}$$

This serves as a definition for the $\phi_{m,k}$ and the $\mu_{m,k}$. The small part $(\mathcal{L}^m)_\partial$ of $\mathcal{L}^m$ is now defined by using the $\mu_{m,k}$ and $\phi_{m,k}$. Note that we have to distinguish between $(\mathcal{L}^m)_\partial$ and $(\mathcal{L}_\partial)^m$.

8.b A formula for $\rho_{\mathbf{ess}}(\mathcal{L})$

¿From now on we fix an admissible wavelet g such its Fourier transform, $\widehat{g}(\xi)$, is compactly supported by $\{\pi \leq |\xi| \leq 2\pi\}$. Therefore the wavelet coefficients of a any distribution s on $\mathbb{T}^n$ vanish for $a \geq 1$

$$a \geq 1 \;\Rightarrow\; \mathcal{W}_g s(b,a) = 0. \tag{8.7}$$

The reproducing kernel $P_g = \mathcal{W}_g g(b,a)$ is therefore supported by the strip $\{1/2 \leq a \leq 2\}$. Let $\chi_j(b,a)$ denote the characteristic function of $\{a \leq 4^{-j}\}$. We clearly want to use the results of section 3. As family of Π_j we now simply set

$$\Pi_j = \Pi M[\chi_j] \Pi. \tag{8.8}$$

We now want to prove that the essential spectrum of $\mathcal{L}$ is given by its restriction on its small scale behavior. As we may we will work with $\mathcal{L}^\sharp$ instead of $\mathcal{L}$. However we have to let act $\mathcal{L}^\sharp$ not in the whole of $B(\mathbb{M}^n)$ but only in the image of $\mathcal{W}_g$; that is the closed supspace $\Pi B(\mathbb{M}^n)$.

Theorem 8.2 *Let $\mathcal{L}$ be acting in a Banach space $B(\mathbb{T}^n)$ coming from a regular Banach space $B(\mathbb{H}^n)$. Then we have*

$$\begin{aligned}
\rho_{ess}(\mathcal{L}) &= \rho_{ess}(\mathcal{L}^\sharp) = \rho_{ess}(\mathcal{L}_\partial) && (8.9)\\
&= \lim_{m\to\infty}\limsup_{j\to\infty}\left\|(\mathcal{L}_\partial)^m\Pi_{\alpha(j)}\right\|^{1/m} = \lim_{m\to\infty}\liminf_{j\to\infty}\left\|\Pi_{\alpha(j)}(\mathcal{L}_\partial)^m\right\|^{1/m} && (8.10)\\
&= \lim_{m\to\infty}\limsup_{j\to\infty}\left\|(\mathcal{L}^m)_\partial\Pi_{\alpha(j)}\right\|^{1/m} = \lim_{m\to\infty}\liminf_{j\to\infty}\left\|\Pi_{\alpha(j)}(\mathcal{L}^m)_\partial\right\|^{1/m}. && (8.11)
\end{aligned}$$

for any sequence $\alpha(j) \in \mathbb{N}$ such that $\alpha(j+1)-\alpha(j) \to \infty$ as $j \to \infty$. The operator norms are understood as operators acting in $\Pi B(\mathbb{M})$.

Remark. As the proof will show, the same theorem holds in the case where $S(\mathbb{H}^n)$ is not dense in $B(\mathbb{H}^n)$ but only in $B'(\mathbb{H}^n)$.

This theorem shows that it is the asymptotic small-scale behavior of the dynamic that determines the essential spectrum. Note that this illustrates nicely the interpretation given by A. Connes in the context of non-commutative geometry, where compact operators play the role of infinitesimals [Con95].

Proof. Clearly $\rho_{\mathrm{ess}}(\mathcal{L}) = \rho_{\mathrm{ess}}(\mathcal{L}^\sharp)$. Now, since $\mathcal{L}^\sharp - \mathcal{L}_\partial$ is smoothing from $B(\mathbb{M}^n)$ to $B(\mathbb{M}^n)$ and hence compact, we have that $\rho_{\mathrm{ess}}(\mathcal{L}^\sharp) = \rho_{\mathrm{ess}}(\mathcal{L}_\partial)$.

To show the other equations we proceed in 5 steps.

Step 1: The Π_j satisfy the conditions of theorem 3.1. Indeed, since they are given by bounded wavelet multipliers, they are clearly uniformly continuous, $\|\Pi_j\| \le c$ by condition (A4) on $B(\mathbb{H}^n)$. By theorem 6.4 we have that $\mathbb{1}-\Pi_j$ is compact. By the support property of $\widehat{g}$ we have that

$$\Pi_j\Pi_{j+1} = \Pi_{j+1}\Pi_j = \Pi_{j+1}. \tag{8.12}$$

Step 2: Property (iii) of theorem 3.1 holds. Indeed, since for $s \in S(\mathbb{H}^n)$ we have $\Pi_j s \to 0$ in $S(\mathbb{H}^n)$ this holds a forteriori in $B(\mathbb{H}^n)$ too. By hypothesis $S(\mathbb{H}^n)$ is dense in $B(\mathbb{H}^n)$ and hence thanks to lemma 5.8 $S(\mathbb{M}^n)$ is dense in $B(\mathbb{M}^n)$,

