# Cohomology and rigidity of Fuchsian groups. *

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# 1 Introduction

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a cocompact Fuchsian subgroup, acting by orientation preserving projective transformations on the unit circle $S^1$. D.Sullivan ([1]) proved that the action of $\Gamma$ on $S^1$ is structurally stable. Namely, if $\tilde{\Gamma}$ is the image of a representation of $\Gamma$ into $Diff^1(S^1)$ such that the generators of $\tilde{\Gamma}$ are sufficiently $C^1$ close to those of $\Gamma$, then there is a homeomorphism $h$ of $S^1$ which conjugates the two actions: $\tilde{\Gamma} = h^{-1}\Gamma h$. The conjugacy $h$ is in general only Hölder continuous. In fact, if we perturb $\Gamma$ within $PSL(2,\mathbb{R})$, the conjugacy $h$ is absolutely continuous with respect to the Lebesgue measure if and only if $h$ itself is projective. This is the Mostow rigidity ([2]). The perturbation space is exactly the local Techmüller space.

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In [3], E.Ghys proved that if $\tilde{\Gamma} \subset Diff^{\infty}(S^1)$ is a perturbation of $\Gamma$ into $C^{\infty}$ diffeomorphisms of the circle such that the generators of $\tilde{\Gamma}$ are sufficiently $C^1$ close to those of $\Gamma$, then there exists another cocompact Fuchsian group $\Gamma_1$ and a $C^{\infty}$ diffeomorphism $h$ such that $\tilde{\Gamma} = h^{-1}\Gamma_1 h$. Ghys's argument relies on a suspension construction, which translates the action on the circle to an Anosov flow on $PSL(2,\mathbb{R})/\Gamma$. Then he shows that the weak stable foliation of the Anosov flow preserves a projective structure, which can be translated back on the circle.

We realized that the obstruction to the existence of projective structure invariant under the perturbed action is its Schwarzian cocycle, and smooth rigidity is equivalent to this cocycle being a coboundary. The vanishing of the cohomology of this twisted cocycle is, in turn, an elementary corollary of one of the authors duality constructions ([4], [5]).

Hence follows the main theorem:

**Theorem 1.1** *Let $\Gamma \subset PSL(2,\mathbb{R})$ be a cocompact lattice. Let $\tilde{\Gamma}$ be a representation of $\Gamma$ into $Diff^3(S^1)$ which is sufficiently $C^1$ close to $\Gamma$ among a set of finitely many generators. Then there exists another cocompact lattice $\Gamma_1 \subset PSL(2,\mathbb{R})$ and a $C^3$ diffeomorphism $h \in Diff^3(S^1)$ such that $\tilde{\Gamma} = h^{-1}\Gamma_1 h$.*

We learned that recently Ghys ([6]) has extended his result to a global one. The main motivation for giving our proof is to exhibit the intricate geometric structure associated to these actions by a simple cocycle. The advantage of our approach is the chance to generalize to higher dimensional case, which might be pursued in a forthcoming work.

# 2 The Schwarzian cocycle.

Let $x(t)$ and $y(t)$ be two $C^3$ curves in $S^1$ which differ by a transformation $PSL(2,\mathbb{R})$:

$$y(t) = \frac{ax(t)+b}{cx(t)+d} \tag{1}$$

where $ad - bc = 1$. To find a differential invariant, let us eliminate constants $a$ and $b$ from the relations resulting from the differentiation of ( 1).

$$y'(t) = \frac{x'(t)}{(cx(t)+d)^2}$$

$$\frac{y''(t)}{y'(y)} = \frac{x''(t)}{x'(t)} - \frac{2}{x(t)+d/c}.$$

Hence,

$$\frac{x+d/c}{2} = \frac{x'y'}{x''y' - x'y''}.$$

Differentiate once more to eliminate all explicit dependence on the constants. The result is the following relation:

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2.$$

Definition: For a diffeomorphism $f \in Diff^3(S^1)$, its Schwarzian derivative $S(f)$ is defined by

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \qquad (2)$$

The Schwarzian derivative was already known to Lagrange ([7]) and Klein ([8]) and plays a key role in the theory of functions of one complex variable. It owes its universality to the following properties:

1. It is projectively invariant; in fact, $S(f) = S(g)$ if and only if

   $$g(x) = \frac{af(x)+b}{cf(x)+d}$$

   for some numbers $a, b, c, d$ such that $ad - bc = 1$.

