1 Introduction

We consider the 2-D surface quasi-geostrophic (QGS) equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^{\alpha}\theta = 0, \quad 0 \le \alpha \le 1$$

where $\kappa > 0$ in the case of the dissipative equations and $\kappa = 0$ for the non-dissipative equations. Here the velocity $u = (u_1, u_2)$ is determined from θ by a stream function ψ:

$$(u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right)$$

where ψ satisfies

$$(-\Delta)^{\frac{1}{2}}\psi = -\theta$$

These equations have been under intensive investigations because of mathematical importance and potential applications in meteorology and oceanography ([3],[9],[10]). As pointed out in [3], the 2-D dissipative and non-dissipative QGS equations are strikingly analogous to the 3-D Navier-Stokes and the Euler equations. Inviscid limit results for the Navier-Stokes equations with smooth and rough initial data have been established ([1],[2], [4],[5],[6]). Naturally the inviscid limit problem for the solutions of the dissipative QGS equations rises and so far we have seen no work in this direction.

We know from [3] that the QGS equations (defined on ^{2} or ^{2}) with smooth initial data admit unique classical solutions for short times and the quantity

$$\int_0^T \|\nabla \theta(\cdot, s)\|_{L^\infty} ds$$

is responsible for possible singularity formation. The control of this quantity is also important in our proof of the inviscid limit results for smooth solutions.

Both dissipative and non-dissipative QGS equations on ^{2} with L^2 initial data have weak solutions in the distribution sense ([11]). Inviscid limits for weak solutions are in general hard to obtain and in this case we only obtain a weak L^2 result without a rate. We believe that explicit rates can be given in certain negative Sobolev norms.

The second part is devoted to regularity estimates for the dissipative QGS equations. Although we use the ideas of Foias-Temam [8] and Doering-Titi [7] developed for the Navier-Stokes equations, our methods of estimating the nonlinear terms is significantly different from theirs because of the special structure of the QGS equations. We treat the norms $\|e^{\gamma t \Lambda^{\alpha}} \Lambda^{\beta} \theta\|_{L^2}$ (see notations in Section 3) and a special consequence of our estimates for $\|e^{\gamma t \Lambda^{\alpha}} \Lambda^{\beta} \theta\|_{L^2}$ is that as long as $\|\Lambda^\beta\theta\|_{L^2}^2$ remains bounded, the Fourier spectrum decays exponentially at high wave numbers. In a similar fashion as Doering and Titi [7] argue for the Navier-Stokes equations, these exponential decay estimates can be used to obtain bounds on small length scale defined through the exponential decay rate.

Finally, I would like to thank Professor Peter Constantin for his suggestions and help.

2 Inviscid Limits

We consider the 2-D dissipative ($\kappa > 0$) and the non-dissipative ($\kappa = 0$) quasi-geostrophic equations on ^{2} or ^{2}:

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa \Lambda^{2\alpha} \theta = 0, \qquad 0 \leq \alpha \leq 1$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Riesz potential operator and the velocity u is defined from the stream function $\psi = -\Lambda^{-1}\theta$ by

$$u = (u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right)$$

We use ^{2} to denote either ^{2} or ^{2}. We first consider the smooth initial data case: $\theta_0 \in H^k(^2)(k \geq 3)$. As shown by Constantin, Majda and Tabak [3], the QGS equations with smooth initial data have local (in time) smooth solutions and the Beale-Kato-Majda type blowup conditions have been obtained. More precisely,

If the initial data $\theta|_{t=0} = \theta_0 \in H^k(^2)$ for some $k \geq 3$, then both the QGS and the dissipative QGS equations have a unique smooth solution for a small time interval, respectively. Furthermore, the solution θ^{QG} of the QGS equations satisfies

$$\int_0^t \|\nabla \theta^{QG}(\cdot, s)\|_{L^\infty} ds < \infty,$$

$$\int_0^t \|\theta^{QG}(\cdot, s)\|_k^2 ds < \infty$$

for any t belonging to the existence interval $[0, T^*)$.