Step 3: $\mathcal{L}_\partial$ satisfies condition (iv) of theorem 3.1 holds and hence the first pair of formulas holds. We have to show that we have

$$\lim_{j\to\infty}\left\|(\mathbb{1}-\Pi_{j'})\mathcal{L}_\partial\Pi_j\right\| = \lim_{j\to\infty}\left\|\Pi_j\mathcal{L}_\partial(\mathbb{1}-\Pi_{j'})\right\| = 0, \tag{8.13}$$

whenever $j - j' \to \infty$. We even have

$$\mathcal{L}_\partial\Pi_j = \Pi_{j-k}\mathcal{L}_\partial\Pi_j, \quad \Pi_j\mathcal{L}_\partial = \Pi_j\mathcal{L}_\partial\Pi_{j+k} \tag{8.14}$$

for some k large enough. This follows from the fact that all kernels in $\mathcal{L}_\partial$ are supported by a strip in the half-space and that Π_j acts like the reproducing kernel in $\{a \le 4^{-j-2}\}$, whereas in $\{a \ge 4^{-j+2}\}$ it acts like the zero operator. Now we may write

$$\left\|(\mathbb{1}-\Pi_{j'})\mathcal{L}_\partial\Pi_j\right\| \le \left\|(\mathbb{1}-\Pi_{j'})\Pi_{j-k}\right\|\,\left\|\mathcal{L}_\partial\Pi_j\right\| \tag{8.15}$$

and

$$\left\|\Pi_j\mathcal{L}_\partial(\mathbb{1}-\Pi_{j'})\right\| \le \left\|\Pi_j\mathcal{L}_\partial\right\|\,\left\|\Pi_{j+k}(\mathbb{1}-\Pi_{j'})\right\|. \tag{8.16}$$

Both expressions are zero for $j - j'$ large enough.

This shows that the second pair of equations holds.

Step 4: The difference $(\mathcal{L}_\partial)^m - (\mathcal{L}^m)_\partial$ is smoothing from $B(\mathbb{H}^n)$ to $aB(\mathbb{H}^n)$ and from $a^{-1}B(\mathbb{H}^n)$ to $B(\mathbb{H}^n)$. Indeed, we have

$$\mathcal{L}^\sharp = \mathcal{L}_\partial + \mathcal{R}_1 \tag{8.17}$$

and

$$(\mathcal{L}^\sharp)^m = (\mathcal{L}^m)_\partial + \mathcal{R}_2 \tag{8.18}$$

where $\mathcal{R}_i$, $i = 1, 2$ are smoothing from $a^k B(\mathbb{H}^n)$ to $a^{k+1}B(\mathbb{H}^n)$ for all $k \in \mathbb{Z}$. Therefore upon taking the m-th power and taking the difference, the statement follows.

Step 5: Hence $\|((\mathcal{L}_\partial)^m - (\mathcal{L}^m)_\partial)\Pi_j\|$ and $\|\Pi_j((\mathcal{L}_\partial)^m - (\mathcal{L}^m)_\partial)\|$ tend to zero as $j \to \infty$. Indeed, the mapping Π_j from $aB(\mathbb{H}^n)$ to $B(\mathbb{H}^n)$ has a norm that is $\leq c4^{-j}$. Indeed, $\Pi_j s(b,a) = 0$ for $a > 4^{-j+2}$ and thus

$$\|\Pi_j s\|_{B(\mathbb{H}^n)} = \left\|aa^{-1}\Pi_j s\right\|_{B(\mathbb{H}^n)} \leq 4^{-j+2}\left\|a^{-1}\Pi_j s\right\|_{B(\mathbb{H}^n)} = 4^{-j+2}\|s\|_{aB(\mathbb{H}^n)}. \tag{8.19}$$

The same estimate shows that Π_j as mapping from $B(\mathbb{H}^n)$ to $a^{-1}B(\mathbb{H}^n)$ has norm $\leq c4^{-j}$. Therefore,

$$\lim_{j\to\infty}\|\Pi_j((\mathcal{L}_\partial)^m - (\mathcal{L}^m)_\partial)\| = 0 = \lim_{j\to\infty}\|(((\mathcal{L}_\partial)^m - (\mathcal{L}^m)_\partial)\Pi_j\|. \tag{8.20}$$

This shows the theorem. □

8.c A more explicit formula for $\rho_{\mathbf{ess}}(\mathcal{L})$

We now can actually use the previous theorem to compute the essential spectral radius in terms of the tangential maps. We use the notations introduced in section 2. In particular recall the definition of γ_K (see (5.52)) and the notations $F((s_k))$ (see 5.54) and $\bigoplus A_k$ (see (2.18)).

Theorem 8.3 *Let $B(\mathbb{R}^n)$ be a completely regular Banach space. Let $B(\mathbb{T}^n)$ be the induced Banach space over $\mathbb{T}^n$. Then the transfer operator acts continuously on $B(\mathbb{T}^n)$. Its essential spectral radius can be computed as follows in terms of the norms of the linearized maps $\mu_{m,k}(\phi_{m,k}^{-1}x)\partial\Phi[\phi_{m,k}, x]$, $k = 1, \ldots, K^m$ acting on $B(\mathbb{R}^n)$.*

$$\rho_{ess}(\mathcal{L}) = \lim_{m\to\infty}\sup_{x\in\mathbb{T}^n}\left\|\bigoplus_{k=1}^{K^m}\mu_{m,k}(\phi_{m,k}^{-1}x)\partial\Phi[\phi_{m,k}, \phi_{k,m}^{-1}x]\right\|^{1/m} \tag{8.21}$$

Note that $\mathcal{L}$ is acting on periodic functions, whereas the norm on the right hand side is computed by the action of $\partial\Phi[\phi_{m,k}]$ on functions over $\mathbb{R}^n$. This formula is easy to understand. It states, that the small scale norm of $\mathcal{L}$ and all its derivatives is produced by the strongest local deformation measures by the norm $\|\partial\Phi[\phi_{m,k}, x]\|$ of the tangential map at x acting in $B(\mathbb{R}^n)$.

Proof. The main idea of the proof is as follows. Consider a small neighborhood of $x \in \mathbb{R}^n$. The points that map to x are $\phi_k^{-1}(x)$. At small scales the dynamic can be described by the tangential map. Since the local regions can be varied independently and since by homogeneity we can localize around the $\phi_k^{-1}(x)$ we make appear the norm of the sum of the tangential maps as stated in the theorem.