2. $S(f) = 0$ if and only if $f \in SL(2, \mathbb{R})$.

3. Cocycle property: $S(f \circ g)(x) = S(f)(g(x))[g'(x)]^2 + S(g)(x)$.

We observe the following fact.

**Proposition 2.1** *A diffeomorphism $f \in Diff^3(S^1)$ is conjugate to a projective transformation $g \in SL(2, \mathbb{R})$ via a $C^3$ diffeomorphism $h$ if and only if its Schwarzian $S(f)$ is cohomologous to 0, i.e. there exists a continuous function $\sigma(x)$ on $S^1$ such that*

$$S(f)(x) = \sigma(f(x))[f'(x)]^2 - \sigma(x) \tag{3}$$

Proof: 1. If $f = hgh^{-1}$ for some $h \in Diff^3(S^1)$, then $fh = hg$. Taking the Schwarzian derivative on both sides gives

$$S(f)(h(x))[h'(x)]^2 = S(h)(g(x))[g'(x)]^2 - S(h)(x).$$

Hence,

$$S(f)(x) = \frac{S(h)(h^{-1}f(x))}{(h'(h^{-1}f(x)))^2}[f'(x)]^2 - \frac{S(h)(h^{-1}(x))}{(h'(h^{-1}(x)))^2}.$$

2. If ( 3) is true. Consider the second order linear ordinary differential equation on $S^1$

$$u'' + \frac{1}{2}\sigma(x)u = 0. \tag{4}$$

By the existence and uniqueness theorem, there exists a unique solution $u(t) \in C^2(S^1)$ with the initial condition $u(0) = 1$, $u'(0) = 0$. Let

$$h(x) = \int_0^x u^{-2}(t)dt.$$

Then, $h(x) \in Diff^3(S^1)$ and, moreover,

$$S(h)(x) = \sigma(x), \frac{S(h^{-1})(x)}{((h^{-1})'(x))^2} = -\sigma(h^{-1}(x)).$$

Therefore,

$$\frac{S(hfh^{-1})(x)}{((h^{-1})'(x))^2} = S(f)(h^{-1}(x)) + \frac{S(h^{-1}(x))}{((h^{-1})'(x))^2} + S(h)(fh^{-1}(x))[f'(h^{-1}(x))]^2 =$$

$$= S(f)(h^{-1}(x)) + \sigma(h^{-1}(x)) + \sigma(fh^{-1}(x))[f'(h^{-1}(x))]^2 = 0.$$

Hence, $hfh^{-1}$ is a projective transformation. $\triangle$

Now, let $\Gamma \subset PSL(2,\mathbb{R})$ be a cocompact lattice. And let $\tilde{\Gamma} \subset Diff^3(S^1)$. If $\tilde{\Gamma}$ is sufficiently $C^1$ close to $\Gamma$ among a set of finitely many generators, then by [1] there exists a homeomorphism $h \in Homeo(S^1)$ such that $\tilde{\Gamma} = h^{-1}\Gamma h$. Via the conjugacy $h$ the Schwarzian cocycle $S$ of $\tilde{\Gamma}$ can be pulled back to a twisted cocycle $T$ of the $\Gamma$ action: $\Gamma \times S^1 \to \mathbb{R}$,

$$T(\gamma, x) = S(h^{-1}\gamma h, h^{-1}(x)),$$

which satisfies

$$T(\gamma_1\gamma_2, x) = T(\gamma_2, x) + T(\gamma_1, \gamma_2(x))[\tilde{\gamma_2'}(h^{-1}(x))]^2 =$$

$$= T(\gamma_2, x) + T(\gamma_1, \gamma_2(x))[\gamma_2'(x)]^2\delta(\gamma_2, x),$$

where $\delta(\gamma, x) = [\tilde{\gamma}'(h^{-1}(x))/\gamma'(x)]^2$ is a $\Gamma$ cocycle (i.e. $\delta(\gamma_1\gamma_2, x) = \delta(\gamma_1, \gamma_2 x)\delta(\gamma_2, x)$). Moreover, if the perturbation $\tilde{\Gamma}$ is $C^1$ close to $\Gamma$ among its finitely many generators, then the cocycle $\delta$ is close to 1 among those generators.