This proposition admits the possibility of finite-time singularity formation and consequently, the inviscid limit results for the smooth solutions are valid only for the time period before the possible breakdown. We need further estimates on the solutions.

Let θ^{QG} and θ^{DQG} be the smooth solutions of the QGS and the dissipative QGS equations with the same initial data $\theta_0 \in H^k(^2)(k \geq 3)$. Respectively, u^{QG} and u^{DQG} are the corresponding velocities and ψ^{QG} and ψ^{DQG} are the stream functions. Then for any t in the maximal time interval $[0, T^*)$ (when the smooth solutions exist),

1. The solution θ^{QG} of the QGS equation satisfies:

$$\int_{^2} G(\theta^{QG}(x,t))dx = \int_{^2} G(\theta_0)dx,$$

where G is a continuous function with $G(0) = 0$. Especially,

$$\|\theta^{QG}(\cdot, t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad 1 \leq p \leq \infty,$$

furthermore,

$$\|u^{QG}(\cdot,t)\|_{L^2} = \|\theta^{QG}(\cdot,t)\|_{L^2} = \|\theta_0\|_{L^2},$$

$$\|u^{QG}(\cdot,t)\|_{L^q} \le C_q \|\theta^{QG}(\cdot,t)\|_{L^q}, \qquad 1 < q < \infty,$$

where C_q is a constant depending on q.

2. The solution θ^{DQG} of the dissipative QGS equation obeys

$$\|\theta^{DQG}(\cdot,t)\|_{L^p} \le \|\theta_0\|_{L^p}, \qquad 1 \le p \le \infty,$$

$$\|u^{DQG}(\cdot,t)\|_{L^q} \le C_q \|\theta^{DQG}(\cdot,t)\|_{L^q}, \qquad 1 < q < \infty.$$

Proof. The proof of this proposition is classical. The L^q estimates are the classical inequality for the Calderon-Zygmund singular integrals while the L^p bounds come from energy estimates. We omit details.

vspace.27in We now state the inviscid limit theorem for smooth solutions.

Let θ^{QG} and θ^{DQG} be the smooth solutions of the QGS equations and the dissipative QGS equations with the same initial data $\theta_0 \in H^k(^2)(k \ge 3)$. If $[0, T^*)$ is the maximal time interval of smooth existence, then for any $t < T^*$,

$$\|\theta^{QG}(\cdot,t) - \theta^{DQG}(\cdot,t)\|_{L^2(^2)} \le C\kappa,$$

where C is a constant depending on θ_0 and T^* only.

The convergence rate $O(\kappa)$ is optimal. In the proof we only treat the case $^2 = ^2$ and the case $^2 = ^2$ is easier.

Proof of Theorem 2. Consider the difference

$$\theta(x,t) = \theta^{DQG}(x,t) - \theta^{QG}(x,t)$$

between the solutions of the QGS and the dissipative QGS equations and let $u(x,t)$ be the corresponding velocity difference. This difference θ satisfies

$$\frac{\partial \theta}{\partial t} + u^{DQG} \cdot \nabla\theta + u \cdot \nabla\theta^{QG} + \kappa\Lambda^{2\alpha}(\theta + \theta^{QG}) = 0$$

where $\Lambda^{2\alpha} = (-\Delta)^\alpha$. Multiplying by θ and integrating in space

$$\frac{1}{2}\frac{d}{dt}\int \theta^2 dx + \kappa \int (\Lambda^{2\alpha}\theta)\theta dx = I + II + III$$

where

$$I = \int (u^{DQG} \cdot \nabla\theta)\theta dx, \quad II = \int (u \cdot \nabla\theta^{QG})\theta dx$$

