LOW TEMPERATURE STOCHASTIC DYNAMICS FOR AN ISING MODEL WITH ALTERNATING FIELD.

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Dedicated to the memory of Roland L'vovich Dobrushin

Abstract

We study, in the low temperature regime, a 2D stochastic Ising system with nearest neighbour ferromagnetic interaction subject to a positive (negative) magnetic field on even (odd) horizontal lines. We find, for different values of the magnetic fields, very different dynamical behaviours which are strictly related to the peculiar features of the expected equilibrium phase diagram at low temperature. In particular we analyse in detail the metastable behaviour (lifetime of the metastable equilibrium, typical paths during the transition to the stable equilibrium) in the vicinities of a "low-temperature critical point".

Keywords : stochastic Ising model, metastability, nucleation, large deviations.

Section 1. Introduction.

In the recent years many efforts have been devoted to the description of metastability and nucleation. This dynamical phenomenon, appearing in thermodynamic systems close to a first order phase transition, has been analysed in several papers, with the rigorous methods of mathematical physics, in the framework of Glauber dynamics for various lattice spin systems ([NS1], [NS2], [KO1], [KO2],[CO]). By Glauber dynamics we mean a single spin-flip Markovian time evolution which is reversible w.r.t the Gibbs measure corresponding to the interaction characterizing a lattice spin model. We refer to [PL1], [PL2], [CGOV] for a general discussion on metastability and, for instance, to [NS1], [CO] for a more specific discussion.

In this paper we study a stochastic Ising model with alternating field in the low temperature region. We consider discrete time Metropolis dynamics (see (2.6) below) for a 2D ferromagnetic Ising system with nearest neighbours interaction (coupling constant $J > 0$) enclosed in a sufficiently large but finite torus Λ and subject to a positive (negative) external magnetic field $h_1 > 0$, $(-h_2 < 0)$ on the even (odd) horizontal lines.

The phase diagram in the (h_1, h_2)-plane at $T = 0$ is discussed below. Mainly three different configurations appear as ground states in different regions: all pluses $+\underline{1}$, all minuses $-\underline{1}$ and an alternate configuration $\pm\underline{1}$ with all pluses (all minuses) on even (odd) horizontal lines. They are separated by the coexistence lines $h_1 = h_2$; $h_1 = 2J$; $h_2 = 2J$ which converge in the triple point $\widetilde{P} \equiv (2J, 2J)$ (see Fig. 2.1).

There are strong arguments leading to the conjecture that for positive temperature T the phase diagram changes completely (phase $\equiv$ infinite volume Gibbs measure). It is expected that for $T > 0$ the coexistence lines $h_1 = 2J$; $h_2 = 2J$ disappear since the system, in that region, behaves like a collection of almost independent one-dimensional standard zero field Ising systems. On the other hand for $h_1 = h_2 < 2J$ we expect that at low enough temperature T we have coexistence of two phases and that, given such a T, on the $h_1 = h_2$ line the coexistence ends at a critical point $P^* \equiv (2J - \delta, 2J - \delta)$ with $\delta \sim \exp(-\text{const}\frac{1}{T})$. It is not hard to realize that on the segment $0 < h_1 = h_2 < 2J - \delta$ our model shares many features with a zero field ferromagnetic anisotropic Ising model with coupling constants, along horizontal and vertical directions given, respectively, by $J_1 = J$, $J_2 = J - \frac{(h_1+h_2)}{4}$ (see (3.4) below). It becomes more and more anisotropic as we approach the triple point $\widetilde{P}$.

In the framework of our stochastic dynamics we will analyse the metastable behaviour by choosing the parameters h_1, h_2 close to the line $0 < h_1 = h_2 < 2J$ or close to the line $h_2 = 2J,\ h_1 > 2J$ and taking a large but finite volume Λ in the limit of very small $T = 1/\beta$. We will always assume $h_1 \geq h_2$; because of the symmetry this is not a loss of generality.

Mathematically the above described asymptotic regime will correspond to the study of some large deviation problem for a Markov chain with transition probabilities exponentially small in a large parameter β: the so called Freidlin-Wentzell regime (see [C], [FW], [OS1], [OS2]).

One expects that the special equilibrium behaviour of our model and in particular the disappearance, for an arbitrarily small but strictly positive T of two coexistence lines, the appearance of a critical ending point of the $0 < h_1 = h_2$ coexistence line and the extreme anisotropical behaviour when approaching $\widetilde{P}$ must have consequences in the dynamical behaviour at low temperature in the vicinities of these ($T = 0$) coexistence lines. Indeed our results provide the dynamical counterparts of the above described static behaviour confirming the above mentioned peculiarities; also some other unexpected features show up.

We will consider the regions:

$$0 < h_1 < 2J;\ \ 0 \leq \varepsilon := h_1 - h_2 \tag{1.1}$$

and

$$h_1 > 2J, 0 < \mu = 2J - h_2 \tag{1.2}$$

corresponding to the unique ground state $+\underline{1}$. Starting from $-\underline{1}$ (or $\pm\underline{1}$) we will consider the first hitting time $\tau_{+\underline{1}}$ to the configuration $+\underline{1}$ (see (2.18) below), compute its asymptotics for large β and describe the typical mechanism of transition (first excursion) from $-\underline{1}$ (or $\pm\underline{1}$) to $+\underline{1}$. Our main results are:

1. Near the $T = 0$ coexistence line given by: $h_2 = 2J, h_1 > 2J$ (disappearing at $T > 0$) we find absence of metastability in the sense that the protocritical droplet (see [NS1], [KO1], [KO2]) is microscopic and the "lifetime" does not diverge as we approach the coexistence line.

2. Near the $T = 0$ coexistence line given by: $0 < h_1 = h_2 < 2J$ we have "usual" metastable behaviour. The transition is driven by the formation of a critical nucleus whose size diverges as we approach the coexistence line. The unexpected behaviour concerns the shape of the critical nucleus as well as the nucleation pattern. We call Wulff shape the equilibrium $T = 0$ shape of a droplet namely the shape minimizing the energy for fixed volume. It turns out that it is given by a rectangle with vertical and horizontal sides in

the ratio

$$\frac{l_2}{l_1} = \frac{J - (\frac{h_1+h_2}{4})}{J} \tag{1.3}$$

This ratio becomes $\ll 1$ near the triple point $\widetilde{P}$. We find that the shape characterizing the critical nucleus and all the droplets appearing before its formation is rectangular but non-Wulff. It is characterized by a ratio

$$\frac{l_2}{l_1} \sim 2 \tag{1.4}$$

Notice that the non-Wulff nature of the nucleation pattern is even enhanced w.r.t. what has been found ([KO1]) in the anisotropic Ising model. The relevant rectangles during the nucleation actually have a vertical side *larger* (almost twice as much) than the horizontal one whereas for the Wulff shape we have the opposite situation and the horizontal side becomes even arbitrarily larger than the vertical one if we are sufficiently close to $\widetilde{P}$. This fact cannot be easily predicted on static grounds but it is rather related to a genuine dynamic effect.
3. In the vicinity of $\widetilde{P}$ the mechanism of nucleation depends on the the angle from which we approach it. In particular it turns out that given a halfline emerging from $\widetilde{P}$, lying inside the regions (1.1), (1.2) and forming an angle α with the line $h_2 = 2J$, no matter how close we go to $\widetilde{P}$, staying on this line, the size of the critical droplet remains constant: it depends on α and tends to infinity when α tends to the value $\frac{3}{4}\pi$ corresponding to the line $h_1 = h_2$.

The paper is organized as follows: in Section 2 we give the definitions and state our main result namely Theorem 1 which refers to the region (1.1). In Section 3 we analyse the local minima of the energy and their (generalized) basins of attraction. In Section 4 we prove Theorem 1. In Section 5 we give further results concerning the typical trajectories of our process during the first excursion. Finally in Section 6 we briefly analyse the other regions of parameters.

Section 2. Definitions and results.

We start by introducing the model that we want to study.

The configuration space is

$$\mathcal{S}_\Lambda = \{-1,+1\}^\Lambda \tag{2.1}$$

where $\Lambda\ = \Lambda_N \equiv [1,\ldots,N]\times[1,\ldots,N]$ is a two-dimensional torus (a square with periodic boundary conditions) of side N; this N is assumed to be even.

A configuration σ is a function :

$$\sigma\ :\ \Lambda_N\ \to\ \{-1,+1\}$$

We write $\sigma \prec \sigma'$ if σ is pointwise smaller than σ':

$$\sigma \prec \sigma' \iff \sigma(x) \le \sigma'(x)\ \forall x \tag{2.2}$$

Let $\mathcal{L}_1$ ($\mathcal{L}_2$) be the set of horizontal even (odd) rows in $\mathbf{Z^2}$:

$$\mathcal{L}_1 = \{x = (x_1, x_2) \in \Lambda_N \quad | \quad x_2 \quad \text{is even} \quad \}$$

$$\mathcal{L}_2 = \Lambda_N \backslash \mathcal{L}_1$$

To any configuration $\sigma \in \mathcal{S}_\Lambda$ we associate the energy:

$$H(\sigma) = -\frac{J}{2} \sum_{<x,y>\subset\Lambda_N} \sigma(x)\sigma(y) - \frac{h_1}{2}\sum_{x\in\mathcal{L}_1}\sigma(x) + \frac{h_2}{2}\sum_{x\in\mathcal{L}_2}\sigma(x) \tag{2.3}$$

where $J,\ h_1,h_2$ are positive constants and $<x,y>$ denotes a generic pair of nearest neighbour sites in Λ_N.

Thus (2.3) describes an Ising model with alternating field given by $h_1 > 0$ in $\mathcal{L}_1$ and by $-h_2 < 0$ in $\mathcal{L}_2$: it will be called h_1h_2-model.

The associated *Gibbs measure* in the torus Λ_N is given by:

$$\mu_{\Lambda_N}\ =\ \frac{\exp(-\beta H(\sigma))}{Z_{\Lambda_N}} \tag{2.4}$$

where β represents the inverse temperature and Z_{Λ_N} is the normalization factor called *partition function.*

We want now to introduce a dynamics in our model . It is a disctrete time Glauber dynamics namely a Markov chain with state space $\mathcal{S}_{\Lambda_N}$ and transition probabilities $P(\sigma,\eta)$ allowing only transitions between pairs σ,η of configurations differing by a single spin-flip and *reversible* w.r.t. the Gibbs measure μ_{Λ_N} for the h_1h_2-model, namely such that:

$$\mu_{\Lambda_N}(\sigma)P(\sigma,\eta)\ =\ \mu_{\Lambda_N}(\eta)P(\eta,\sigma). \tag{2.5}$$

Our choice is the so called Metropolis algorithm, where $P(\sigma,\eta)$ for $\sigma\neq\eta$, is defined as

$$P(\sigma,\eta)=\begin{cases}\frac{e^{-\beta[H(\eta)-H(\sigma)]^+}}{|\Lambda_N|} & \text{if } \exists\, x\in\Lambda_N:\ \sigma^{(x)}=\eta\\ \\ 0 & \text{otherwise}\end{cases} \tag{2.6}$$

where

$$\sigma^{(x)}(y)=\begin{cases}\sigma(y) & \forall\, y\neq x\\ -\sigma(y) & y=x\end{cases}. \tag{2.7}$$

and :

$$a^+=\begin{cases}a & \text{if } a\geq 0\\ 0 & \text{if } a<0\end{cases}. \tag{2.8}$$

The *space of trajectories* of the process is

$$\Omega\equiv\left(\mathcal{S}_{\Lambda_N}\right)^{\mathbf{N}}.$$

An element in Ω is denoted by ω; it is a function

$$\omega:\mathbf{N}\to\mathcal{S}_{\Lambda_N}.$$

We often write $\omega=\sigma_0,\sigma_1,\dots,\sigma_t,\dots$.