The central argument to prove the lower bound is given by the next two lemmas.

Lemma 8.4 *For $s = \mathcal{W}_g r$, $r \in S_0(\mathbb{R}^n)$, $\|s\| = 1$ consider the family of dilated and translated functions $s_{\beta,\alpha} = \mathcal{W}_g T_\beta D_\alpha r = T^\sharp_\beta D^\sharp_\alpha s$. We then have for $\mathcal{F} = \Pi Op[F]\Pi$, with F given by (compare 8.3)*

$$F(b,a;b',a') = \mu(b')\frac{1}{a'^n} D\left(b'; \frac{b-\phi(b')}{a'}, \frac{a}{a'}\right), \tag{8.22}$$

with $D(b';b,a) = \mathcal{W}[g, \partial\Phi[\phi,b']g](b,a)$ that

$$\lim_{\alpha\to 0} \|\mathcal{F}s_{\beta,\alpha}\| \,/\, \|s_{\beta,\alpha}\| = |\mu(\beta)|\, \|\partial\Phi[\phi,\beta]r\|_{B(\mathbb{R}^n)} \tag{8.23}$$

Proof. By homogeneity of the norm of $B(\mathbb{H}^n)$ we may suppose that $\beta = 0$ and that $\phi(0) = 0$. Suppose $\|s\| = 1$ and let $s_\alpha = s_{0,\alpha} = D^\sharp_\alpha s$. We write $\partial\phi$ for $\partial\phi_0$. We then may write, using again the homogeneity

$$\|\mathcal{F}s\| = a^\gamma \left\|(D^\sharp_\alpha)^{-1}\mathcal{F}D^\sharp_\alpha s\right\|_{B(\mathbb{H}^n)}. \tag{8.24}$$

Now $\mathcal{F}$ may be factorized as follows

$$\mathcal{F} = A[P]\Omega[D']\Psi[\phi]M[\mu] = \Omega[D']\Psi[\phi]M[\mu], \tag{8.25}$$

with $D'(x;b,a) = D(\phi^{-1}(x);b,a)$. We now have three elementary steps.

Step 1: The family of operators $\mathcal{F}_\alpha = (D^\sharp_\alpha)^{-1}\mathcal{F}D^\sharp_\alpha$, $0 < \alpha \leq 1$ is uniformly continuous from $S(\mathbb{H}^n)$ to $S(\mathbb{H}^n)$ (it maps highly localized function into functions having the same localization). Clearly it is enough to check this for each factor since the symbols satisfying these estimates constitute an algebra (= the algebra of highly regular Calderón Zygmund operators). For this let us denote for any operator $\mathcal{G}$ by $\mathcal{G}_\alpha$ the rescaled operator $(D^\sharp_\alpha)^{-1}\mathcal{G}$

$\mathrm{D}^\sharp_\alpha$. We may write

$$\mathcal{F}_\alpha = \Omega[D']_\alpha \Psi[\phi]_\alpha M[\mu]_\alpha \tag{8.26}$$

Suppose that $\mathcal{G} = Op[G]$ with G satisfying with some $v \in S(\mathbb{H}^n)$ (compare lemma 6.1)

$$|G(b,a;b',a')| \leq \frac{1}{a'^n} v\left(\frac{b-b'}{a'}, \frac{a}{a'}\right). \tag{8.27}$$

Then we may find G_α with $\mathcal{G}_\alpha = Op[G_\alpha]$ where the G_α satisfies the same estimate with the same function v. In addition these operators are uniformly continuous in $S(\mathbb{H}^n)$. Therefore, $\Omega[D']_\alpha$ is uniformly continuous in $S(\mathbb{H}^n)$. Now we have

$$M[\mu]_\alpha = M[\mu_\alpha], \quad \mu_\alpha(b,a) = \mu(\alpha b, a) \tag{8.28}$$

thereby showing that $M[\mu]_\alpha$ is uniformly continuous in $S(\mathbb{H}^n)$. Finally we may write

$$\Psi[\phi]_\alpha = \Psi[\phi_\alpha], \quad \phi_\alpha(x) = \alpha\phi(x/\alpha). \tag{8.29}$$

Now, since $|\phi^{-1}(x)| \geq c\,|x|$ we have with the same c that $|(\phi_\alpha)^{-1}(x)| \geq c\,|x|$. Therefore again the localization of $\Psi[\phi]_\alpha s$ may be estimated in terms of the localization of s, uniformly in α.

This shows that there is some function $u \in S(\mathbb{H}^n)$ such that

$$|\mathcal{F}_\alpha s(b,a)| \leq u(b,a). \tag{8.30}$$

Step 2: We have pointwise with $W(b,a) = D'(0;b,a)$

$$\mathcal{F}_\alpha s(b,a) \to \mu(0)A[W]\Psi[\partial\phi]s(b,a) \quad (\alpha \to 0). \tag{8.31}$$

Indeed, we have pointwise $\Psi[\phi]_\alpha M[\mu]_\alpha s \to \mu(0)\Psi[\partial\phi]s$. Now $\Psi[\phi]_\alpha M[\mu]_\alpha s$ is uniformly localized by step 1. Therefore, by standard Lebesgue dominated convergence we have pointwise

$$\Omega[D']_\alpha \Psi[\phi]_\alpha M[\mu]_\alpha s(b,a) \to \mu(0)A[W]\Psi[\partial\phi]s(b,a). \tag{8.32}$$

Step 3: The family $\mathcal{F}_\alpha s(b,a)$ is uniformly continuous on compact sets. Indeed, thanks to corollary 6.6 uniform localization of $v_\alpha = \Omega[D']_\alpha \Psi[\phi]_\alpha M[\mu]_\alpha s$ implies unorm localization of all partial derivatives of $\mathcal{F}_\alpha s(b,a) = v_\alpha(b,a) = P * v_\alpha(b,a)$. We may evoke lemma 5.4 to conclude for the convergence of the norms. □

In exactly the same way we can show, the next lemma.