$$III = \kappa \int (\Lambda^{2\alpha}\theta^{QG})\theta dx$$

We estimate these three terms and start with the first one. Clearly the estimates in Proposition 2 guarantee that I is integrable. We show that it is actually zero. Let χ be a smooth cut-off function: $\chi(x) = 1$ if $|x| < 1$ and $\chi(x) = 0$ if $|x| > 2$ and $\chi_r = \chi(\frac{x}{r})$ for $r > 0$. Using the Dominated Convergence Theorem and the divergence theorem,

$$I = \lim_{r\to\infty} \int (u^{DQG} \cdot \nabla\theta)\theta\chi_r(x)dx = -\lim_{r\to\infty} \frac{1}{2r} \int \chi' \cdot u^{DQG}\theta^2 dx$$

since the last integral is bounded,

$$|\int \chi' \cdot u^{DQG}\theta^2 dx| \leq \int |u^{DQG}|\theta^2 dx \leq \|\theta^{DQG}\|_{L^2}\|\theta\|_{L^4}^2 \leq 4\|\theta_0\|_{L^2}\|\theta_0\|_{L^4}^2$$

we obtain $I = 0$. II and III can be estimated by using Proposition 2,

$$|II| \leq \|\nabla\theta^{QG}\|_{L^\infty}\|u\|_{L^2}\|\theta\|_{L^2} = \|\nabla\theta^{QG}\|_{L^\infty}\|\theta\|_{L^2}^2$$

$$|III| \leq \frac{\kappa^2}{2}\int (\Lambda^{2\alpha}\theta^{QG})^2 dx + \frac{1}{2}\int \theta^2 dx$$

Collecting these estimates,

$$\frac{d}{dt}\int \theta^2 dx + \kappa\|\Lambda^\alpha\theta\|_{L^2}^2 dx \leq \mathcal{P}(\sqcup)\|\theta\|_{\mathcal{L}^\in}^{\in} + \kappa^{\in}\|\theta^{\mathcal{QG}}\|_{\in\alpha}^{\in}$$

where

$$\mathcal{P}(\sqcup) = \in\|\nabla\theta^{\mathcal{QG}}(\cdot,\sqcup)\|_{\mathcal{L}^\infty} + \infty$$

By Gronwall's inequality,

$$\|\theta\|_{L^2}^2 \leq e^{\int_0^t \mathcal{P}(\int)\lceil\int}\|\theta_0\|_{L^2}^2 + \kappa^2\int_0^t e^{\int_\tau^t \mathcal{P}(\int)\lceil\int}\|\theta^{QG}\|_{2\alpha}^2 d\tau$$

Noting that $\theta_0 = 0$ and using the result of Proposition 2, especially,

$$\int_0^t \|\nabla\theta^{QG}(\cdot,s)\|_{L^\infty} ds < \infty, \quad \int_0^t \|\theta^{QG}(\cdot,s)\|_k^2 ds < \infty$$

we obtain

$$\|\theta\|_{L^2} \leq C\kappa,$$

which completes the proof of Theorem 2.

We now turn to weak solutions of these equations corresponding to L^2 initial data. We restrict ourselves to the periodic domain $^2 = [0,L]\times[0,L]$. We quote the result of Resnick [11] on the existence of weak solutions.

Let $\theta_0 \in L^2(^2)$ and $T > 0$ be arbitrarily fixed. Then there exist weak solutions $\theta^{QG} \in L^\infty([0,T]; L^2(^2))$ and $\theta^{DQG} \in L^\infty([0,T]; L^2(^2)) \cap L^2([0,T]; H^\alpha(^2))$ of the QGS and the dissipative QGS equations, respectively. That is, for each test function $\phi \in C^\infty(^2)$,

$$\int \theta^{QG}\phi dx - \int \theta_0 \phi dx - \int_0^T \int_2 \theta^{QG}(u^{QG} \cdot \nabla\phi) dx dt = 0,$$

$$\int \theta^{DQG}\phi dx - \int \theta_0 \phi dx - \int_0^T \int_2 \theta^{DQG}(u^{DQG} \cdot \nabla\phi) dx dt = 0,$$

where u^{QG} and u^{DQG} are the velocities corresponding to θ^{QG} and θ^{DQG}.