We will call *path* an *allowed* trajectory namely: $\omega=\sigma_0,\sigma_1,\dots,\sigma_t,\dots$ is a path iff σ_j and σ_{j+1} $\forall j$ are *connected* in the sense that $\sigma_{j+1}=\sigma_j^{(x)}$ for some $x\in\Lambda_N$. We use the notation $\omega:\sigma\to\eta$ to denote a path ω joining σ to η.

A path $\omega=\sigma_0,\sigma_1,\dots,\sigma_n$ is called *downhill* (*uphill*) iff $H(\sigma_{j+1})\leq H(\sigma_j)$ $(H(\sigma_{j+1})\geq H(\sigma_j))$ $\forall j=0,1,\dots,n-1$. We will use the convention that a downhill path can (and will) end only in a local minimum for H.

Given $V>0$ a path $\omega=\sigma_0,\sigma_1,\dots,\sigma_n$ is called *V-downhill* if

$$\text{either}\qquad \forall\, i\in(0,\dots,n-1)\qquad \mathrm{H}(\sigma_i)>\mathrm{H}(\sigma_{i+1})$$

$$\text{or}\quad \forall\, i\in(0,\dots,n-2)\quad \mathrm{H}(\sigma_i)+V=\mathrm{H}(\sigma_{i+1})\qquad\text{and}\qquad \mathrm{H}(\sigma_i)>\mathrm{H}(\sigma_{i+2})$$

A set D of configurations, $D\in\mathcal{S}_{\Lambda_N}$, is said to be *connected* iff for every pair of configurations $\sigma,\eta\in D$, $\exists$ a path $\omega:\sigma\to\eta$ such that $\omega\subset D$.

We say that a configuration σ is *downhill connected* (*V-downhill connected*) to η iff there exists a downhill (V-downhill) path $\omega:\sigma\to\eta$.

We say that a path $\omega = \sigma_0, \dots, \sigma_t, \dots$ is *almost downhill* if the energy does not exceed the initial value, namely:

$$H(\sigma_i) \leq H(\sigma_0), \quad i = 0, 1, \dots$$

We will denote by M the set of all locally stable configurations namely the set of all local minima of the energy. More precisely: $\sigma \in M$ iff for every $x \in \Lambda_N$, the corresponding increment in energy, given by

$$\Delta_x H(\sigma) := H(\sigma^{(x)}) - H(\sigma) \tag{2.9}$$

is positive.

We will choose the parameters J, h_1, h_2 so that the quantity $\Delta_x H(\sigma)$ will be always non-zero and this will somehow simplify some arguments.

Given $D \subset \mathcal{S}_{\Lambda_N}$ we define the (outer) boundary ∂D of D as the set:

$$\partial D \equiv \partial^+ D := \{\sigma \notin D : \exists \sigma' \in D : P(\sigma', \sigma) > 0\} \ ,$$

namely

$$\partial D := \{\sigma \notin D : \exists x \in \Lambda_N \ \text{ such that } \sigma' = \sigma^{(x)} \in D\} \ . \tag{2.10}$$

Similarly we define the inner boundary $\partial^- D$ as

$$\partial^- D := \{\sigma \in D : \exists x \in \Lambda_N \ \text{ such that } \sigma' = \sigma^{(x)} \notin D\} \ . \tag{2.11}$$

We denote by $F = F(D)$ the "bottom" of D namely the set of all minima of the energy on D:

$$F(D) := \{\zeta \in D : \min_{\sigma \in D} H(\sigma) = H(\zeta)\} \ . \tag{2.12}$$

We denote by $U = U(D)$ the set of all minima of the energy on the boundary ∂D of D:

$$U(D) \equiv F(\partial D) := \{\zeta \in \partial D : \min_{\sigma \in \partial D} H(\sigma) = H(\zeta)\} \tag{2.13}$$

and we define $H(U(D)) := H(\xi)$ with $\xi \in U(D)$.

Given a stable state $\sigma \in M$ i.e. a local minimum for the energy, we define the following *basins* for σ:

i) the *basin of attraction of* σ given by:

$$B(\sigma) := \{\zeta : \text{every downhill path starting from } \zeta \text{ ends in } \sigma\} \ , \tag{2.14}$$

$B(\sigma)$ can be seen as the usual basin of attraction of σ with respect to the $\beta = \infty$ dynamics.
ii) $\bar{B}(\sigma) =$ the *strict basin of attraction of* σ given by :

$$\bar{B}(\sigma) := \{\zeta \in B(\sigma) \ : \ H(\zeta) < H(U(B(\sigma)))\} \ . \tag{2.15}$$

iii) finally, given $V > 0$, we define the V-*basin* of σ as the (possibly empty) set:

$$B^{(V)}(\sigma) := \{\zeta : \text{every downhill path starting from } \zeta \text{ ends}$$

$$\text{in a local minimum } \xi \in M \text{ which is } V\text{-downhill connected to } \sigma\} \ ; \tag{2.16}$$

We introduce now the useful notion of *cycle.* A connected set A which satisfies:

$$\max_{\sigma \in A} H(\sigma) = \bar{H} < \min_{\zeta \in \partial A} H(\zeta) = H(U(A))$$

is called *cycle.* Notice that every local minimum for the energy is a (trivial) cycle.

The following simple properties of the cycles are true. Their proof is immediate (see, for instance [OS1]).

1. Given a state $\bar{\sigma} \in \mathcal{S}_{\Lambda_N}$ and a real number c the set of all σ's connected to $\bar{\sigma}$ by paths with energy always below c either coincides with $\mathcal{S}_{\Lambda_N}$ or it is a cycle A with

$$H(U(A)) \geq c \ .$$

2. Given two cycles $A_1, \ A_2$, either i) $A_1 \cap A_2 = \emptyset$ or ii) $A_1 \subset A_2$ or, vice-versa, $A_2 \subset A_1$.

For each pair of states $\sigma, \eta \in \mathcal{S}_{\Lambda_N}$ we define their minimal saddle $S(x,y)$ as the set of states corresponding to the solution of the following minimax problem: let, for any path ω

$$\hat{H}(\omega) \ := \max_{\zeta \in \omega} H(\zeta), \qquad \bar{H}_{\sigma,\eta} \ := \ \min_{\omega : \sigma \to \eta} \hat{H}(\omega) \ ,$$

find

$$S(\sigma,\eta) := \{\zeta : \ H(\zeta) = \bar{H}_{\sigma,\eta}; \ \exists \ \omega : \ \sigma \to \eta, \ \omega \ni \zeta, \ \hat{H}(\omega) = \bar{H}_{\sigma,\eta}\} \ .$$

$\bar{H}_{\sigma,\eta}$ is called *communication height* between σ and η.

A saddle $S(\sigma,\eta)$ such that $S(\sigma,\eta) \cap \{\sigma,\eta\} \neq \emptyset$ is called *trivial.*

Given a connected set $D \subset \mathcal{S}_\Lambda$, consider the (possibly empty) set $\widetilde{S}(D)$ of configurations in ∂D downhill connected to some configurations in $D^c \equiv \mathcal{S}_\Lambda \setminus D$. We call *set of minimal saddles* in the boundary of D the (possibly empty) set:

$$S(D) = \widetilde{S}(D) \cap U(D) \tag{2.17}$$

A *global saddle point* is any configuration

$$\bar{\sigma} \ \in S(-\underline{1},+\underline{1}) \ .$$

One immediately verifies that a strict basin of attraction of a local minimum is a cycle with nontrivial saddles in its boundary. This case corresponds to a "one well" structure. More general cases involve the presence of "internal saddles" and correspond to a "several wells" situation.

Given any set of configurations $A \subset \mathcal{S}_{\Lambda_N}$, we use τ_A to denote the *first hitting time* to A:

$$\tau_A := \inf\{t \geq 0 \ : \sigma_t \in A\} \ . \tag{2.18}$$

We use $P_\eta(\cdot)$ to denote the probability distribution over the process starting at $t=0$ from the configuration η.

We denote by $+\underline{1}$ $(-\underline{1})$ the configuration where $\sigma(x) = +1 \ \forall\, x \in \Lambda_N$ $(\sigma(x) = -1 \ \forall\, x \in \Lambda_N)$.

We want to define now a class of configurations in $\mathcal{S}_{\Lambda_N}$. Let r_i be the i-th row of Λ_N

$$r_i = \{x = (x_1, x_2) \in \Lambda_N \quad | \quad x_2 = i \quad \} \tag{2.19}$$

and let ν be a subset of indices contained in $[1, \cdots, N]$. Given ν let :

$$\sigma_\nu = \begin{cases} +1 & \forall\, x \in r_i \ \forall\, i \in \nu \\ -1 & \forall\, x \in r_i \ \forall\, i \in [1, \cdots, N] \backslash \nu \end{cases} \tag{2.20}$$

Notice that for instance if

$$\nu = [1, \ldots, N] \qquad \Longrightarrow \qquad \sigma_\nu = +\underline{1}; \qquad\qquad \nu = \emptyset \qquad \Longrightarrow \qquad \sigma_\nu = -\underline{1}$$

$$\nu = 2[1, \ldots, \frac{N}{2}] \Longrightarrow \sigma_\nu = \begin{cases} +1 & \text{if} \quad \mathrm{x} \in \mathcal{L}_1 \\ -1 & \text{if} \quad \mathrm{x} \in \mathcal{L}_2 \end{cases} \tag{2.21}$$

The above configuration with $+1$ in $\mathcal{L}_1$ and -1 in $\mathcal{L}_2$ is denoted by $\pm\underline{1}$.

Now let $k \in \mathbf{N}$, $\{n_i, m_i\}_{i=1,\ldots,k}$, be such that both $n_i \geq 0$ and $m_i \geq 0$ are even numbers; let us define ν as the union of intervals:

$$\nu = [n_1, n_1 + m_1] \cup [n_2, n_2 + m_2] \cup, \dots, \cup [n_k, n_k + m_k] \tag{2.22}$$

with $n_i + m_i < n_{i+1}$. The configurations σ_ν defined by equations (2.20), (2.22) are given by unions of bands of + spins of heights $m_i + 1$, which start and end in $\mathcal{L}_1$. They are called *stable bands*.

The bands of height one corresponding to $m_i = 0$ are rows belonging to $\mathcal{L}_1$.

Given h_1, h_2 we denote by: $\mathcal{M}_{h_1,h_2} = F(\mathcal{S}_{\Lambda_N})$ the set of ground states (absolute minima) for H.