To finish the proof of Theorem 1.1 all we have to do now is to prove that $T$ is a continuous coboundary.

# 3 Triviality of the Schwarzian.

We start with recalling some definitions and introducing some notations.

Recall that if a group $G$ acts on a space $X$ by continuous transformations then the continuous cocycle of this action with coefficients in $\mathbb{R}$ is a map $\alpha : G \times X \to \mathbb{R}$ such that $\alpha(g, m)$ is continuous in $m$ for every $g \in G$ and

$$\alpha(g_1g_2, m) = \alpha(g_1, g_2(m))\alpha(g_2, m)\forall g_1, g_2 \in G, m \in X.$$

Two cocycles $\alpha$ and $\beta$ will be called cohomologous if there exists a continuous function $P : X \to \mathbb{R}$ such that

$$\beta(g, m) = P(gm)^{-1}\alpha(g, m)P(m).$$

By $\mathrm{H}(G, X)$ we will denote a set of equivalence classes of cocycles.

Fix $\alpha \in \mathrm{H}(G, X)$. A map $\beta(g, m) : G \times M \to \mathbb{R}$ will be called continuous twisted cocycle if $\beta(g, m)$ is continuous in $m$ for every $g \in G$ and

$$\beta(g_1g_2, m) = \beta(g_2, m) + \alpha(g_2, m)^{-1}\beta(g_1, g_2m), \forall g_1, g_2 \in G, m \in X.$$

Two cocycles $\beta$ and $\beta_1$ will be called cohomologous if and only if there exists continuous function $P : X \to \mathbb{R}$ such that

$$\beta(g, m) - \beta_1(g, m) = P(m) - \alpha(g, m)^{-1} P(gm), \forall g \in G, m \in X.$$

We will denote the set of equivalence classes of twisted cocycles by $\mathrm{H}(G, M, \alpha)$. It is easy to check that if $\alpha_1$ and $\alpha_2$ are cohomologous then $\mathrm{H}(G, M, \alpha_1)$ and $\mathrm{H}(G, M, \alpha_2)$ are isomorphic.

We put on $\mathrm{H}(G, X)$ ($\mathrm{H}(G, M, \alpha)$) factor topologies induced from the set of all cocycles (twisted cocycles) endowed with the topology of uniform convergence on compact subsets of $G \times X$.

## 3.1 Duality Theorem.

Suppose that $G$ acts on $X_1$ and $X_2$ in such a way that the action on $X_2$ is a factor of the action on $X_1$. Every cocycle of the action on $X_2$ can be uniquely lifted to a cocycle of the action on $X_1$. By $\mathrm{H}^{tr}(G, X_2)$ we will denote a set of those elements of $\mathrm{H}(G, X_2)$ that lift to a trivial element in $\mathrm{H}(G, X_1)$. Analogously, we define $\mathrm{H}^{tr}(G, X_2, \alpha)$ for $\alpha \in \mathrm{H}(G, X_2)$.

Let $G$ be a topological group (Lie group), $P$ and $Q$ its subgroups. Let $P$ act on $G/Q$ from the left, and $Q$ act on $P\backslash G$ from the right. Lift both actions to the actions of $P$ and $Q$ on the whole group $G$. In [4] and [5] the following results are proved:

**Theorem 3.1** *1. There is a naturally defined, continuous isomorphism*

$$K : \mathrm{H}^{tr}(Q, P\backslash G) \to \mathrm{H}^{tr}(P, G/Q).$$

*2. For any $\Delta \in \mathrm{H}^{tr}(Q, P\backslash G)$ there is a naturally defined, continuous linear isomorphism*

$$K_1 : \mathrm{H}^{tr}(Q, P\backslash G, \Delta) \to \mathrm{H}^{tr}(P, G/Q, K(\Delta)).$$

**Theorem 3.2** *Let G be a Lie group, $\Gamma$ a discrete subgroup, acting on G from the left. Then* $\mathrm{H}(\Gamma, G) = 0$.