Lemma 8.5 *For K functions $s_k = \mathcal{W}_g r_k$, $r_k \in S_0(\mathbb{R}^n)$, and diffeomorphisms ϕ_k, operators $\mathcal{F}_k$ associated with them as in lemma 8.4 and pairwise distinct points β_k with $\phi_k\beta_k = \beta$ we have*

$$\lim_{\alpha\to 0}\left\|\sum_{k=1}^K \mathcal{F}_k s_{k;\beta_k,\alpha}\right\| / \left\|\sum s_{k;\beta_k,\alpha}\right\| = \left\|\sum_{k=1}^K \mu_k(\beta_k)\partial\Phi[\phi_k,\beta_k]r_k\right\| / F((s_k)) \tag{8.33}$$

Proof. We only have to note that thanks to formula 5.54 we have

$$\lim_{\alpha\to 0}\alpha^{-\gamma}\left\|\sum s_{k;\beta_k,\alpha}\right\| = \lim_{\alpha\to 0}\left\|\sum s_{k;\beta_k/\alpha,1}\right\| = F((s_k)). \tag{8.34}$$

where the exponent γ is given by hypothesis 5.47. □

Until now we have considered functions over $\mathbb{R}^n$. However if we introduce the periodization operator $\mathcal{P}$ via ($s \in S(\mathbb{H}^n)$)

$$\mathcal{P}s = \sum_{k\in\mathbb{Z}^n} s(\cdot + k, \cdot), \tag{8.35}$$

then $\mathcal{P}s \in S(\mathbb{M}^m)$. Then if we set $s^\circ_{\beta,\alpha} = \mathcal{P}s_{\beta,\alpha}$ we have again that (supposing $\|s\| = 1$)

$$\lim_{\alpha\to 0}\left\|\mathcal{F}s^\circ_{\beta,\alpha}\right\| / \|s_{\beta,\alpha}\| = |\mu(\beta)| \, \|\partial\Phi[\phi_k,\beta]r\|_{B(\mathbb{R}^n)} \tag{8.36}$$

We now come back to the transfer operator $\mathcal{L}_\partial = \sum_{k=1}^K \mathcal{L}_k$, $\mathcal{L}_k = Op[L_k]$ (compare 8.3). In the same as before we may prove the next lemma. We may the details to the reader.

Lemma 8.6 *Consider the operator $\mathcal{L}_\partial = \sum_{k=1}^K \mathcal{L}_k$ acting on $S(\mathbb{M}^n)$. Let $x \in \mathbb{T}^n$ and let $\beta_k = \phi_k^{-1}(x) \in \mathbb{T}^n$ be the preimages of the diffeomorphism ϕ_k. Then for $s_k \in \Pi S(\mathbb{H}^n)$, $s_k = \mathcal{W}_g r_k$ consider the functions $s_\alpha = \sum_{k=1}^K \mathcal{P}s_{k;\beta_k,\alpha}$. Then we have*

$$\lim_{\alpha\to 0}\|\mathcal{L}_\partial s_\alpha\|_{B(\mathbb{M}^n)} / \|s_\alpha\|_{B(\mathbb{M}^n)} = \left\|\left(\bigoplus_{k=1}^K A_k\right)(r_k)\right\|_{B(\mathbb{R}^n)} / F((s_k)) \tag{8.37}$$

where $A_k = \mu(\beta_k)\partial\Phi[\phi_k,\beta_k]$.

We shall show now that the lower bound of the theorem holds. Pick a sequence $x_m \in \mathbb{T}^n$ for which

$$\left\|\bigoplus_{k=1}^K \mu_k(\phi_k^{-1}(x_m))\partial\Phi[\phi_k,\phi_k^{-1}(x_m)]\right\| \to \sup_{x\in\mathbb{T}^n}\left\|\bigoplus_{k=1}^K \mu_k(\phi_k^{-1}(x))\partial\Phi[\phi_k,\phi_k^{-1}(x)]\right\|, \quad (m\to\infty). \tag{8.38}$$

Now, for each m, pick a sequence of K-touples of functions $s_{k,l,m} \in \Pi S(\mathbb{H}^n)$, $s_{k,l,m} = \mathcal{W}_g r_{k,l,m}$, $k = 1,\ldots,K$, $l = 1,2,\ldots$, such that

$$F((s_{k,l,m})_{k=1,\ldots,K}) = 1, \tag{8.39}$$

and

$$\left\|\sum_{k=1}^{K} \mathcal{L}_k s_{k,l,m}\right\|_{B(\mathbb{H}^n)} \to \left\|\bigoplus_{k=1}^{K} \mu_k(\phi_k^{-1}(x_m))\partial\Phi[\phi_k, \phi_k^{-1}(x_m)]\right\|, \quad (m\to\infty) \tag{8.40}$$

Such a sequence exists by density. In addition we may suppose that each $s_{k,l,m}$ is supported in some (different) strip $\{1/c \le a \le c\}$. Pick a sequence $\alpha_p > 0$, $\alpha_p \to 0$ as $p \to \infty$. Consider the dilated and periodized functions

$$s^\circ_{k,l,m,p} = \sum_{q\in\mathbb{Z}^n} \alpha_p^\gamma s_{k,l,m}([\cdot - \phi_k^{-1}(x_m) + q]/\alpha_p, \cdot/\alpha_p) = \alpha_p^{n+\gamma}\mathcal{P}T^\sharp_{\phi_k^{-1}(x_m)} D^\sharp_{\alpha_p} s_{k,l,m}, \tag{8.41}$$

and set $u^\circ_{l,m,p} = \sum_{k=1}^K s^\circ_{k,l,m,p}$ Then $u^\circ_{l,m,p} \in S(\mathbb{M}^n)$ and

$$\left\|u^\circ_{l,m,p}\right\|_{B(\mathbb{M}^n)} \to 1, \quad (l\to\infty) \tag{8.42}$$

and

$$\left\|\mathcal{L}_\partial u^\circ_{l,m,p}\right\|_{B(\mathbb{M}^n)} \to \left\|\bigoplus_{k=1}^{K} \mu_k(\phi_k^{-1}(x_m))\partial\Phi[\phi_k, \phi_k^{-1}(x_m)]\right\|, \quad (p\to\infty) \tag{8.43}$$