It is easy to see that:

if $h_1 = h_2 < 2J$	$\mathcal{M}_{h_1,h_2} \equiv \{-\underline{1}, +\underline{1}\}$. In this case $\mathcal{S}_{\Lambda_N}$ has a symmetric double well structure.
if $h_1 < h_2$ and $0 < h_1 < 2J$	The symmetry between $-\underline{1}$ and $+\underline{1}$ is broken and $\mathcal{M}_{h_1,h_2} = -\underline{1}$.
if $h_1 > h_2$ and $0 < h_2 < 2J$	$\mathcal{M}_{h_1,h_2} = +\underline{1}$.
if $h_2 > 2J$ and $h_1 > 2J$	$\mathcal{M}_{h_1,h_2} = \sigma_\nu$ where $\nu = 2\,[1, \dots, \frac{N}{2}]$.
if $h_2 = 2J$ and $h_1 > 2J$	the ground state is multiply degenerate; $\mathcal{M}_{h_1,h_2} = \{\sigma_\nu \text{ such that } \nu \supset 2\,[1, \dots, \frac{N}{2}]\}$.
if $h_1 = 2J$ and $h_2 > 2J$	the ground state is multiply degenerate; $\mathcal{M}_{h_1,h_2} = \{\sigma_\nu \text{ such that } \emptyset \subset \nu \subset 2\,[1, \dots, \frac{N}{2}]\}$.
if $h_2 = h_1 = 2J$	the ground state is multiply degenerate; $\mathcal{M}_{h_1,h_2}$ is the set of all possible unions of stable bands σ_ν given by (2.20), (2.22)

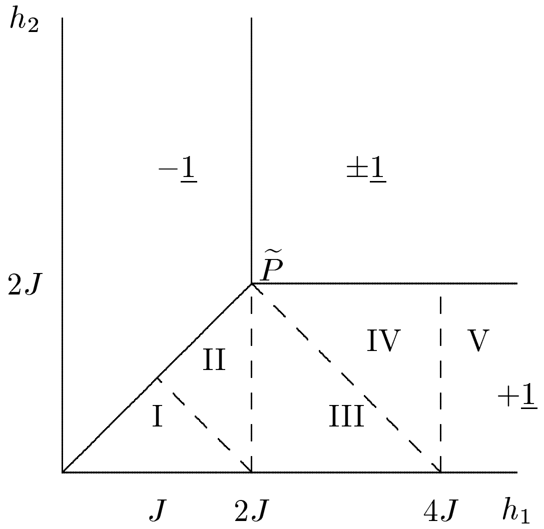
It will turn out that, for our purposes, the most interesting regions of the parameters are the following ones:

$$\mathrm{I} \;:=\; h_2 \;<\; h_1 \;<\; 2J - h_1$$

$$\mathrm{II} \;:=\; 2J - h_1 \;<\; h_2 \;<\; h_1\,; \quad J \;<\; h_1 \;<\; 2J$$

We will also consider the regions

$$\mathrm{III} \;:=\; 0 \;\le\; h_2 \;\le\; 4J - h_1\,; \;\; 2J \;\le\; h_1 \;\le\; 4J$$



$$\mathrm{IV} \;:=\; 0 \le \; 4J - h_1 \;\le\; h_2 \;<\; 2J$$

$$\mathrm{V} \;:=\; 4J \;\le h_1 \;; \quad h_2 \;<\; 2J$$

We want now to define a set $\mathcal{P}$ of configurations (see Fig. 2.2, 2.3) that will be called *protocritical.* The crucial part of the present work will be to prove that it is precisely the set $S(-\underline{1},+\underline{1})$ of global saddles. Following a general strategy already adopted in other cases ([KO1], [KO2], [CO]) we shall prove, and this is the content of Theorem 1 below, that the first excursion from $-\underline{1}$ to $+\underline{1}$ typically passes through a configuration from $\mathcal{P}$ and the time needed for this to happen is of the order $\exp(\beta\Gamma)$ where

$$\Gamma \;:=\; H(\mathcal{P}) - H(-\underline{1}) \;\; . \tag{2.23}$$

We first define two critical lenths whose meaning will appear clear below:

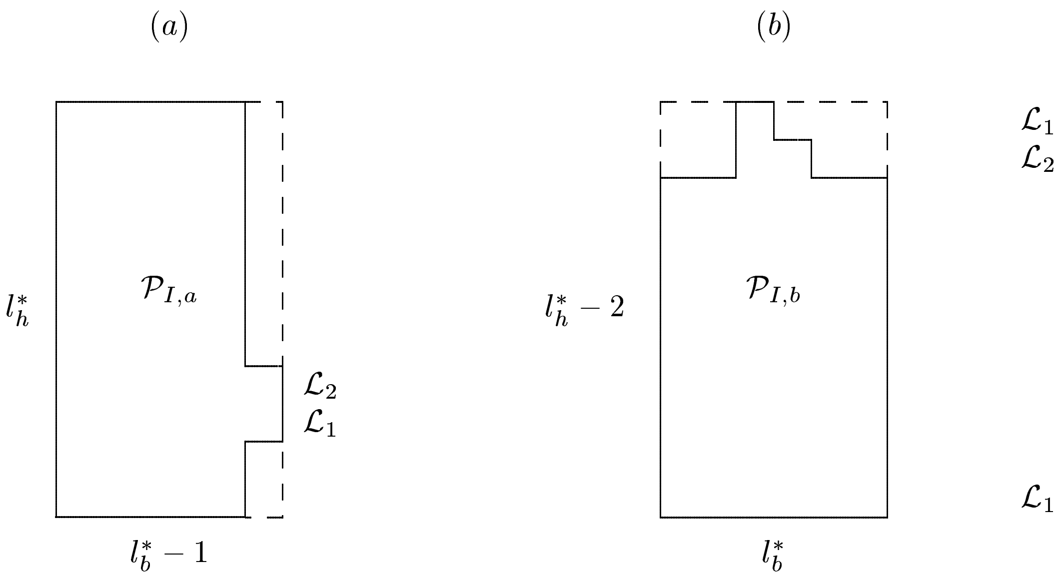
$$l_b^* = \left[\frac{\mu}{\varepsilon}\right] + 1 \tag{2.24}$$

and

$$l_h^* = 2l_b^* - 1 \tag{2.25}$$

For $x \in \mathbf{R}^+$ we denote by $[x]$ the largest integer $\le\; x$. We suppose that $\frac{\mu}{\varepsilon}$ is not integer.

Given two integers $l_1, l_2 \le N-2$, let $= R(l_1,l_2)$ be the set of configurations where the plus spins are precisely the ones on the sites internal to the rectangle, drawn on the dual lattice $\mathbf{Z}^2 + (1/2,1/2)$, with horizontal and vertical sides equal, respectively, to l_1, l_2.



Let $\mathcal{P}_{\mathrm{I},a}$ be the set of configurations where the plus spins are precisely the ones contained in a contour obtained by adding to $R(l_b^*-1, l_h^*)$ a protuberance given by two adjacent unit squares as indicated in Fig. 2.2(a). Let $\mathcal{P}_{\mathrm{I},b}$ be the set of configurations where the plus spins are precisely the ones contained in a contour obtained by adding to $R(l_b^*, l_h^*-2)$ a protuberance given by three adjacent unit squares as indicated in Fig. 2.2(b).

Similarly we call $\mathcal{P}_{\mathrm{II},a}$,$\mathcal{P}_{\mathrm{II},b}$ the configurations depicted in Fig. 2.3(a), 2.3(b), respectively.

It turns out that $\mathcal{P}_{\mathrm{I},a}$, $\mathcal{P}_{\mathrm{I},b}$,$\mathcal{P}_{\mathrm{II},a}$, $\mathcal{P}_{\mathrm{II},b}$ are particular cases of the classes of configurations $\sigma_3, \sigma_4, \bar\sigma_3, \bar\sigma_4$ that will be defined in Section 3.

For $(h_1, h_2) \in \mathrm{I}$ we set

$$\mathcal{P} := \mathcal{P}_{\mathrm{I},a} \cup \mathcal{P}_{\mathrm{I},b}$$

Whereas for $(h_1, h_2) \in \mathrm{II}$ we set

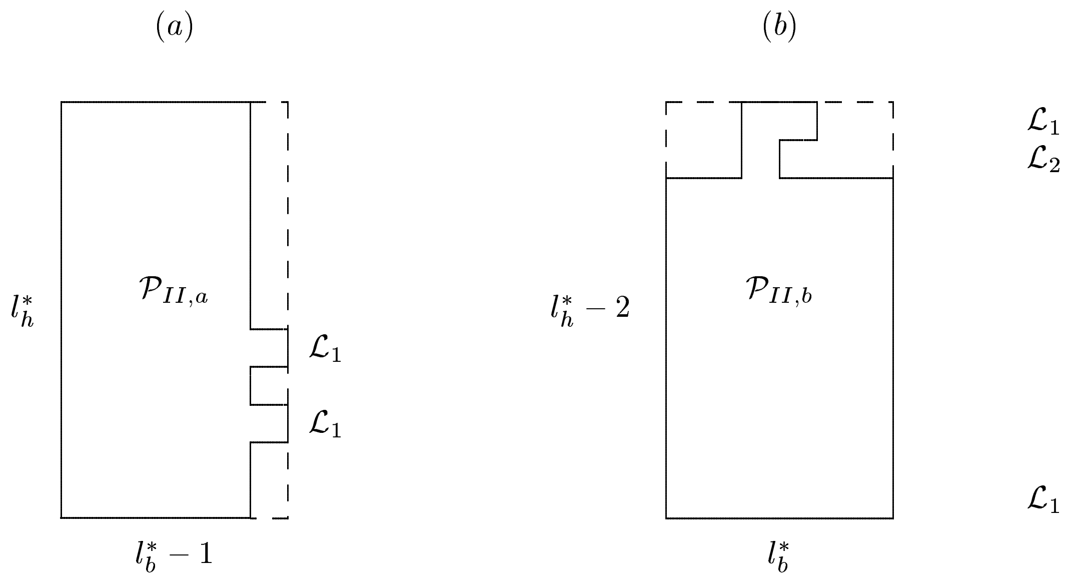
$$\mathcal{P} := \mathcal{P}_{\mathrm{II},a} \cup \mathcal{P}_{\mathrm{II},b}$$

Now we are ready to state our Theorem 1.

Theorem 1.

Let $\bar\tau_{-\underline{1}}$ be the last instant in which $\sigma_t = -\underline{1}$ before $\tau_{+\underline{1}}$:

$$\bar\tau_{-\underline{1}} := \max\{t < \tau_{+\underline{1}} : \sigma_t = -\underline{1}\} \ . \tag{2.26}$$



Let

$$\bar\tau_{\mathcal{P}} := \min\{t > \bar\tau_{-\underline{1}} : \sigma_t = \mathcal{P}\} \;\; ; \tag{2.27}$$

for every $\delta > 0$*:*

i)

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\bar\tau_{\mathcal{P}} < \tau_{+\underline{1}}) = 1 \;\; ; \tag{2.28}$$

ii)

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\exp[\beta(\Gamma - \delta)] < \tau_{+\underline{1}} < \exp[\beta(\Gamma + \delta)]) = 1 \;\; . \tag{2.29}$$

The proof will be given in Section 5 and it will be based on various propositions stated and proved in Sections 3,4 below.

Section 3. Local minima of the energy.

In this section we want to analyse the local minima of the energy ($\equiv$ *stable configurations*) especially from a geometric point of view; in particular we want to discuss some particularly interesting, "more stable" minima corresponding to a class of rectangles.

To describe more general configurations we need some more definitions.

For any configuration $\sigma \in \mathcal{S}_\Lambda$ we denote by $c^+(\sigma)$, $c^-(\sigma)$ the union of all closed unit squares centered at sites $x \in \Lambda$ with $\sigma(x)$ respectivly equal to $+1, -1$. $c^+(\sigma)$, $c^-(\sigma)$ decompose into maximal connected components ($\equiv$ clusters) c_j^+, $j = 1, \ldots, k^+$, c_j^-, $j = 1, \ldots, k^-$.

In fact the centers of c_j^+, c_j^- form a $\star$–cluster in the sense of sites percolation, namely they are maximally connected components in the sense of the next nearest neighbours but we will simply call them clusters.

To any cluster c_j^+ (c_j^-) we associate its *rectangular envelope* $R(c_j^+)$ ($R(c_j^-)$) defined as the minimal closed rectangle (with sides on the dual lattice $\mathbf{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)$) containing it; if none of the rectangles $R(c_j^+)$ is winding around the torus, we call the corresponding configuration *acceptable*. In any such configurations it is well defined the "sea" of minuses.

Given an acceptable configuration σ its *monotonic envelope*, denoted by $\gamma(\sigma)$, is the smallest monotonic cluster containing $c^+(\sigma)$; where a cluster c is said to be *monotonic* if its boundary ∂c is a unique "monotonic" contour in the sense that it has the same length as its rectangular envelope.

Given an acceptable configuration σ containing a unique cluster we also write $R(\sigma)$ for its rectangular envelope.

Remark

We often interchange an acceptable configuration σ giving rise to a unique cluster (of pluses) $C(\sigma)$ with $C(\sigma)$ itself. $C(\sigma)$, a closed subset of $\mathbf{R}^2$, in turn, can be identified with the set of its internal $\mathbf{Z}^2$ sites. Moreover we will often assimilate (and even denote by the same symbol with an abuse of notation) the cluster $C(\sigma)$ with the equivalence class of all the clusters obtained from $C(\sigma)$ by suitable translations on the lattice (arbitrary in horizontal direction and even in vertical direction). Actually, for clusters like σ_i, $\bar{\sigma}_i, i = 1, \ldots, 4$ below (see Fig. 3.1, 3.2) we will assimilate $C(\sigma)$ with the family of configurations obtained by displacements, along the sides, of some protuberances. The reader will easily deduce the meaning of our statements and symbols from the context.