These weak solutions are constructed by using classical Galerkin approximations. The weak L^2 inviscid limit result is an easy consequence of this construction method. Let $\theta_0 \in L^2(^2)$ and θ^{QG} and θ^{DQG} be the weak solutions of the QGS and the dissipative QGS equations with the same initial data θ_0. Then for any arbitrarily fixed $T > 0$ and any $\phi \in L^2(^2)$,

$$\limsup_{\kappa \to 0} \left(\theta^{DQG}(\cdot, t) - \theta^{QG}(\cdot, t), \phi\right) = 0, \qquad \text{for any } t \leq T, \tag{1}$$

Proof Consider the n-th Galerkin approximation $\{\theta_n^{QG}\}$ and $\{\theta_n^{DQG}\}$, which are in the space S_n spanned by the Fourier modes e^{imx} with $0 < |m| \leq n$ and satisfy

$$\frac{\partial \theta_n}{\partial t} + P_n(u_n \cdot \nabla\theta_n) + \kappa\Lambda^{2\alpha}\theta_n = 0,$$

$$\theta_n|_{t=0} = P_n\theta_0,$$

where P_n is the orthogonal projection from L^2 onto S_n and $\kappa = 0$ in the case of θ^{QG}. As we know from [11], for some subsequences

$$\theta_n^{DQG} \rightharpoonup \theta^{DQG}, \quad \theta_n^{QG} \rightharpoonup \theta^{QG} \quad \text{weakly in } L^2(^2),$$

So we have for any $\epsilon > 0$ by taking large n,

$$\begin{aligned} |(\theta^{DQG}(\cdot, t) - \theta^{QG}(\cdot, t), \phi)| &\leq \epsilon + |(\theta_n^{DQG} - \theta_n^{QG}, \phi)| \\ &\leq \epsilon + \|\phi\|_{L^2}\|\theta_n^{DQG} - \theta_n^{QG}\|_{L^2} \leq \epsilon + C_n\kappa \end{aligned} \tag{2}$$

which implies (1). Here we've applied the inviscid limit result for smooth solutions to $\theta_n^{DQG} - \theta_n^{QG}$.

Since the constants C_n in the inequality (2) depends on n, we obtain no convergence rate. An explicit rate may exist in negative Sobolev norms.

3 Regularity Estimates

We consider the 2-D dissipative QGS equations with smooth initial data on the torus $^2 = [0, L] \times [0, L]$, which admits a unique classical local solution. Let θ be this solution. We will estimate the quantity

$$\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}^2$$

where the operators Λ^β and $e^{\lambda \Lambda^\alpha}$ are defined through the Fourier transform

$$\Lambda^\beta f = L^{-2} \sum_k e^{ik\cdot x} |k|^\beta \widehat{f}(k)$$

$$e^{\lambda \Lambda^\alpha} f = L^{-2} \sum_k e^{ik\cdot x + \lambda |k|^\alpha} \widehat{f}(k)$$

with $\widehat{f}(k)$ being the k-th Fourier mode of f,

$$\widehat{f}(k) = \int_{^2} e^{-ik\cdot x} f(x) dx.$$

It is easy to see from these notations that $e^{\lambda \Lambda^\alpha}$ commutes with Λ^β and partial derivatives for periodic boundary conditions considered here.