Moreover for l large enough we have $u^\circ_{l,m,p} \in \Pi_j S(\mathbb{M}^n)$. Therefore we can find a diagonal sequence $u^\circ_j \in \Pi_j B(\mathbb{M}^n)$ for which

$$\left\|u^\circ_j\right\|_{B(\mathbb{M}^n)} \to 1 \tag{8.44}$$

and

$$\left\|\mathcal{L}_\partial u^\circ_j\right\|_{B(\mathbb{M}^n)} \to \sup_{x\in\mathbb{T}^n}\left\|\bigoplus_{k=1}^{K} \mu_k((\phi_k^{-1}(x))\partial\Phi[\phi_k, \phi_k^{-1}(x)]\right\|. \tag{8.45}$$

This shows that

$$\|\mathcal{L}_\partial\| \ge \sup_{x\in\mathbb{T}^n}\left\|\bigoplus_{k=1}^{K} \mu_k(\phi_k^{-1}(x))\partial\Phi[\phi_k, \phi_k^{-1}(x)]\right\|. \tag{8.46}$$

The same argument applies to $(\mathcal{L}^m)_\partial$ which shows the lower bound of the essential spectral radius.

We now prove the upper bound for $\mathcal{L}_\partial$. We now shall prove that

$$\|\mathcal{L}_\partial \Pi_j\|_{\Pi B(\mathbb{M}^n)\to B(\mathbb{M}^n)} \le c \sup_{x\in\mathbb{T}^n}\left\|\bigoplus_{k=1}^{K} \mu_k((\phi_k^{-1}(x))\partial\Phi[\phi_k, \phi_k^{-1}(x)]\right\| \tag{8.47}$$

with a c not depending on $\mathcal{L}$. The theorem follows by looking at $(\mathcal{L}^m)_\partial$.

We need some notation. For each $l \in \mathbb{N}$ consider the grid $\Gamma_l = 2^{-l}\mathbb{Z}^n \cap \mathbb{T}^n$. Γ_l is an abelian group in the obvious way (quotient of $\mathbb{Z}^n$ by $2^l\mathbb{Z}^n$). Here we have again identified the torus with $[0,1)^n$. Consider the hypercubes of equal size centered localizationese grid points. They cover $\mathbb{T}^n$ and they intersect only on sets of measure zero. Call these cubes Q_σ, $\sigma \in \Gamma_l$ and their centers x_σ. Note that they depend on l but we do not write this dependency explicitly for notational simplicity. Now consider the set P_σ, of pre images

$$P_\sigma = \bigcup_{k=1}^{K} \phi_k^{-1} Q_\sigma \tag{8.48}$$

(we work modulo $[0,1)^n$). The sets P_σ have pair-wise disjoint interiors. Now fix $j \in \mathbb{N}$. We define sets in $\mathbb{M}^n$ as follows. The set of points with positions in $b \in P_\sigma$ and scales $0 < a \leq 2^{-j}$ shall be denoted by $\tilde{P}_\sigma$ and analogously for $\tilde{Q}_\sigma$. Note that these sets depend on j and l but we do again not write it for the sake of notational simplicity. In the same vain we denote the associated projection multipliers given by the characteristic functions, with the same symbols. Thus e.g. $\tilde{P}_\sigma$ stands for the multiplier with the characteristic function of the set $\tilde{P}_\sigma$.

Step 1: For σ, $\sigma' \in \Lambda_l$ that are not next or next nearest neighbors, we have

$$\lim_{j\to\infty} \left\| \tilde{Q}_{\sigma'} \mathcal{L}_\partial \tilde{P}_\sigma \right\| = 0. \tag{8.49}$$

This follows easily from the locality property of the kernel of $\mathcal{L}_\partial$. Therefore, we need only consider the following diagonalized operator

$$\sum_{\sigma\in\Gamma_l} \sum_{\rho\in\{-1,0,+1\}^n} \tilde{Q}_{\sigma+\rho} \mathcal{L}_\partial \tilde{P}_\sigma, \tag{8.50}$$

Note that every $\sigma \in \Gamma_l$ can be written in a unique way as $\lambda + \rho$ with $\lambda \in \{0,1\}^n$ and $\rho \in \Gamma_{l-1}$.

Step 2: We have

$$\begin{aligned} \left\| \sum_{\sigma\in\Gamma_l} \sum_{\rho\in\{-1,0,+1\}^n} \tilde{Q}_{\sigma+\rho} \mathcal{L}_\partial \tilde{P}_\sigma \right\| &\leq 6^n \max_{\delta\in\{0,1\}^n} \max_{\rho\in\{-1,0,+1\}^n} \max_{\sigma\in\Gamma_{l-1}} \left\| \tilde{Q}_{\sigma+\rho+\alpha} \mathcal{L}_\partial \tilde{P}_{\sigma+\delta} \right\| && (8.51) \\ &\leq c \max_{\delta\in\{0,1\}^n} \max_{\sigma\in\Gamma_{l-1}} \left\| \mathcal{L}_\partial \tilde{P}_{\sigma+\delta} \right\| && (8.52) \end{aligned}$$