In the following we will consider the whole region of parameters:

$$h_2 \ < \ h_1 \ ; \ h_2 \ < \ 2J \tag{3.1}$$

corresponding to regions I, II, III, IV, V in Fig. 2.1. We will first consider "extended " minima and then, only for h_1, h_2 varying in I, II the "localized" ones.

We start by giving a list of all possible spin-flips with the corresponding increment in energy $\Delta_x H := H(\sigma^{(x)}) - H(\sigma)$.

In the following we draw all the initial configuration σ at the site x and in the four nearest neighbours when $\sigma(x) = -1$; moreover we specify the layer, $\mathcal{L}_1$ or $\mathcal{L}_2$, to which it belongs. We associate to each configuration the corresponding value for $\Delta_x H$. To complete the list it suffices to take into account all the reciprocal spin-flips: this amounts to consider all the remaining cases in which $\sigma(x) = +1$; the corresponding $\Delta_x H$ are just the opposite of the previous ones.

$$\begin{array}{lccc} \mathcal{L}_2 & & - & \\ \mathcal{L}_1 & - & - & - \\ \mathcal{L}_2 & & - & \end{array} \qquad \Delta_x\mathrm{H} = 4J - h_1 \qquad \begin{array}{lccc} \mathcal{L}_1 & & - & \\ \mathcal{L}_2 & - & - & - \\ \mathcal{L}_1 & & - & \end{array} \qquad \Delta_x\mathrm{H} = 4J + h_2$$

$$\begin{array}{lccc} \mathcal{L}_2 & & - & \\ \mathcal{L}_1 & + & - & - \\ \mathcal{L}_2 & & - & \end{array} \qquad \Delta_x\mathrm{H} = 2J - h_1 \qquad \begin{array}{lccc} \mathcal{L}_1 & & - & \\ \mathcal{L}_2 & + & - & - \\ \mathcal{L}_1 & & - & \end{array} \qquad \Delta_x\mathrm{H} = 2J + h_2$$

$$\begin{array}{lccc} \mathcal{L}_2 & & - & \\ \mathcal{L}_1 & + & - & - \\ \mathcal{L}_2 & & + & \end{array} \qquad \Delta_x\mathrm{H} = -h_1 \qquad \begin{array}{lccc} \mathcal{L}_1 & & - & \\ \mathcal{L}_2 & + & - & - \\ \mathcal{L}_1 & & + & \end{array} \qquad \Delta_x\mathrm{H} = +h_2$$

$$\begin{array}{lccc} \mathcal{L}_2 & & - & \\ \mathcal{L}_1 & + & - & + \\ \mathcal{L}_2 & & + & \end{array} \qquad \Delta_x\mathrm{H} = -(2J + h_1) \qquad \begin{array}{lccc} \mathcal{L}_1 & & - & \\ \mathcal{L}_2 & + & - & + \\ \mathcal{L}_1 & & + & \end{array} \qquad \Delta_x\mathrm{H} = -(2J - h_2)$$

$$\begin{array}{lccc} \mathcal{L}_2 & & + & \\ \mathcal{L}_1 & + & - & + \\ \mathcal{L}_2 & & + & \end{array} \qquad \Delta_x\mathrm{H} = -(2J + h_1) \qquad \begin{array}{lccc} \mathcal{L}_1 & & + & \\ \mathcal{L}_2 & + & - & + \\ \mathcal{L}_1 & & + & \end{array} \qquad \Delta_x\mathrm{H} = -(2J - h_2)$$

It immediately follows from the previous list that the local minima are characterized by the following four properties:

1) Any spin $+1 \in \mathcal{L}_1$ must have at least two n.n. $+1$ spins

2) Any spin $+1 \in \mathcal{L}_2$ must have at least three n.n. $+1$ spins

3) Any spin $-1 \in \mathcal{L}_1$ must have at least three n.n. -1 spins

4) Any spin $-1 \in \mathcal{L}_2$ must have at least two n.n. -1 spins

It follows from the above properties that in a stable configuration the "plus angles" must belong to $\mathcal{L}_1$ whereas the "minus" angles must belong to $\mathcal{L}_2$:

$$
\begin{array}{lccc}
\mathcal{L}_2 & & - & \\
\mathcal{L}_1 & - & + & + \\
\mathcal{L}_2 & & + &
\end{array}
\qquad\qquad
\begin{array}{lccc}
\mathcal{L}_1 & & + & \\
\mathcal{L}_2 & + & - & - \\
\mathcal{L}_1 & & - &
\end{array}
$$

We immediately get that any band σ_ν defined by (2.20), (2.22), starting and ending in $\mathcal{L}_1$, with any height $2p+1,\ p = 0, 1, 2 \ldots$, is a local minimum in regions I, II, III, IV and V.

It is easy to convince oneself that these bands are the only local minima in regions III, IV, V whereas in I, II there are other, localized minima.

We notice that in regions I and II there are stable rectangles $R(l_1, l_2)$ but not all of them are stable. For that it is necessary and sufficient that $l_1 \geq 2, l_2 \geq 3$ and $R(l_1, l_2)$ starts and ends in $\mathcal{L}_1$. In particular l_2 must be odd .

Always in regions I and II there are many other minima beyond the stable rectangles and the stable bands but, as we will see, they are in a sense less stable.

We want now to analyse the "different degree of stability" of different local minima.

It follows from the list of spin-flips that in regions I and II the minimal positive increment of energy V_1 is given by:

$$V_1 \;=\; \min\{h_2, 2J - h_1\} \tag{3.2}$$

Notice that in region I we have: $V_1 \;=\; h_2$ whereas in region II we have: $V_1 \;=\; 2J - h_1$.

Given a minimum σ we call it V_1-unstable if there exists a V_1-downhill path starting from σ and leading to another minimum. Otherwise it is called V_1-stable.

We have the following:

3.1 Proposition

Let $h_1, h_2 \in \{\mathrm{I} \cup \mathrm{II}\}$. Any minimum σ is V_1-unstable if and only if it has a minus angle on $\mathcal{L}_2$ like:

$$\begin{array}{lccc}
\mathcal{L}_1 & & + & \\
\mathcal{L}_2 & + & - & - \\
\mathcal{L}_1 & & - &
\end{array}$$

Proof. It easily follows from the above properties 1), 2), 3), 4) that if a local minimum σ contains the above minus angle then it must also contain:

$$\begin{array}{lcccc}
\mathcal{L}_1 & + & + & + & \\
\mathcal{L}_2 & + & + & - & - \\
\mathcal{L}_1 & + & + & - & - \\
\mathcal{L}_2 & & & - &
\end{array}$$

Starting from this situation it is easy to costruct a V_1-dowhill path leading in two steps to a configuration σ' with $H(\sigma') - H(\sigma) = h_2 - h_1 < 0$: we flip the minus spin at the angle, say x and the n.n. minus spin below it, say y. In region I we first flip $\sigma(x)$ then $\sigma(y)$. Vice-versa in region II we first flip $\sigma(y)$ then $\sigma(x)$.

On the other hand it is easily seen that a stable configuration without any minus angle in $\mathcal{L}_2$ is V_1-stable.

∎

It is immediately seen that the stable bands and the stable rectangles are the only stable configurations which do not contain minus angles belonging to $\mathcal{L}_2$ and therefore they are the only V_1-stable configurations.

Consider the strict basin of attraction $\bar{B}(R)$ of a stable rectangle R and the set $S(\bar{B}(R))$ of its minimal saddles (see (2.17)). It is easily seen that

$$H(S(\bar{B}(R))) - H(R) =: V_2 = \min\{h_1, \mu\}. \tag{3.3}$$

We want now to analyse the tendency of a stable configuration consisting only of an isolated cluster to grow or shrink. It will turn out that the less stable between such minima, corresponding to a non-rectangular shape, will typically evolve towards a suitable rectangle in a time scale much smaller than the typical time needed for a rectangle to grow or shrink. In other words after an initial transient leading to a rectangle the tendency to grow or shrink is established starting from this rectangle.

Let us now give a (naive) static argument that could be used in order to discriminate subcriticality (tendency to shrink) or supercriticality (tendency to grow) of a given rectangle.

The energy formation of a stable rectangle $R(l_1, l_2)$ with sides l_1, l_2 is:

$$H(R(l_1,l_2)) - H(-\underline{1}) = 2J(l_1+l_2) - h_1 l_1 \left(\frac{l_2+1}{2}\right) + h_2 l_1 \left(\frac{l_2-1}{2}\right) =$$

$$= 2J(l_1+l_2) - \frac{\varepsilon}{2}(l_2+1)l_1 - l_1 h_2 \tag{3.4}$$

where we recall that:

$$\varepsilon = h_1 - h_2 \tag{3.5}$$

It is interesting to compute the $T = 0$ Wulff shape namely the shape minimizing the energy as a function of the real variables l_1, l_2 for fixed area. It is easily seen that this Wulff shape is characterized by the following ratio between the sides:

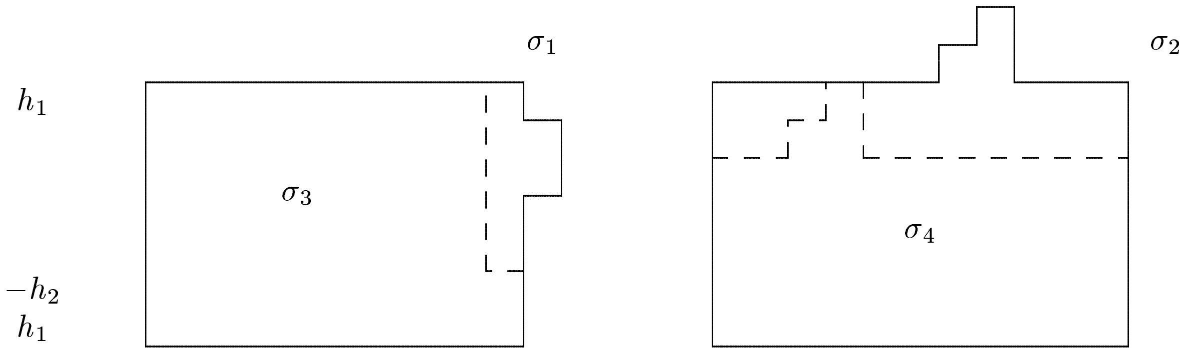
$$\frac{l_1}{l_2} = \frac{4J}{4J - \varepsilon - 2h_2} \tag{3.6}$$

Looking at the expression (3.4) one would be lead to the conclusion that the critical droplet is the unique saddle point corresponding to:

$$l_1 = \frac{4J}{\varepsilon}, \qquad l_2 = \frac{4J - \varepsilon - 2h_2}{\varepsilon} \tag{3.7}$$

defining a particular Wulff shape rectangle. We will see that this conclusion is erroneous. ¿From a dynamical point of view we have to carefully analyse the different ways of shrinking and growing. It will turn out that the crucial point is the comparison between four different times which constitute the typical scales for the relevant mechanisms.