We obtain bounds for the quantity $\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}^2$, which lead to the exponential decay of the Fourier spectrum of θ. The precise estimates are

Consider the 2-D dissipative QGS equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa \Lambda^{2\alpha} \theta = 0, \qquad \kappa > 0, \quad \frac{1}{2} < \alpha \le 1 \tag{3}$$

on the 2-D torus $^2 = [0, L] \times [0, L]$. Let the initial data $\theta_0 \in H^k(^2)$ and θ be the unique smooth solution. We take β :

$$\beta > 0, \qquad \beta + 2\alpha > 2.$$

Then θ satisfies for any $\gamma > 0$

$$\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}^2 \le \frac{e^{\frac{2\gamma^2 t}{\kappa}} \|\Lambda^\beta \theta_0\|_{L^2}^2}{\left(1 - C(\|\Lambda^\beta \theta_0\|_{L^2}^2)^{N-1} \kappa^{-(M-1)} \gamma^{-2} (e^{\frac{2(N-1)\gamma^2 t}{\kappa}} - 1)\right)^{\frac{1}{N-1}}}, \tag{4}$$

which is finite for $t \in [0, t^*)$,

$$t^* = \frac{\kappa}{2(N-1)\gamma^2} \log\left(1 + \frac{\kappa^{M-1}\gamma^2}{C\|\Lambda^\beta \theta_0\|_{L^2}^{2(N-1)}}\right)$$

Here C is a constant and M, N are given by

$$M = \frac{\alpha+\sigma}{\alpha-\sigma}, \quad N = 1 + \frac{\alpha}{\alpha-\sigma}, \quad \sigma = \begin{cases} 1-\alpha, & \text{if } \beta \geq 1 \\ 2-\beta-\alpha, & \text{if } \beta \leq 1 \end{cases}$$

A special consequence of this theorem is that each Fourier mode amplitude can be individually controlled. In fact, a rough estimate gives

$$e^{2\gamma t|k|^\alpha}|k|^{2\beta}|\theta(k,t)|^2 \leq \sum_k e^{2\gamma t|k|^\alpha}|k|^{2\beta}|\theta(k,t)|^2 = L^2\|e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\|_{L^2}^2$$

Thus the k-th mode is bounded by

$$|\theta(k,t)|^2 \leq \frac{L^2}{|k|^{2\beta}} \frac{e^{\frac{\gamma^2 t}{\kappa}-2\gamma t|k|^\alpha}\|\Lambda^\beta\theta_0\|_{L^2}^2}{\left(1 - C(\|\Lambda^\beta\theta_0\|_{L^2}^2)^{N-1}\kappa^{-(M-1)}\gamma^{-2}(e^{\frac{2(N-1)\gamma^2 t}{\kappa}}-1)\right)^{\frac{1}{N-1}}}$$

for $t \in [0, t^*)$.

Doering and Titi [7] establish the exponential decay of power spectrum for the flow field of the 3-D Navier-Stokes equations. Similar analysis based on these bounds can be made to conclude that if $\|\Lambda^\beta\theta\|_{L^2}^2$ is bounded uniformly in time, then after a transient time of length $t^*/2$ the Fourier spectrum of θ decays exponentially at high wave numbers. Furthermore, the associated decay length can be defined and estimated in terms of the dissipation rates.

The main difficulty in proving the estimate (4) is how to bound the nonlinear term properly. We need the inequalities for the Calderon-Zygmund type singular integrals. We'll also use the following lemma concerning the operator Λ^s, which is proved in [11],[12].

For $s > 0$ and $1 < r < p \leq \infty$,

$$\|\Lambda^s(uv)\|_{L^r} \leq C(\|u\|_{L^p}\|\Lambda^s v\|_{L^q} + \|v\|_{L^p}\|\Lambda^s u\|_{L^q})$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and C is a constant.