Indeed, we may write

$$\sum_{\sigma\in\Gamma_l} \sum_{\rho\in\{-1,0,+1\}^n} \tilde{Q}_{\sigma+\rho} \mathcal{L}_\partial \tilde{P}_\sigma = \sum_{\delta\in\{0,1\}^n} \sum_{\rho\in\{-1,0,+1\}^n} \sum_{\sigma\in\Gamma_{l-1}} \tilde{Q}_{\sigma+\rho+\delta} \mathcal{L}_\partial \tilde{P}_{\sigma+\delta}. \tag{8.53}$$

Therefore we have, since the first sums are over $2^n 3^n$ terms

$$\left\| \sum_{\sigma\in\Gamma_l} \sum_{\rho\in\{-1,0,+1\}^n} \tilde{Q}_{\sigma+\rho} \mathcal{L}_\partial \tilde{P}_\sigma \right\| \leq 6^n \max_{\delta\in\{0,1\}^n} \max_{\rho\in\{-1,0,+1\}^n} \left\| \sum_{\sigma\in\Gamma_{l-1}} \tilde{Q}_{\sigma+\rho+\delta} \mathcal{L}_\partial \tilde{P}_{\sigma+\delta} \right\|. \tag{8.54}$$

Now for fixed l but j small enough, the sets $\tilde{P}_{\sigma+\delta}$ have pairwise disjoint bases for fixed δ. The same holds for the sets $\tilde{Q}_{\sigma+\rho+\delta}$ for fixed δ, ρ. We now may invoke lemma 5.6 to conclude.

The next lemma allows us then to conclude.

Lemma 8.7 *Let ϕ_k, $k = 1, \ldots, K$ be uniformly smooth diffeomorphisms. For $\beta \in \mathbb{R}^n$, define β_k through $\phi_k(\beta_k) = \beta$. Let $\Pi_{k,\epsilon}$ denote the multiplier with the characteristic function of $\{|b - \beta_k| \leq \epsilon, 0 < a \leq \epsilon\}$ and set $\Pi_\epsilon = \sum_k \Pi_{k,\epsilon}$. Let $\mathcal{F} = Op[L]$ with L given by (compare 8.3)*

$$L(b,a;b',a') = \sum_{k=1}^K L_k(b,a;b',a') = \sum_{k=1}^K \mu_k(b') \frac{1}{a'^n} D_k \left(b'; \frac{b - \phi_k(b')}{a'}, \frac{a}{a'} \right), \tag{8.55}$$

with $D_k(b';b,a) = \mathcal{W}[g, \partial\Phi[\phi_k, b']g](b,a)$. It follows that

$$\|\mathcal{F}\Pi_\epsilon\| \leq \left\| \bigoplus_k Op[\Lambda_k] \right\| + \rho(\epsilon) \tag{8.56}$$

where $\rho(\epsilon) \to 0$ as $\epsilon \to 0$ and ρ may be chosen independent of the β. The symbols Λ_k are given by

$$\Lambda_k(b,a;b',a') = \frac{\mu_k(\beta_k)}{a'^n} D_k \left(\beta_k; \frac{b - \beta - \partial[\phi_k, \beta_k](b' - \beta_k)}{a'}, \frac{a}{a'} \right) \tag{8.57}$$

Proof. Around each β_k we may use lemma 7.3 to write

$$L_k(b,a;b',a') = \frac{\mu_k(\beta_k)}{a'^n} D_k\left(\beta_k; \frac{b-\phi_k(b')}{a'}, \frac{a}{a'}\right) + |b-\beta_k|\, R_k(b,a;\phi_k(b'),a'). \tag{8.58}$$

The remainder $r_k = |b-\beta_k|\, R_k$ is such that

$$\|Op[r_k]\Pi_{k,\epsilon}\| \le \epsilon \,\|Op[r_k]\| \le c\epsilon. \tag{8.59}$$

Here c may be chosen uniformly for all β_k. We therefore need only consider the first term in 8.58 which we call T_k. We have the following factorization into a convolution operator with $P_k(b,a) = \mu_k(\beta_k)D_k(\beta_k;b,a)$ and the diffeomorphism acting on the b variable only

$$Op[T_k] = P_k * \Psi[\phi_k] = A[P_k]\Psi[\phi_k] \tag{8.60}$$

If we write $\delta\phi_k$ for the linearization of ϕ_k around β_k, then we have

$$Op[T_k]\Pi_{k,\epsilon} = A[P_k]\Psi[\delta\phi_k]\Psi[(\delta\phi_k)^{-1}\phi_k]\Pi_{k,\epsilon} \tag{8.61}$$

Now the mapping

$$\Gamma_\epsilon : B(\mathbb{H}^n)^K \to B(\mathbb{H}^n)^K, \quad (s_k) \mapsto (\Psi[(\delta\phi_k)^{-1}\phi_k]\Pi_{k,\epsilon} s_k) \tag{8.62}$$

satisfies by lemma 5.5

$$\|\Gamma_\epsilon\| \le 1 + \rho'(\epsilon), \tag{8.63}$$

where $\rho'(\epsilon) \to 0$ as $\epsilon \to 0$ independent of β. If we denote by Ξ the operator $(s_k) \mapsto \sum_k s_k$, and by Λ the operator with components Λ_k, then we have

$$\left\| \bigoplus_{k=1}^{K} Op[T_k]\Pi_{k,\epsilon} \right\| = \|\Xi\Lambda\Gamma_\epsilon\| \le \|\Xi\Lambda\|\,\|\Gamma_\epsilon\|, \tag{8.64}$$

and the proof is finished. □

This shows the theorem. □

8.d A simpler upper bound.