Let $\sigma_1(l_1, l_2)$ be the configuration obtained from $R(l_1, l_2)$ by adding to one of its vertical edges a 1×2 rectangular protuberance as it is shown in Fig. 3.1; $\sigma_2(l_1, l_2)$ is obtained from



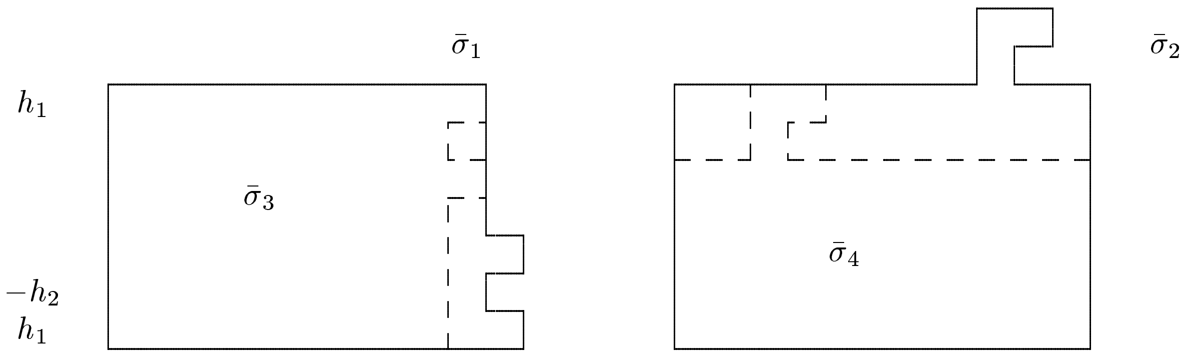
$R(l_1, l_2)$ by adding to one of its horizontal sides a"triangular" protuberance containing three unit squares as shown in Fig. 3.1. $\sigma_3(l_1, l_2)$ is obtained from $R(l_1, l_2)$ by removing all the unit squares contained in a vertical $1 \times l_2$ column except for a 1×2 rectangle as shown in Fig. 3.1. Finally $\sigma_4(l_1, l_2)$ is obtained from $R(l_1, l_2)$ by removing all the unit squares contained in a horizontal $l_1 \times 2$ strip except for a triangle as shown in Fig. 3.1 The corresponding typical times $t_i,\ i = 1, \dots, 4$ needed to reach $\sigma_i(l_1, l_2)$ starting from $R(l_1, l_2)$ namely the exponentials of β times the heigths of the corresponding energy barriers are:

$$\begin{array}{ll} t_1 = e^{\beta(2J - h_1 + h_2)} & t_2 = e^{\beta(4J + 2h_2 - h_1)} \\ t_3 = e^{\beta(\frac{\varepsilon}{2}(l_h - 3) + h_1)} & t_4 = e^{\beta(\varepsilon(l_b - 2) + h_1)} \end{array}, \tag{3.8}$$

Let us now introduce, similarly to what we did above, the configurations $\bar{\sigma}_i(l_1, l_2),\ i = 1, \dots 4$ depicted in Fig. 3.2.

The difference w.r.t.the previous configurations $\sigma_i(l_1, l_2),\ i = 1, \dots 4$ is just in the shape of the protuberances. The 1×2 rectangle on the vertical side is substituted by two unit squares at distance one whereas the triangle on the horizontal side is "inverted". The corresponding typical times are:

$$\begin{array}{ll} \bar{t}_1 = e^{\beta 2(\mu - \varepsilon)} & \bar{t}_2 = e^{\beta(2J + h_2 + 2(\mu - \varepsilon))} \\ \bar{t}_3 = e^{\beta(\frac{\varepsilon}{2}(l_h - 3) + \mu)} & \bar{t}_4 = e^{\beta(\varepsilon(l_b - 2) + \mu)} \end{array}, \tag{3.9}$$



It will be shown later on that, starting from $R(l_1, l_2)$, to determine the most likely future evolution in the sense of the first new rectangle visited after leaving $R(l_1, l_2)$ one has just to make a comparison between the four times $t_i, i = 1, \dots, 4$ in region I and between $\bar{t}_i,\ i = 1, \dots, 4$ in region II. This will follow from the fact that the minimal saddles on the boundary of a suitably defined generalized basin of attraction of $R(l_1, l_2)$ can always be found among the configurations $\sigma_i(l_1, l_2)$ or among the $\bar{\sigma}_i(l_1, l_2)$, in regions I and II, respectively.

One immediately realizes that it is always $t_1 < t_2$, $\bar{t}_1 < \bar{t}_2$ whereas the relation between the other pairs of times $t_i, t_j,\ (i, j) \neq (1, 2)$, or $\bar{t}_i, \bar{t}_j,\ (i, j) \neq (1, 2)$ depends on the sides of $R(l_1, l_2)$. In order for a rectangle to have a tendency to grow, in region I, we must have $t_1 < t_3$ and $t_1 < t_4$ namely:

$$l_h > \frac{2\mu - \varepsilon}{\varepsilon} \qquad\qquad l_b > \frac{\mu}{\varepsilon} \ \ ; \tag{3.10}$$

where we recall that:

$$\mu := 2J - h_2, \quad \varepsilon = h_1 - h_2 \tag{3.11}$$

and that we suppose that $\frac{\mu}{\varepsilon}$ is not integer. Then the minimal supercritical rectangle, in region I, has sides

$$l_b^* = \left[\frac{\mu}{\varepsilon}\right] + 1 \qquad l_h^* = 2l_b^* - 1 \tag{3.12}$$

If either $l_1 < l_b^*$ or $l_2 < l_b^*$ the corresponding stable rectangle has a tendency to shrink (it is subcritical); namely a stable rectangle $R(l_1, l_2)$ is supercritical if and only if it contains in its interior a rectangle $R(l_b^*, l_h^*)$.

In region II again by characterizing the critical lengths l_b^*, l_h^* (sides of the minimal supercritical rectangle) as the minimal integers such that $\bar{t}_1 < \bar{t}_3$ and $\bar{t}_1 < \bar{t}_4$ one has (for different reasons) exactly the same results:

$$l_b^* = \left[\frac{\mu}{\varepsilon}\right] + 1 \qquad l_h^* = 2l_b^* - 1 \tag{3.13}$$

Notice that the critical shape is non-Wulff similarly to what it has been found for the anisotropic Ising model (see [KO1]); but now this effect is even enhanced: for the critical rectangle the vertical side is larger (almost the double) than the horizontal one whereas in the Wulff rectangle the horizontal side is larger than the vertical one (and tends to be infinitely larger as we approach the triple point $h_1 = h_2 = 2J$).

We will say that a stable rectangle has *standard shape* if its sides l_1, l_2 satisfy the relation

$$l_1 = 2l_2 - 1. \tag{3.14}$$

Notation

We denote by $\mathcal{B}(R)$ the V_1-basin of attraction of R.

It is easy to see that given any configuration in $\partial(\mathcal{B}(R)\ )$ giving rise to a single cluster either

i) its rectangular envelope $R(\sigma)$ coincides with R and in this case if σ is V_1-downhill connected to a V_1-stable configuration $\eta \neq R$ then η must consist of one or more stable rectangles strictly contained in R ; or

ii) $R(\sigma)$ strictly contains R and in this case the stable rectangles $\bar{R} \neq R$ reachable from σ with a V_1-downhill path strictly contain R.

Other situations are impossible.

We want now to determine the set of minimal saddles $S(\mathcal{B}(R))$. To this end we need some definitions.

We say that a saddle in $\widetilde{S}(\mathcal{B}(R))$ (see definition before (2.17)) is *towards the growth (contraction)* if $R(\sigma) \supset R$ $(R(\sigma) = R)$.

Consider a configuration η giving rise to a unique cluster $C(\eta)$.

Let $\gamma(C(\eta))$ be the monotone envelope of $C(\eta)$. We can write:

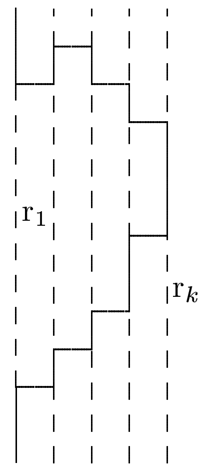
$$\gamma(C(\eta))\backslash C(\eta) = \bigcup A_i \ \cup \ \bigcup B_j \tag{3.15}$$

where A_i are (possibly empty) sets which are not connected to the sea of minuses and $\partial C(\eta) \bigcap \partial A_i = \partial A_i$ if non-empty is made by one or more closed curves. B_j, if non-empty, are connected to the sea of minuses and $\partial C(\eta) \bigcap \partial B_i \subset \partial B_i$ is an open curve. The A_i's are called *(internal) lakes* the B_i's are called *gulfs*.

A gulf is said to be *vertical (horizontal)* if the segment

$$\delta(B_i) := \partial\big(B_j\big) \setminus \partial\big(C(\eta)\big)$$

is vertical (horizontal).



If we have a vertical (horizontal) gulf B_j, we can consider a line r_1 which contains $\delta(B_j)$.

Let us consider a line r_2 parallel to r_1, at a distance one from it in the direction of the interior of B_j. We can then recursively define $r_i, i > 2$, as the line parallel to r_{i-1}, at distance one from it in the same direction as the one leading from r_1 to r_2. We stop at step k when r_k intersetcs B_j, but r_{k+1} does not intersect B_j. The line r_k obtained with this procedure is called *innermost line of the gulf* B_j (see Fig. 3.3).

Before stating Proposition 3.3, the main result of this Section, concerning the characterization of the set $S(\mathcal{B}(R(l_1, l_2)))$ we give two Lemmata which will be used in its proof.

3.2 Lemma

Let σ be a cluster with at least one vertical gulf B_j or with an internal lake A_i or with a horizontal gulf such that

$$dist(\delta(B_j), r_k) \begin{cases} \geq 2 & or \\ = 1 & and\ B_j \in \mathcal{L}_1 \end{cases}$$

Then there exists $\eta \succ \sigma$ with $\gamma(\eta) = \gamma(\sigma)$ and $H(\eta) < H(\sigma)$.

Proof

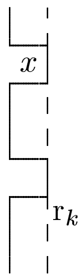
We will show that either there esists a site x such that $\sigma(x) = -1$ and

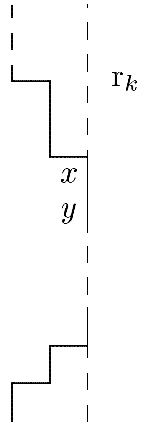
$$H(\sigma^{(x)}) - H(\sigma) < 0$$

and then we can choose $\eta \equiv \sigma^{(x)}$; or there exist two sites x y such that $\sigma(x) = \sigma(y) = -1$ and η is obtained from σ by flipping the spins in those sites: $\eta = \left(\sigma^{(x)}\right)^{(y)}$ and we have:

$$H(\eta) - H(\sigma) < 0$$

Given a vertical gulf B_j we consider the innermost line r_k of B_j. If $\partial B_j \cap r_k$ contains only length-one segments (see Fig. 3.4) then, adjacent to these segments in B_j, we have only isolated sites. Let x be any of those, then





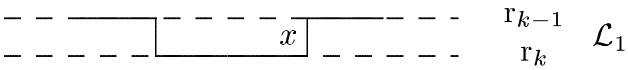
$$\Delta H = H(\sigma^{(x)}) - H(\sigma) = \begin{cases} -2J - h_1 & \text{if } x \in \mathcal{L}_1 \\ \\ -2J + h_2 & \text{if } x \in \mathcal{L}_2 \end{cases}$$

If the intersection contains segments whith length ≥ 2 (see Fig. 3.5), we consider the top angle and the sites x, y immediately below as shown in Fig. 3.5.

Let $\eta = \left(\sigma^{(x)}\right)^{(y)}$ then

$$H(\eta) - H(\sigma) \leq +h_2 - h_1 < 0.$$

The case with an internal lake A_i can be discussed with arguments similar to those used for the vertical gulf, by taking a vertical line r that intersects the lake and divides it into two vertical gulfs.

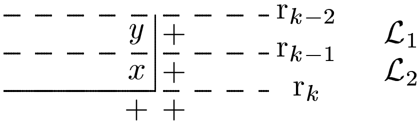


Given a horizontal gulf if the row between r_{k-1} and r_k belongs to $\mathcal{L}_1$, we take a site x, say, in the right hand angle. We get a situation like in Fig. 3.6

$$\Delta H = H(\sigma^{(x)}) - H(\sigma) \leq -h_1$$

If the row between r_{k-1} and r_k belongs to $\mathcal{L}_2$ then, by hypothesis, the one between r_{k-2} and r_{k-1} belongs to $\mathcal{L}_1 \cap B_j$.