Proof of Theorem 3.1. Using the equation (3),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta|^2 dx &= \int \left(e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\right)\left(\gamma e^{\gamma t\Lambda^\alpha}\Lambda^\alpha\Lambda^\beta\theta + e^{\gamma t\Lambda^\alpha}\Lambda^\beta\frac{\partial\theta}{\partial t}\right)dx \\ &= I + II + III\end{aligned}$$

where

$$I = \gamma\int \left(e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\right)\left(e^{\gamma t\Lambda^\alpha}\Lambda^\alpha\Lambda^\beta\theta\right)dx$$

$$\leq \frac{\gamma^2}{\kappa}\|e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\|_{L^2}^2 + \frac{\kappa}{4}\|e^{\gamma t\Lambda^\alpha}\Lambda^{\alpha+\beta}\theta\|_{L^2}^2,$$

$$II = -\int \left(e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\right)\left(e^{\gamma t\Lambda^\alpha}\Lambda^\beta(u\cdot\nabla\theta)\right)dx,$$

$$III = -\kappa\int \left(e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\right)\left(e^{\gamma t\Lambda^\alpha}\Lambda^{2\alpha+\beta}\theta\right)dx = -\kappa\|e^{\gamma t\Lambda^\alpha}\Lambda^{\alpha+\beta}\theta\|_{L^2}^2$$

Now we deal with the second term II. Since the operators commute,

$$II = -\int \left(\Lambda^{(\alpha+\beta)}e^{\gamma t\Lambda^\alpha}\theta\right)\left(R\Lambda^{\beta-\alpha+1}e^{\gamma t\Lambda^\alpha}(u\cdot\theta)\right)dx$$

where $R = (\partial_{x_1}\Lambda^{-1}, \partial_{x_2}\Lambda^{-1})$ are Riesz transforms. For brevity, we'll use the notations

$$\tilde{u} = e^{\gamma t\Lambda^\alpha}u, \qquad \tilde{\theta} = e^{\gamma t\Lambda^\alpha}\theta$$

To obtain further estimates, we break up the term $e^{\gamma t\Lambda^\alpha}(u\cdot\theta)$. Its structure can be better seen from the Fourier transform.

$$e^{\gamma t\Lambda^\alpha}(u\cdot\theta) = L^{-2}\sum_{k'+k''=k} e^{\gamma t|k|^\alpha}\widehat{u}(k')\widehat{\theta}(k'')$$

For $0 \leq \alpha \leq 1$,

$$|k|^\alpha = |k'+k''|^\alpha \leq |k'|^\alpha + |k''|^\alpha$$

So this term is bounded by

$$L^{-2}\sum_{k'} e^{\gamma t|k'|^\alpha}|u(k')| \cdot \sum_{k''} e^{\gamma t|k''|^\alpha}|\theta(k'')|$$

Now using Hölder's inequality and the boundedness of Riesz transform,

$$|II| \leq C\|\tilde{\theta}\|_{H^{\alpha+\beta}}\|\Lambda^{\beta+1-\alpha}(\tilde{u}\cdot\tilde{\theta})\|_{L^2}$$

where C is a constant.

Now we take $q = 2$ if $\beta > 1$ and $q = \frac{2}{\beta}$ if $\beta \leq 1$. Choose p such that

$$\frac{2}{p} + \frac{2}{q} = 1$$

and $\sigma = 2 - \frac{2}{q} - \alpha$. The condition that $\beta + 2\alpha > 2$ implies that $0 < \sigma < \alpha$. We apply Lemma 2.2 to obtain

$$|II| \leq C\|\tilde{\theta}\|_{H^{\alpha+\beta}}\left(\|\tilde{u}\|_{L^p}\|\Lambda^{\beta+1-\alpha}\tilde{\theta}\|_{L^q} + \|\tilde{\theta}\|_{L^p}\|\Lambda^{\beta+1-\alpha}\tilde{u}\|_{L^q}\right) = II_1 + II_2$$