Let $\gamma_K : \mathbb{R}_+^K \to \mathbb{R}_+$ denote the function for which $F((s_k)) = \gamma_K((\|s_k\|))$. As we have seen, it defines a norm on $\mathbb{R}^K$ via $|x| = \gamma_K((|x_k|))$. Denote by γ'_K the dual norm, defined through ($x = (x_k)$)

$$\gamma'_K((x_k)) = \sup_{\gamma_K(y)=1} |(x \mid y)| \tag{8.65}$$

We then have clearly under the same hypothesis as in theorem 8.3

$$\rho_{ess}(\mathcal{L}) \le \limsup_{m\to\infty} \sup_{x\in\mathbb{T}^n} \gamma'_{K^m}\left(\left(\left|\mu_{m,k}(\phi^{-1}_{m,k}(x))\right| \left\|\partial\Phi[\phi_{m,k},\phi^{-1}_{k,m}x])\right\|^{1/m}\right.\right.$$

(8.66)

In particular this proves theorem 2.2.

9 The case $\Lambda_\gamma(\mathbb{T}^n)$.

The theorem that we have proved in the previous section applies a priori to $\lambda_\gamma(\mathbb{T}^n)$ but not to $\Lambda_\gamma(\mathbb{T}^n)$ because in the latter space $S_0(\mathbb{T}^n)$ is not dense. Therefor in particular condition (*iii*) of theorem 3.1 does not hold. However we have

Theorem 9.1 *The formula of theorem 8.3 holds for $\Lambda_\gamma(\mathbb{T}^n)$, $\gamma \in \mathbb{R}$.*

Note that in the case of Λ_γ we have that

$$\gamma((\|s_k\|)) = \max_k \|s_k\| = . \tag{9.1}$$

Proof. The upper bound holds without any change as the proof of theorem 3.1 and the proof of theorem 8.3 show. To proof the lower bound we invoke theorem 3.3. We therefore have to construct the spaces V_j^m. We will do this first for the family of operators $(\mathcal{L}^m)_\partial$ and we will argue why the same spaces work for $(\mathcal{L}_\partial)^m$, which differs only by a small scale smoothing operator. Denote for fixed x and m the linearized diffeomorphism in wavelet space by

$$\partial\mathcal{L}_k = \partial\mathcal{L}_k(x,m) = \mu_{m,k}(\phi_{m,k}^{-1}(x))\partial\Phi[\phi_{m,k},\phi_{m,k}^{-1}(x)]^\sharp. \quad k = 1,\ldots,K^m. \tag{9.2}$$

That is, $\partial\mathcal{L}_k = Op[D_k]$ with

$$D_k(b,a;b',a') = \mu_{m,k}(\phi_{m,k}^{-1}(x))\frac{1}{a'^n}P\left(\frac{b-\delta\phi_{k,m}(b')}{a'},\frac{a}{a'}\right) \tag{9.3}$$

where $\delta\phi_{k,m}$ is the linearization of ϕ around $\phi_{m,k}^{-1}(x)$ and $P(b,a) = \mathcal{W}[g,\partial\Phi[\phi_{m,k},\phi_{m,k}^{-1}(x)]g](b,a)$.

Step 1: There is a constant $c > 0$ such that for all m and all x there is a sequence of vectors $s_{k,l} \in \Pi S(\mathbb{H}^n)$, each being supported by some strip $\{1/c_l \leq a \leq c_l\}$ such that

$$\max_k \|s_{k,l}\| = 1, \tag{9.4}$$

and that

$$\lim_{l\to\infty}\left\|\sum_{k=1}^{K^m}\partial\mathcal{L}_k s_{k,l}\right\| \to c\left\|\bigoplus_{k=1}^{K^m}\partial\mathcal{L}_k\right\|. \tag{9.5}$$

Since $S(\mathbb{H}^n)$ is not dense in $\Lambda_\gamma(\mathbb{H}^n)$ we need an extra argument. However, there clearly is a sequence $r_{k,l} \in \Pi\Lambda_\gamma(\mathbb{H}^n)$, $l = 1,2,\ldots$, $k = 1,2,\ldots,K^m$ with $\max_k \|r_{k,l}\| = 1$ such that

$$\left\|\sum_{k=1}^{K^m}\partial\mathcal{L}_k r_{k,l}\right\| \to \left\|\bigoplus_{k=1}^{K^m}\partial\mathcal{L}_k\right\|, \quad (l\to\infty). \tag{9.6}$$

Let P_w be the multiplier with the characteristic function of $\{a + 1/a + |b| \leq w\}$ and let $E_w = \Pi P_w \Pi$. We now set

$$v_{k,l,w} = E_w r_{k,l}. \tag{9.7}$$

Clearly $v_{k,l,w} \in \Pi S(\mathbb{H}^n)$, and moreover, it is supported by some strip in the halfspace thanks to the localization of the reproducing kernel. We have $\|E_w\| \leq c_1$ and thus

$$\|v_{k,l,w}\| \leq c_1 \|r_{k,l}\| \tag{9.8}$$

Again by the locality of the reproducing kernel we have for all functions $f \in \Pi\Lambda_\gamma(\mathbb{H}^n)$ that pointwise

$$\lim_{w\to\infty} E_w f(b,a) = f(b,a). \tag{9.9}$$

Now, since $\|f\| = \sup a^{-\gamma} f$ we have

$$\liminf_{w\to\infty} \|v_{k,l,w}\| = \liminf_{w\to\infty} \|E_w r_{k,l}\| \geq \|r_{k,l}\|. \tag{9.10}$$

For the same reason, since the kernel of $\partial\mathcal{L}_k$ are localized too we have

$$\liminf_{w\to\infty} \left\| \sum_{k=1}^{K^m} \partial\mathcal{L}_k v_{k,l,w} \right\| \geq \left\| \sum_{k=1}^{K^m} \partial\mathcal{L}_k r_{k,l} \right\|. \tag{9.11}$$

We may extract a diagonal sequence $s_{k,p}$ from $v_{k,l,w}$, normalize it in such a way that $\max_k \|s_{k,p}\| = 1$ and that

$$\lim_{p\to\infty} \left\| \sum_{k=1}^{K^m} \partial\mathcal{L}_k s_{k,p} \right\| = c \left\| \bigoplus_{k=1}^{K^m} \mu_{m,k}(\phi_{m,k}(x)) \partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)] \right\|. \tag{9.12}$$

Step 2: There is a sequence of functions $s_l \in \Pi S(\mathbb{H}^n)$, such that the following holds.