Let $G = PSL(2, \mathbb{R})$, $P$ be the subgroup consisting of upper triangular matrices, and $\Gamma$, as before, a cocompact Fuchsian subgroup. Then the action by projective transformations is isomorphic to the right action of $\Gamma$ on $P\backslash G$. Let $\Delta_1 = (\gamma'(x))^{-2}$ be the inverse of the square of the derivative cocycle for the $\Gamma$ action. Let, $\Delta = \delta(\gamma, x)^{-1}\Delta_1$ — the pull back of the inverse of the square of the derivative cocycle for the $\tilde{\Gamma}$ action. Then the Schwarzian's pull back is an element of $\mathrm{H}(\Gamma, P\backslash G, \Delta)$.

By the Theorem 3.2 we have $\mathrm{H}(\Gamma, G, \Delta_1) = \mathrm{H}(\Gamma, G) = 0$. Thus, we have

$$\mathrm{H}(\Gamma, P\backslash G, \Delta) = \mathrm{H}^{tr}(\Gamma, P\backslash G, \Delta) = \mathrm{H}^{tr}(P, G/\Gamma, K(\Delta)).$$

We will prove that if $\Delta$ is any continuous cocycle close enough to $\Delta_1$ — the square of the derivative cocycle, then $\mathrm{H}(P, G/\Gamma, K(\Delta)) = 0$, and thus $\mathrm{H}(\Gamma, P\backslash G, \Delta) = 0$.

First of all we will calculate $K(\Delta_1)$.

## 3.2 Calculation of the dual to $\Delta_1$.

To do it we will have to outline how the map $K$ is constructed. Lift $\Delta \in \mathrm{H}(\Gamma, P\backslash G)$ to a cocycle $\tilde{\Delta} \in \mathrm{H}(\Gamma, G)$. By the Theorem 3.2 there exists a continuous function $F : G \to \mathbb{R}$, such that $\tilde{\Delta}(\gamma, g) = F(g)(F(g\gamma^{-1})^{-1})$, then $a(p, g) = F(pg)(F(g)^{-1}), p \in P, g \in G$ is $\Gamma$ right invariant (in $g$) and thus defines some cocycle $K(\Delta) \in \mathrm{H}(P, G/\Gamma)$. In [4] it is proved that this correctly defines a map $K$ that satisfies the conclusion of the Theorem 3.1(1).

$P\backslash G$ can be identified with $K$ – maximal compact subgroup of $G$. Let $\mathcal{G} = \mathcal{P} \bigoplus \mathcal{K}$ be the decomposition of the Lie algebra of $G$ into algebras tangent to $P$ and $K$ correspondingly. Define a bilinear form $B$ on $\mathcal{G}$ so that it is positive definite on $\mathcal{K}$, zero on $\mathcal{P}$, and $\mathcal{P}$ and $\mathcal{K}$ are orthogonal. Spread it on the whole $G$ by right translations. Restricting it to $K$ we get a Riemanian structure on $K$, invariant with respect to both left and right translations.

Denote $B(u, u) = \|u\|^2$. Let $v$ be a unit vector tangent to $K$ at the identity. Renormalizing $B$, we can assume that $v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then we have $\tilde{\Delta_1}(g, g_1) = = F(g_1)(F(g_1g^{-1})^{-1})$, where $F(g) = \|Ad(g^{-1})(v)\|^2$. And, we have

$$K(\Delta_1)(p, g) = F(pg)F(g)^{-1} = \|Ad(g^{-1}p^{-1})v\|^2\|Ad(g^{-1})(v)\|^{-2} = \|Ad(p^{-1})(v)\|^2.$$

Thus, we see that $K(\Delta_1)$ depends only on $p \in P$, and, thus, is just a representation of $P$. More specifically, for $p = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$, we have:

$$Ad(p^{-1})(v) = \begin{pmatrix} x^{-1} & -y \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} xy & y^2 + x^{-2} \\ -x^2 & -xy \end{pmatrix} =$$

$$= x^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} xy & y^2 + x^{-2} - x^2 \\ 0 & -xy \end{pmatrix}.$$

And since the second summand belongs to $\mathcal{P}$ we have $K(\Delta_1)(p) = \|Ad(p^{-1}(v)\|^2 = x^4$. Thus, we have calculated $K(\Delta_1)$.