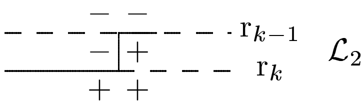
Let us consider the angle on the right of $r_k \cap \partial B_j$: if it is of the type shown in Fig. 3.7



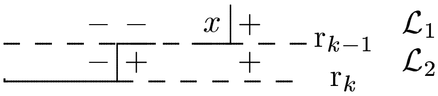
then we can simply flip the spins at x and y, and the energy cost will be

$$\Delta H \leq +h_2 - h_1$$

Otherwise, if it is of the type shown in Fig. 3.8



i.e. if we can immediately find another angle, then we continue to follow the boundary until we find a third angle, as shown in Fig. 3.9.



Let x be the site in the minus angle in $\mathcal{L}_1$; we have:

$$\Delta H = H(\sigma^{(x)}) - H(\sigma) \leq -h_1$$

By taking $\eta = \sigma^{(x)}$ we conclude the proof ∎

3.3 Lemma.

Among the configurations σ with a single cluster $C(\sigma)$ having a given (stable) rectangular envelope R the one with minimal energy is given precisely by $C(\sigma) \equiv R$.

Proof

It is not totally immediate like in the standard Ising model. It follows from the following argument . Suppose first that $C(\sigma)$ is monotonic. We can decompose $R \setminus C(\sigma)$ into a union of rectangles R_j having a nonempty part of their boundary in common with the boundary of R. Since R starts and ends in $\mathcal{L}_1$, if n is the number of sites in $(R \setminus C(\sigma)) \cap \mathcal{L}_1$ and m is the number of sites in $(R \setminus C(\sigma)) \cap \mathcal{L}_2$, we have

$$n \geq m$$

Then, since the J-contribution to the energy in R and in $C(\sigma)$ is the same, we have:

$$H(\sigma) - H(R) = nh_1 - mh_2 \geq 0$$

If $C(\sigma)$ is not monotonic, by repeatedly applying Lemma 3.2 we can easily reduce ourselves to the case of a configuration η with the same monotonic envelope as σ, with lower energy and being monotonic except for some heigth-one horizontal gulfs in $\mathcal{L}_2$. Also in this case we can apply the procedure of the monotonic case and decompose $R \setminus C(\sigma)$ into a union of rectangles R_j having a nonempty part of their boundary in common with the boundary of R. We get

$$H(\sigma) - H(R) \geq nh_1 - mh_2 \geq 0$$

with

$$n \geq m$$

This concludes the proof. ■

Now, given a stable rectangle $R = R(l_b, l_h)$, consider the previously defined configurations $\sigma_i(l_b, l_h), \bar{\sigma}_i(l_b, l_h), i = 1, \ldots, 4$ (see Fig. 3.1, 3.2). We have the following

3.4 Proposition.

The set of minimal saddles in $S(\mathcal{B}(R(l_b, l_h)))$ towards the growth is $\sigma_1(l_b, l_h)$, $(\bar{\sigma}_1(l_b, l_h))$ in region I (II).

The set of minimal saddles in $S(\mathcal{B}(R(l_b,l_h)))$ towards the contraction are:
In region I

$$\sigma_3(l_b,l_h) \ \ if \ \ l_h < 2l_b-1, \qquad \sigma_4(l_b,l_h) \ \ if \ \ l_h > 2l_b-1$$

$$\sigma_3(l_b,l_h)\cup\sigma_4(l_b,l_h) \ \ if \ \ l_h = 2l_b-1 \tag{3.16}$$

In region II

$$\bar\sigma_3(l_b,l_h) \ \ if \ \ l_h < 2l_b-1, \qquad \bar\sigma_4(l_b,l_h) \ \ if \ \ l_h > 2l_b-1$$

$$\bar\sigma_3(l_b,l_h)\cup\bar\sigma_4(l_b,l_h) \ \ if \ \ l_h = 2l_b-1 \tag{3.17}$$

Proof.

Consider the configurations $\sigma_1=\sigma_1(l_b,l_h)$, $\bar\sigma_1=\bar\sigma_1(l_b,l_h)$. It is immediate to verify that they belong to $\widetilde S(\mathcal{B}(R(l_b,l_h)))$. Let :

$$\Delta H^* := \min\{H(\sigma_1)-H(R), H(\bar\sigma_1)-H(R)\} \equiv \min\{2J-\varepsilon, 2(\mu-\varepsilon)\}. \tag{3.18}$$

We have that any configuration σ with $H(\sigma)-H(R) > \Delta H^*$ cannot belong to $S(\mathcal{B}(R(l_b,l_h)))$.

Now we notice that, given any configuration in $S(\mathcal{B}(R(l_b,l_h)))$, giving rise to a single cluster it can only have a unique gulf being a unit square vertical gulf belonging to $\mathcal{L}_1$ and no lakes. Indeed it is immediate to verify that to create a lake starting from R one needs at least an energy $4J-h_2 > \Delta H^*$ since either an internal plus spin has to flip into minus ($\Delta H=4J-h_2$) or an external protuberance enclosing a minus has to be formed ($\Delta H>4J-h_2$). Similarly we can exclude the formation of gulfs made by single sites in $\mathcal{L}_2$ ($\Delta H>2J+h_1$) or by more than one site in $\mathcal{L}_2$, ($\Delta H>\min\{2J-h_2+h_1, 4J-2h_2\}$).

On the other hand for similar reasons a configuration σ in $S(\mathcal{B}(R(l_1,l_2)))$ cannot consist of more than one cluster.

We want now to show that if there are minimal saddles in $S(\mathcal{B}(R(l_1,l_2)))$ towards the growth they must be either σ_1 or $\bar\sigma_1$ (see Fig. 3.1). Let σ be one of these saddles in $S(\mathcal{B}(R(l_1,l_2)))$.

We know that σ must contain a unique cluster. Moreover we have: $R(\sigma)=R'\supset R$ and by Lemma 3.3 $\sigma\cap R=R$. Necessarily either $\sigma=\sigma_2(R)$, or $\sigma=\bar\sigma_2(R)$ or $\sigma=\sigma_1(R)$ or $\sigma=\bar\sigma_1(R)$ as it easily follows by looking at the minimizing configurations in $R'\setminus R$ among the ones giving rise to a saddle in $\widetilde S(\mathcal{B}(R(l_1,l_2)))$. Since

$$H(\sigma_2)-H(R) > \Delta H^*, \qquad H(\bar\sigma_2)-H(R) > \Delta H^*$$

necessarily the minimal saddles towards the growth are either σ_1 or $\bar{\sigma}_1$.

Now we want to show that the minimal saddles in $S(\mathcal{B}(R(l_1,l_2)))$ towards the contraction are given by (3.16), (3.17) in regions I, II, respectively.

Let ζ be one of these saddles. We know that ζ is *almost monotonic* in the sense that it is monotonic except at most for one vertical unit gulf in $\mathcal{L}_2$.

Consider $\sigma \in \mathcal{B}(R)$ such that $\sigma^{(x)} = \xi$ with $\xi \in \widetilde{S}(\mathcal{B}(R))$ and almost monotonic ($\sigma \in \partial^-\mathcal{B}(R)$, see (2.11)) and with $\sigma_x = +1$, $\xi_x = -1$, $H(\sigma) < H(R)$. Necessarily also σ is almost monotonic. Starting from σ it is easy to construct an almost monotonic configuration $\eta \in \partial^-\mathcal{B}(R)$ being stable except possibly for the unit vertical gulf y, such that

$$\eta \prec \sigma, \qquad H(\eta) \leq H(\sigma) \tag{3.19}$$

Indeed suppose first that σ is monotonic ; if σ is stable we take $\eta \equiv \sigma$. If σ is not a local minimum nonetheless it must be sufficiently "fat" to contain a local minimum as we immediately get by a comparison of $H(\sigma) - H(R)$ with ΔH^*. We start eroding from the angles the plus spins in $\mathcal{L}_2 \cap \eta$; in other words we flip into minus all the plus spins belonging to rows in $\mathcal{L}_2 \cap \eta$ touching the sea of minuses (this operation is well defined since η is monotonic). It is immediately seen that, in this way, we end up with a stable configuration $\eta \in \partial^-\mathcal{B}(R)$ satisfying the above requirements (3.19). By the argument of proof of Lemma 3.3 we get:

$$H(\eta) - H(R) > \varepsilon m$$

where m is an integer ≥ 0 . If ξ is almost monotonic with a vertical unit gulf in $y \in \mathcal{L}_2$ we have necessarily $y \neq x$ and $\sigma^{(y)}$ monotonic . We can apply to σ the above procedure of angle erosion of pluses in $\mathcal{L}_2$. The spin at y , having three plus neighbours in $\mathcal{L}_2 \cap \sigma$, is not changed so we end up with η such that $\eta^{(y)}$ is a monotonic local minimum. It follows that

$$H(\eta) - H(R) = \varepsilon m + 2J - h_2 = \varepsilon m + \mu$$

$m \in \mathbf{Z}^+$. Consider the configuration $\eta^{(x)}$. Obviously we have $\eta^{(x)} \in S(\mathcal{B}(R(l_1,l_2)))$. Suppose first that $x \in \mathcal{L}_1$. Since our construction involved only sites in $\mathcal{L}_2$ we did not touch $\sigma(x)$. We have either

$$H(\eta^{(x)}) - H(R) \geq \varepsilon m + h_1 \tag{3.20}$$

or

$$H(\eta^{(x)}) - H(R) \geq \varepsilon m + \mu + h_1 \tag{3.21}$$

with $m \in \mathbf{Z}^+$. The case of (3.21) can be excluded since

$$\mu + h_1 > \Delta H^*$$

If $x \in \mathcal{L}_2$ again in our construction we did not touch x since we only decreased the energy. If η is monotonic we have:

$$H(\eta^{(x)}) - H(R) = \varepsilon m + 2kJ - h_2 \tag{3.22}$$

with $m \in \mathbf{Z}^+$, $k = 0, 1, 2$. It is immediately seen that we cannot have $k = 0$; indeed we must have $\eta_x = +1$ but since σ is monotonic and $x \in \mathcal{L}_2$ our construction should have given $\eta_x = -1$. It is immediate to verify that $k = 2$ imply an increment in energy strictly larger than ΔH^* an then it can be neglected. Therefore we have only to consider the case $k = 1$. In this case $\eta^{(x)}$ is not monotonic but it has a unit vertical gulf at x. We get:

$$H(\eta^{(x)}) - H(R) = \varepsilon m + \mu \tag{3.23}$$

Similarly if, in the above situation, η is only almost monotonic in the sense that it contains a unit vertical gulf in $\mathcal{L}_2$ we get

$$H(\eta^{(x)}) - H(R) = \varepsilon m + 2\mu > \Delta H^* \tag{3.24}$$

By a global comparison between the above cases we see that we are led to find the minimal possible values of m in the r.h.s of (3.20), (3.23), respectively. If $\eta^{(x)}$ is downhill connected only to minima V_1-downhill connected to a unique rectangle we easily get that this minimal value is given by:

$$m = \min\Big\{\frac{l_h - 3}{2},\ l_b - 2\Big\}. \tag{3.25}$$

If $\eta^{(x)}$ is downhill connected to at least a minimum V_1-downhill connected to a V_1-stable configuration containing more than one stable rectangle (splitting) we easily get:

$$m \geq \frac{l_h - 3}{2} + l_b - 2. \tag{3.26}$$

and this concludes the proof of the Proposition. ■

Section 4. Proof of Theorem 1.

Let us assume that there exists a set $\mathcal{G}$ satisfying the following

Property $\mathcal{K}$:

1. $\mathcal{G}$ is connected.
2. $\mathcal{P} \in S(\mathcal{G})$
3. $\exists\, c > 0$ such that $\min_{\sigma \in \partial\mathcal{G} \setminus \mathcal{P}} H(\sigma) \geq H(\mathcal{P}) + c$
4. There exists a path $\omega : \; -\underline{1} \to \mathcal{P}, \; \omega \in \mathcal{G}$ such that $H(\zeta) < H(\mathcal{P}) \quad \forall\, \zeta \in \omega \setminus \mathcal{P}$
5. There exists a path $\omega' : \; \mathcal{P} \to +\underline{1}, \; \omega' \notin \mathcal{G}$ such that $H(\zeta) < H(\mathcal{P}) \quad \forall\, \zeta \in \omega' \setminus \mathcal{P}$
6. $\exists\, \Gamma_1 < \Gamma$ (see (2.23)) such that for all paths $\omega : \mathcal{P} \to +\underline{1}$ such that $H(\zeta) < H(\mathcal{P}) \;\; \forall\, \zeta \in \omega \setminus \mathcal{P}$ we have, $\;\; \forall \sigma \in \omega$:

$$H(S(\sigma, +\underline{1})) - H(\sigma) < \Gamma_1$$

¿From Proposition 3.4 in [OS1] we know that there exists a cycle $\bar{\mathcal{G}}$ satisfying above properties $\mathcal{K}$ 1,2,3.