- Their wavelet transforms have disjoint supports $\Xi_l \subset \mathbb{H}^n$ such that the ratios $a/a' + a'/a$ with $a \in \Xi_l$ and $a' \in \Xi_{l'}$ tend to ∞ as l, $l' \to \infty$ with $l \neq l'$.
- we have $\|s_l\| = 1$
- The following limit holds with c as above

$$\lim_{l\to\infty} \|(\mathcal{L}^m)_\partial s_l\| \to c \sup_{x\in\mathbf{T}^n} \left\| \bigoplus_{k=1}^{K^m} \mu_{m,k}(\phi_{m,k}^{-1}(x)) \partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)] \right\|. \tag{9.13}$$

Indeed, by homogeneity of the norm, we may dilate the $s_{k,l}$ in such a way, that the supports for different l satisfy the above requirement. Then, periodizing a suitable sequence of superpositions as before in the proof of theorem 8.3 will do the job.

Step 3: Construction of the spaces V_j^m. Let us define V_j^m to be the closure in $\Lambda_\gamma(\mathbb{M}^n)$ of the space spanned by those s_l above, for which l is large enough (= that are living at small enough scale) so that $\Pi_j s_l = s_l$. The vectors in V_j^m have supports in a collection of disjoined strips Ξ_l whose relative scale ratio tends to infinity at small scale. If we decompose $r \in V_j^m$ into components r_l, each having support in one of these strips

$$r \in V \Rightarrow r = \sum r_k, \quad \text{with } \operatorname{supp} r_k \subset \Xi_k, \tag{9.14}$$

then we have by lacunarity

$$\|r\| = \sup_k \|r_k\|. \tag{9.15}$$

Now, the transfer operator $(\mathcal{L}^m)_\partial$ respects the band structure in wavelet space. More precisely, it maps the space of functions with support in Ξ_l to a space of functions supported by the strip

$$\Xi_l' = \{a : \exists a' \in \Xi_l, c_2^{-1} a' \leq a \leq c_2 a'\}, \tag{9.16}$$

with c_2 independent of l. Thus for l large enough the bands Ξ'_l have disjoint supports. Therefore we have for $r \in V_j^m$, j large enough

$$\|(\mathcal{L}^m)_\partial r\| = \left\|\sum_k (\mathcal{L}^m)_\partial r_k\right\| = \sup_k \|(\mathcal{L}^m)_\partial r_k\| \tag{9.17}$$

Now, by construction of V_j^m we have for any sequence $r_l \in V_j^m$, $\|r_l\| = 1$, with support in Ξ_l that

$$\lim_{l\to\infty} \|(\mathcal{L}^m)_\partial r_l\| \to c \sup_{x\in\mathbf{T}^n} \left\|\bigoplus_{k=1}^{K^m} \mu_{m,k}(\phi_{m,k}^{-1}(x))\partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)]\right\|. \tag{9.18}$$

Therefore in particular for j large enough we have for all $r_l \in V_j^m$, $\|r_l\| = 1$, with support in Ξ_l that

$$\|(\mathcal{L}^m)_\partial r_l\| \geq (c/2) \sup_{x\in\mathbf{T}^n} \left\|\bigoplus_{k=1}^{K^m} \mu_{m,k}(\phi_{m,k}^{-1}(x))\partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)]\right\|. \tag{9.19}$$

Now suppose we are given a sequence $u_j \in V_j^m$ with $\|u_j\| = 1$. We decompose it into lacunary components $u_j = \sum_k r_{j,k}$ with $r_{j,k}$ supported by Ξ_k. It follows from 9.15 that for all j there is some l such that

$$\|r_{j,l}\| \geq 1/2. \tag{9.20}$$

Therefore, by 9.17 and 9.19 we have for j large enough

$$\|(\mathcal{L}^m)_\partial u_j\| \geq \|(\mathcal{L}^m)_\partial r_{j,l}\| \geq (1/4)c \sup_{x\in\mathbf{T}^n} \left\|\bigoplus_{k=1}^{K^m} \mu_{m,k}(\phi_{m,k}^{-1}(x))\partial\Phi[\phi_{m,k}, \phi_{m,k}^{-1}(x)]\right\|. \tag{9.21}$$

Step 4: Conclusion. We have constructed a space V_j^m for $(\mathcal{L}^m)_\partial$ and not for $(\mathcal{L}_\partial)^m$ as we should in order to use theorem 3.3. However we may use once more step 4 of the proof of theorem 8.2 to conclude. □

10 Extensions.

We end this paper with two remarks concerning possible extension of the established theorems.

first, we can consider Fourier integral operators I, that are more general than mere transfer operators. The only condition we need, is that modulo smoothing we have

$$I^\sharp = \sum_k \Omega[D_k]\Psi[\phi_k]M[\mu_k] \tag{10.1}$$

where D_k may be any smooth mapping $\mathbb{T}^n \to S(\mathbb{H}^n)$.

The second extensions concerns matrix valued multipliers. Se e.g. [CL96] for a detailed discussion and recent results. In that case we are given matrix valued functions $\mathbb{T}^n \to GL(n')$ acting in the obvious way on $B(\mathbb{T}^n)^{n'}$. The theorems remain valid if we replace in all theorems the local linearization at x by

$$\mu_{k,m}(\phi_{k,m}^{-1}(x)) \otimes \partial\Phi[\phi_{m,k}, \phi_{k,m}^{-1}(x)] \tag{10.2}$$

and compute their norms when acting in the obvious way on $B(\mathbb{R}^n)^{n\prime}$. The proofs can be adopted easily and we leave the details to the reader.

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