## 3.3 Trivialization of cocycles twisted by $K(\Delta)$.

Let $\alpha = K(\Delta)$. Let $a = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$. Let $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, $R^{x^{-2}} = \begin{pmatrix} 1 & x^{-2} \\ 0 & 1 \end{pmatrix}$, for arbitrary $x$. Let $Z = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$.

Then we have

1. $K(\Delta_1)(a)^{-1} = 1/16$,
2. $K(\Delta_1)(a^{-1})^{-1} = 16$,
3. $K(\Delta_1)(Z)^{-1} = 4$,
4. $K(\Delta_1)(R^2)^{-1} = 1$.

Since the map $K$ is continuous, by choosing $\Delta$ sufficiently close to $\Delta_1$ we can assure that

1. $0 < |\alpha(a, m)^{-1}| < 1/2$
2. $15 < |\alpha(a^{-1}, m)^{-1}| < 17$,
3. $3 < |\alpha(Z, m)^{-1}| < 5$,
4. $0.5 < |\alpha(R^2, m)^{-1}| < 1.5$, for all $m \in G/\Gamma$.

Then we would have

$$|\alpha(a^k,m)^{-1}| = |\alpha(a,m)^{-1}\alpha(a,am)^{-1}...\alpha(a,a^{k-1}m)^{-1}| < 1/2^k.$$

Now, let $\beta \in \mathrm{H}(P,G/\Gamma,\alpha)$. Then, the series $\sum_{k=0}^{\infty}\alpha(a^k,m)^{-1}\beta(a,a^km)$ converges uniformly to a continuous function $P(m)$. Notice, that $P(m) - P(am)\alpha(a,m)^{-1} = = \beta(a,m)$. Thus, passing to an equivalent cocycle we may assume that $\beta(a,m) = 0$, $\forall m \in G/\Gamma$.

If $a_1$ commutes with $a$ then we have:

$$|\beta(a^ka_1,m)| = |\beta(a_1,m)| = |\beta(a_1,a^km)\alpha(a^k,m)^{-1}| \leq M/2^k,$$

where $M = \max_{m\in G/\Gamma}\beta(a_1,m)$. Thus $\beta(a_1,m) = 0, \forall m \in G/\Gamma$. Therefor, $\beta(p,m) = 0, \forall m \in G/\Gamma$, and $p$ —diagonal.

Then, $X^{-1}RX = R^{x^{-2}}$. And, thus we have:

$$\beta(R^{x^{-2}},m) = \beta(X^{-1}RX,m) = \beta(R,Xm)\alpha(X,m)^{-1}.$$

Denote $f(m) = \beta(R,m)$. Then:

$$\beta(R^4,m) = f(a^{-1}m)\alpha(a^{-1},m)^{-1} =$$

$$= \beta(R^2R^2,m) = \beta(R^2,m) + \beta(R^2,R^2m)\alpha(R^2,m)^{-1} =$$

$$= f(Zm)\alpha(Z,m)^{-1} + f(ZR^2m)\alpha(Z,R^2m)^{-1}\alpha(R^2m)^{-1}.$$

Let $C = max_{m\in G/\Gamma}|f(m)|$. Then,

$$max_{m\in G/\Gamma}|\beta(R^4,m)| = max_{m\in G/\Gamma}|f(a^{-1}m)\alpha(a^{-1},m)^{-1}| \geq 15C.$$

On the other hand,

$$max_{m\in G/\Gamma}|\beta(R^4,m)| = max_{m\in G/\Gamma}|\beta(R^2,m) + \beta(R^2,R^2m)\alpha(R^2,m)^{-1}| =$$

$$= max_{m\in G/\Gamma}|f(Zm)\alpha(Z,m)^{-1} + f(ZR^2m)\alpha(Z,R^2m)^{-1}\alpha(R^2,m)^{-1}| \leq$$

$$\leq 5C + 5\times 1.5C = 12.5C.$$

Therefor, we must have $C = 0$. And thus, $\beta(R,m) = 0, \forall m \in G/\Gamma$, which together with $\beta(p,m) = 0, \forall m \in G/\Gamma$, and $p$ —diagonal, proves that $\beta$ is trivial.

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