¿From Proposition 3.7 in [OS1] we get:

$$\forall \delta \; > \; 0 \qquad \lim_{\beta \to \infty} P_{-\underline{1}}(\tau_{\partial\bar{\mathcal{G}}} < e^{\Gamma(\beta+\delta)}) = 1 \tag{4.1}$$

On the other hand, from Lemma 3.1 in [OS1] we get that:

$$\forall\, \delta \; > \; 0 \qquad \lim_{\beta \to \infty} P_{-\underline{1}}(\tau_{\partial\bar{\mathcal{G}}} > e^{\Gamma(\beta-\delta)}) = 1 \tag{4.2}$$

We know that

$$\exists\, \lambda > 0 \text{ such that } P_{\mathcal{P}}(\tau_{+\underline{1}} < \tau_{-\underline{1}}) > \lambda. \tag{4.3}$$

¿From (4.1), (4.2), (4.3), property $\mathcal{K}$ 6 and Proposition A.1 of [CO] we get the result. ■

Then the proof of Theorem 1 is reduced to the construction of the set $\mathcal{G}$.

Costruction of $\mathcal{G}$.

Let us now start the construction of the set $\mathcal{G}$. To do that we need some more definitions. We want to recall that in the present work we often identify a configuration σ with the union $C(\sigma)$ of the closed unit squares in $\mathbf{R}^2$ centered at sites x with $\sigma(x) = +1$. Then it makes sense to construct configurations as unions or intersections of other configurations and write, for istance,

$$\sigma = \eta \cup \xi, \qquad \sigma' = \eta' \cap \xi'$$

meaning

$$C(\sigma) = C(\eta) \cup C(\xi), \qquad C(\sigma') = C(\eta') \cap C(\xi')$$

Given a set A being a union of unit squares centered at lattice sites in some $\mathcal{A} \subset \Lambda$ we identify A with the configuration σ given by

$$\sigma(x) = \begin{cases} +1 & \forall\, x \in \mathcal{A} \\ -1 & \forall\, x \in \Lambda \setminus \mathcal{A} \end{cases}$$

We write $\sigma = \emptyset$ to denote the configuration $\sigma(x) = -1\ \forall\, x \in \Lambda$.

The basic idea for a successful construction of a set $\mathcal{G}$ satisfying property $\mathcal{K}$ is to think of $\mathcal{G}$ as a sort of lower estimate of the set of subcritical configurations namely the ones which typically for large β visit $-\underline{1}$ before getting to $+\underline{1}$. By "lower estimate" we mean that any σ in $\mathcal{G}$ is certainly subcritical but we can (and we will) even leave some subcritical configurations outside $\mathcal{G}$ provided we still fulfil the crucial property that the minimum of the energy in $\partial\mathcal{G}$ is reached precisely at the set $\mathcal{P}$ of protocritical configurations. We then need a criterion (sufficient condition) for subcriticality which applies to every configuration and not only to the stable rectangles like in the discussion we did in the previous section 3.

We will start our analysis from configurations σ giving rise to a single cluster. The main difficulties arising in our h_1h_2-model are that, contrary to the standard Ising model:

1. a downhill path leading from σ to its rectangular envelope in general does not exist and

2. the protuberances which are able to induce the growth of a rectangle (when attached to one of its sides) are more complicated than the ones (simply unit squares) appearing in the standard Ising model. These protuberances which can be considered as a sort of "one-dimensional protocritical droplets" depend on the direction of the side of the rectangle and are different in the different regions of the parameters h_1h_2. Therefore, as it will appear

clear in the following, given a single cluster σ we will need to introduce a geometrical object associated to σ, denoted by $\rho(\sigma)$ and called *quasi-rectangle*, or quasi-rectangular envelope of σ in terms of which it will be easy to decide the subcriticality or supercriticality of σ. The important properties that we will use are :

$$\rho(\sigma) \succ \sigma, \qquad H(\rho(\sigma)) \leq H(\sigma)$$

Now suppose we are given a (not necessarily stable) rectangle $R = R(l_1, l_2)$ having a given location and being large enough to contain an $R(1,2)$ rectangle. We recall that l_1, l_2 denote the horizontal side (basis) and the vertical side (height), respectively. We want to introduce some useful geometrical notions depending on the location of R.

We call $\tilde{s}_t, \tilde{s}_b$, respectively, the rectangles adjacent from the interior to the top and bottom horizontal edge of R with basis l_1 and height equal to: two if the edge of R to which it is adjacent "lies on $\mathcal{L}_1$"; one if the edge of R to which they are adjacent "lies on $\mathcal{L}_2$". We call $\tilde{c}_l, \tilde{c}_r$, respectively, the rectangles with height l_2 and basis one adjacent from the interior to the left and right vertical edges of R. We set

$$c_l = c_l(R) := \tilde{c}_l \setminus [\tilde{c}_l \cap (\tilde{s}_t \cup \tilde{s}_b)],$$

$$c_r = c_r(R) := \tilde{c}_r \setminus [\tilde{c}_r \cap (\tilde{s}_t \cup \tilde{s}_b)]$$

and call them left and right *column* of R (see Fig. 4.1).

We set

$$s_t = s_t(R) := \tilde{s}_t \setminus [\tilde{s}_t \cap (\tilde{c}_l \cup \tilde{c}_r)],$$

$$s_b = s_b(R) := \tilde{s}_b \setminus [\tilde{s}_b \cap (\tilde{c}_l \cup \tilde{c}_r)]$$

and call them (horizontal) top and bottom *strip* of R (see Fig. 4.1).

We set, for $i = t, b;\ \ j = l, r$

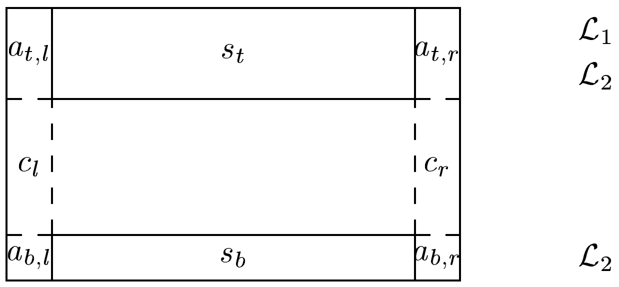
$$a_{i,j} = a_{i,j}(R) = \tilde{s}_i \cap \tilde{c}_j$$

and call them corners of R. They are $R(1,2)$ or $R(1,1)$ rectangles.

The union

$$\mathcal{F} = \mathcal{F}(R) := \tilde{s}_t \cup \tilde{s}_b \cup \tilde{c}_l \cup \tilde{c}_r \equiv s_t \cup s_b \cup c_l \cup c_r \cup a_{t,l} \cup a_{t,r} \cup a_{b,l} \cup a_{b,r}$$

is called the (picture) *frame* of the rectangle R.



Finally the (possibly empty) set

$$\mathcal{H} = \mathcal{H}(R) := R \setminus \mathcal{F}(R)$$

is called *hard core* of R. (see Fig. 4.1)

Now, given a single cluster σ we define its quasi-rectangular envelope $\rho(\sigma)$. We distinguish various cases.

CASE 1 The rectangular envelope $R(\sigma)$ of the cluster σ starts and ends in rows in $\mathcal{L}_1$.

We start preliminarily defining an auxiliary sequence of configurations (clusters). For the two columns $c_i, i = l, r$ and for the two strips $s_j, j = t, b$ we define $\eta_{c_i} = \eta_{c_i}(\sigma)$ and $\eta_{s_j} = \eta_{s_j}(\sigma)$ as follows:

- If the intersection (restriction) of σ with the column, $\sigma \cap c$, contains only one site in $\mathcal{L}_1$ then we set: $\eta_c = \sigma \cap c$, otherwise $\eta_c = c$.
- If the intersection of σ with the strip $\sigma \cap s$ contains one of this two protuberances

then we set $\eta_s = s$ otherwise $\eta_s = \sigma \cap s$.

We follow this procedure for all strips and columns. Now we preliminarily define a configuration $\rho''(\sigma)$:

$$\rho''(\sigma) = \mathcal{H}(R) \cup \eta_{c_l} \cup \eta_{c_r} \cup \eta_{s_t} \cup \eta_{s_b}$$

Let a_{ij} be a corner and let s_i, c_j be the strip and column adjacent to a_{ij}.

The above procedure will give rise to one of the following four different results for η_{s_i} and η_{c_j}:

I both s_i and c_j are filled: $\eta_{s_i} = s_i$ and $\eta_{c_j} = c_j$
II only the strip s_i is filled: $\eta_{s_i} = s_i$ and $\eta_{c_j} = c_j \cap \sigma$
III only the column c_j is filled : $\eta_{s_i} = s_i \cap \sigma$ and $\eta_{c_j} = c_j$
IV neither the strip s_i nor the column c_j are filled: $\eta_{s_i} = s_i \cap \sigma$ and $\eta_{c_j} = c_j \cap \sigma$

According to the different case we are dealing with we now prescribe, for each corner $a = a_{i,j}$, a new configuration η_a in a that subsequently will be united to the previously defined $\rho''(\sigma)$.

CASE I		$\eta_a = a$
CASE II	if $\sigma \cap a = \emptyset$	$\eta_a = \emptyset$
	if $\sigma \cap a \subset \mathcal{L}_1$ and $\sigma \cap c_j = \emptyset$	$\eta_a = \emptyset$
	otherwise	$\eta_a = a \cap c_j$
CASE III	if $a \cap \sigma = \emptyset$	$\eta_a = \emptyset$
	if $a \cap \sigma = a$ and $\sigma \cap s_i = \emptyset$	$\eta_a = a \cap \sigma$
	otherwise	$\eta_a = a \cup s_i$
CASE IV	if $\sigma \cap a = \emptyset$	$\eta_a = \emptyset$
	if $\sigma \cap a \subset \mathcal{L}_1$ and $\sigma \cap c_j = \emptyset$	$\eta_a = a \cap \sigma$
	if $\sigma \cap a \subset \mathcal{L}_1$ and $\sigma \cap c_j \neq \emptyset$	$\eta_a = a \cup s_i \cup c_j$
	if $\sigma \cap a \subset \mathcal{L}_2$	$\eta_a = a \cup s_i \cup c_j$
	if $\sigma \cap a = a$ and $\sigma \cap c_j = \emptyset$ and $\sigma \cap s_i = \emptyset$	$\eta_a = a$
	" and $\sigma \cap c_j \neq \emptyset$ and $\sigma \cap s_i = \emptyset$	$\eta_a = a \cup c_j$
	" and $\sigma \cap c_j = \emptyset$ and $\sigma \cap s_i \neq \emptyset$	$\eta_a = a \cup c_j \cup s_i$

" and $\sigma \cap c_j \neq \emptyset$ and $\sigma \cap s_i \neq \emptyset$ $\eta_a = a \cup c_j \cup s_i$

Now we define:

$$\rho'(\sigma) = \rho''(\sigma) \cup \eta_{a_{t,l}} \cup \eta_{a_{t,r}} \cup \eta_{a_{b,l}} \cup \eta_{a_{b,r}}$$

If $\rho'(\sigma) \cap s_i = s_i$ and $\rho'(\sigma) \cap c_j = c_j$ we set, for $a_{i,j} = \tilde{s}_i \cap \tilde{c}_j$: $\bar{\eta}_{a_{i,j}} = a_{i,j}$ otherwise we set $\bar{\eta}_{a_{i,j}} = \eta_{a_{i,j}}$. Finally we define

$$\rho(\sigma) = \rho'(\sigma) \cup \bar{\eta}_{a_{t,l}} \cup \bar{\eta}_{a_{t,r}} \cup \bar{\eta}_{a_{b,l}} \cup \bar{\eta}_{a_{b,r}}$$

■

CASE 2 The rectangular envelope $R(\sigma)$ begins and ends on rows in $\mathcal{L}_2$

- If $\sigma \cap \tilde{c}_i$ contains only one site that belongs to $\mathcal{L}_1$ then we set $\eta_{c_i} = \sigma \cap c_i$, otherwise $\eta_{c_i} = c_i$.
- For each corner a we set: $\eta_a = \sigma \cap a$. We define

$$\rho(\sigma) = \mathcal{H}(R) \cup \eta_{c_l} \cup \eta_{c_r} \cup (\sigma \cap (s_t \cup s_b)) \cup \eta_{a_{t,l}} \cup \eta_{a_{t,r}} \cup \eta_{a_{b,l}} \cup \eta_{a_{b,r}} \tag{4.4}$$

CASE 3 The rectangular envelope starts on a row in $\mathcal{L}_1$ and ends on a row in $\mathcal{L}_2$ (or vice-versa); then on the row that belongs to $\mathcal{L}_1$ we define the strip, the corner and the configuration η as we did in case 1. On the row that belongs to $\mathcal{L}_2$ the definitions are the same as in case 2.