Using Sobolev imbeddings

$$H^\beta \hookrightarrow H^{\frac{2}{q}} \hookrightarrow L^p, \qquad H^{\beta+\sigma} \hookrightarrow L^p_{\beta+1-\alpha}$$

and the Gagliado-Nirenberg interpolation (since $\sigma < \alpha$):

$$\beta + \sigma = \frac{\sigma}{\alpha}(\beta + \alpha) + \left(1 - \frac{\sigma}{\alpha}\right)\beta$$

we have

$$II_1 \le C\|\tilde\theta\|_{H^{\beta+\alpha}}\|\tilde u\|_{H^\beta}\|\tilde\theta\|_{H^{\beta+\sigma}} \le C\|\tilde\theta\|_{H^\beta}\|\tilde\theta\|_{H^{\beta+\alpha}}\|\tilde\theta\|_{H^{\beta+\sigma}}$$

$$\le C\|\tilde\theta\|^{1+\frac{\sigma}{\alpha}}_{H^{\beta+\alpha}}\|\tilde\theta\|^{2-\frac{\sigma}{\alpha}}_{H^\beta}$$

where C is constant depending on α and β. By Young's inequality

$$II_1 \le \frac{\kappa}{8}\|\tilde\theta\|^2_{H^{\beta+\alpha}} + \frac{C}{\kappa^M}\left(\|\tilde\theta\|^2_\beta\right)^N$$

where $N = \frac{2\alpha-\sigma}{\alpha-\sigma}$ and $M = \frac{\alpha+\sigma}{\alpha-\sigma}$.

A similar estimate results in the same bound for II_2.

Collecting the estimates for I, II and III and reintroducing $\tilde\theta = e^{\gamma t\Lambda^\alpha}\theta$,

$$\frac{d}{dt}\|e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\|^2_{L^2} \le -\kappa\|e^{\gamma t\Lambda^\alpha}\Lambda^{\beta+\alpha}\theta\|^2_{L^2}$$

$$+\frac{2\gamma^2}{\kappa}\|e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\|^2_{L^2} + \frac{C}{\kappa^M}\|e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\|^{2N}_{L^2}$$

Now we let

$$Z(t) = \|e^{\gamma t\Lambda^\alpha}\Lambda^\beta\theta\|^2_{L^2}$$

the differential inequality becomes

$$\frac{dZ}{dt} \le \frac{2\gamma^2}{\kappa}Z + \frac{C}{\kappa^M}Z^N$$

An elementary algebra results

$$\frac{dY}{dt} \le \frac{Ce^{2(N-1)\frac{\gamma^2 t}{\kappa}}}{\kappa^M}Y^N$$

where $Y = e^{\frac{-2\gamma^2 t}{\kappa}}Z$. After a simple calculation,

$$Y \le \frac{Y_0}{\left(1 - CY_0^{N-1}\kappa^{-(M-1)}\gamma^{-2}(e^{\frac{2(N-1)\gamma^2 t}{\kappa}} - 1)\right)^{\frac{1}{N-1}}}$$

Reintroducing $Z(t) = \|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_2^2$ and noting that $Z(0) = \|\Lambda^\beta \theta_0\|_2^2$ that

$$\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_2^2 \leq \frac{e^{\frac{2\gamma^2 t}{\kappa}} \|\Lambda^\beta \theta_0\|_2^2}{\left(1 - C(\|\Lambda^\beta \theta_0\|_2^2)^{N-1} \kappa^{-(M-1)} \gamma^{-2} (e^{\frac{2(N-1)\gamma^2 t}{\kappa}} - 1)\right)^{\frac{1}{N-1}}}$$

This means that $\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_2^2$ is finite on the interval $[0, t^*)$, where

$$t^* = \frac{\kappa}{2(N-1)\gamma^2} \log \left(1 + \frac{\kappa^{M-1} \gamma^2}{C \|\Lambda^\beta \theta_0\|_2^{2(N-1)}}\right)$$

The smaller the initial decay rate $\|\Lambda^\beta \theta_0\|_2$ is , the larger t^*. Similarly, the larger the parameter γ , the shorter t^*.

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