We call the above constructed $\rho(\sigma)$ the *quasi-rectangular envelope* of the cluster σ.

We call *core* of σ and denote by $\mathcal{N}(\sigma)$ the biggest stable rectangle which is contained in $\rho(\sigma)$.

4.1 Proposition

Among all clusters σ that have the same quasi-rectangular envelope ρ the one with minimal energy is $\sigma = \rho(\sigma)$.

Proof.

The proof immediately follows from lemma 3.3 ■

Now we want to introduce some new critical lengths. They will come out by a comparison between typical times of two different classes of competing events, respectively related to contraction and coalescence. Consider two stable rectangles R_1, R_2 at a distance one in the vertical direction; by this we mean that there exists a site $x \in \mathcal{L}_2$ such that the unit square $q(x)$ centered at x has two horizontal sides in common with the boundaries of R_1, R_2, respectively.

The energy necessary to flip the minus spin at x and, in this way, induce the coalescence of the two rectangles R_1, R_2 is:

$$\Delta H = h_2 \tag{4.5}$$

Let us denote by $\Delta H_s,\ \Delta H_c$ the energy barriers necessary to contract a rectangle by removing a horizontal strip of width 2 or a column of width 1, respectively.

By Proposition 3.4 we know that the expressions of these quantities depend on the region of parameters and that in region I we have:

$$\Delta H_s = \varepsilon(l_1 - 2) + h_1 \tag{4.6}$$

$$\Delta H_c = \frac{\varepsilon}{2}(l_2 - 3) + h_1 \qquad (4.7),$$

whereas in region II we have:

$$\Delta H_s = \varepsilon(l_1 - 2) + \mu \tag{4.8}$$

$$\Delta H_c = \frac{\varepsilon}{2}(l_2 - 3) + \mu \tag{4.9}$$

We define l_1^*, l_2^* as the minimal horizontal and vertical lengths such that

$$\Delta H_s \ > \ \Delta H, \qquad \Delta H_c \ > \ \Delta H, \tag{4.10}$$

respectively. We suppose that $\frac{h_2-\mu}{\varepsilon}$ is not integer. We get:
In region I:

$$(l_1^*, l_2^*) \quad = \quad (2,3).$$

In region II:

$$l_1^* = \left[\frac{h_2 - \mu}{\varepsilon} + 3\right] \tag{4.11}$$

and l_2^* is odd and depends on the fractional part $\{\frac{h_2-\mu}{\varepsilon}\}$ of $\frac{h_2-\mu}{\varepsilon}$ (for $x \in \mathbf{R}^+$ we set $\{x\} := x - [x]$).

$$\text{If} \quad \left\{\frac{h_2 - \mu}{\varepsilon}\right\} \geq \frac{1}{2} \quad \Longrightarrow \quad l_2^* = \left[\frac{2(h_2 - \mu)}{\varepsilon} + 4\right], \tag{4.12}$$

$$\text{if}\quad \left\{\frac{h_2-\mu}{\varepsilon}\right\}<\frac{1}{2}\quad\Longrightarrow\quad l_2^*=\left[\frac{2(h_2-\mu)}{\varepsilon}+5\right]. \tag{4.13}$$

In any case we have:

$$l_2^*=2l_1^*-1 \tag{4.14}$$

namely $R(l_1^*,l_2^*)$ has the standard shape.

We notice from (4.11), (4.13) also in region II, if $h_2<\mu$ we have $(l_1^*,l_2^*)\ =\ (2,3)$.

We say that a cluster σ is *ephemere* if its core $\mathcal{N}(\sigma)$ does not contain a recangle $\mathrm{R}(l_1^*,l_2^*)$.

Two clusters σ_1, σ_2 are said to be *interacting* if they verify one of the following conditions:

1) their rectangular envelope $\mathrm{R}(\sigma_1)$ and $\mathrm{R}(\sigma_2)$ intersect
2) there exists a site x in $\mathcal{L}_1$ such that $q(x)\cap(R(\sigma_1)\cup R(\sigma_2))$ is given by two vertical unit segments
3) there exists a site $x\in\mathcal{L}_1$ such that $q(x)\cap(R(\sigma_1)\cup R(\sigma_2))$ is given by two horizontal unit segments
4) there exists a site $x\in\mathcal{L}_2$ such that $q(x)\cap(R(\sigma_1)\cup R(\sigma_2))$ is given by two horizontal unit segments; moreover, supposing that $R(\sigma_1)$ lies over $R(\sigma_2)$ we have:

$$\rho(\sigma_1)\cap s_b(R(\sigma_1))=s_b(R(\sigma_1))\ \text{and}\ \rho(\sigma_2)\cap s_t(R(\sigma_2))=s_t(R(\sigma_2))$$

with both $R(\sigma_1)$ and $R(\sigma_2)$ non-ephemere.

A family of quasi-rectangles $\rho_1,\dots,\rho_n$ is said to form a *chain* $\mathcal{C}$ if for every pair $i,j\in\{1,\dots,n\}$ there exists a sequence $(i_1,\dots,i_l)$ such that ρ_{i_k} and $\rho_{i_{k+1}}$ are interacting and $\rho_{i_1}=\rho_i \qquad \rho_{i_l}=\rho_j$.

To construct $\mathcal{G}$ for our h_1,h_2 model, first of all we define a map $F:\Omega^*\longrightarrow\ \Omega^*$ where $\Omega^*=\{$ acceptable configurations $\}$. We use the notation: $F(\sigma)=\hat\sigma$. F is given in the following steps:

1) starting from σ that consist of some clusters $\sigma_1,\dots,\sigma_n$ we construct $\rho(\sigma_1),\dots,\rho(\sigma_n)$
2) We define *chains of first generation* $\mathcal{C}_1^{(1)},\dots,\mathcal{C}_{k_1}^{(1)}$ the maximal connected chains in which we can decompose the set $\rho(\sigma_1),\dots,\rho(\sigma_n)$. Then for any $\mathcal{C}_j^{(1)}$ we define

$$\rho\big(\mathcal{C}_j^{(1)}\big)=\bigcup_{\rho(\sigma_i)\,\in\,\mathcal{C}_j^{(1)}}\rho(\sigma_i)\quad\cup\quad R\Big(\bigcup_{\rho(\sigma_i)\in\mathcal{C}_j^{(1)}}\mathcal{N}(\sigma_i)\Big) \tag{4.15}$$

3) Having defined the chain of r-th generation $\mathcal{C}_1^{(r)}, \dots, \mathcal{C}_{k_r}^{(r)}$ we consider $\forall \; j \in (1, \dots, k_r)$ $\rho\big(\mathcal{C}_j^{(r)}\big)$ and we construct $\mathcal{C}_1^{(r+1)}, \dots, \mathcal{C}_{k_{r+1}}^{(r+1)}$.

We continue in this way up to a finite maximal order f such that the chains of the f-th generation are non-interacting clusters $\mathcal{C}_1^{(f)}, \dots, \mathcal{C}_{k_f}^{(f)} \equiv \bar{\rho}_1^{(\sigma)}, \dots, \bar{\rho}_m^{(\sigma)}$. We call $F(\sigma) = \hat{\sigma} = \bar{\rho}_1^{(\sigma)}, \dots, \bar{\rho}_m^{(\sigma)}$ these clusters.

We observe that F satisfies the following properties:

1) $\sigma \prec \hat{\sigma}$
2) $H(\hat{\sigma}) \leq H(\sigma)$
3) It is monotone in the sense that $\sigma \prec \eta \Longrightarrow \hat{\sigma} \prec \hat{\eta}$

A cluster σ is called *subcritical* if $\mathcal{N}(\rho(\sigma))$ does not contains a rectangle $R^* = R(l_b^*, l_h^*)$.
Now we are ready to define our set $\mathcal{G}$.

$$\mathcal{G} = \{\sigma \in \Omega^* \text{ such that } \hat{\sigma} = \{\bar{\rho}_1^{(\sigma)}, \dots, \bar{\rho}_m^{(\sigma)}\} \; \forall j \in \{1, \dots, m\} \qquad \bar{\rho}_j^{(\sigma)} \text{ is subcritical } \} \tag{4.16}$$

It easily follows from Proposition 3.4 that properties $\mathcal{K}$ 4,5 are satisfied. Property $\mathcal{K}$ 6 follows from Proposition 3.4 and from the observation that for any cycle A not containing $\{-1, +1\}$ we have $H(U(A)) - H(F(A)) < 2J - h_1 + h_2$.

We have to now prove the fundamental point namely property $\mathcal{K}$ 2.

4.2 Proposition

$$\min_{\sigma \in \partial\mathcal{G}} H(\sigma) = H(\mathcal{P}) \tag{4.17}$$

Proof

We know that, given $\zeta \in \partial\mathcal{G}$, there exists $\sigma \in \mathcal{G}$ such that: $\sigma^{(x)} = \zeta \qquad \sigma_x = -1$. We have:

1) $x \notin \bar{\rho}_j^{(\sigma)} \qquad \forall j = \{1, \dots, m\}$ otherwise $\hat{\sigma}$ would be equal to $\hat{\zeta}$ but we know that $\zeta \notin \mathcal{G}$.

2) In $\hat{\zeta}$ we must have $\bar{\rho}_j^{(\zeta)}$ such that

- $\mathcal{N}(\bar{\rho}_j^{(\zeta)})$ is supercritical; the other $\bar{\rho}_k^{(\zeta)}$ belong to $\hat{\sigma}$ so they are subcritical .
- Let $\tilde{\zeta} = \bar{\rho}_j^{(\zeta)}$, then $H(\tilde{\zeta}) \leq H(\hat{\zeta})$
- $\tilde{\zeta} = \bar{\rho}_j^{(\zeta)}$ contains $q(x)$ and some $\bar{\rho}_1^{(\sigma)}, \dots, \bar{\rho}_m^{(\sigma)}$ that belong to $\hat{\sigma}$.

We denote by $C^{(0)}$ the union of $q(x)$ with the clusters $\bar\rho_1^{(\sigma)},\dots,\bar\rho_m^{(\sigma)}$ which intersect $q(x)$ in some one of its sides. These clusters can be one or two; moreover $\mathcal{N}(\rho(C^{(0)}))$ can be subcritical or supercritical.

In what follows, starting from a generic configuration $\zeta\in\partial\mathcal{G}$ we will often modify it (remaining in $\partial\mathcal{G}$) by decreasing the energy. This will be useful since we are looking for the minimum of the energy on $\partial\mathcal{G}$.

1. $\mathcal{N}(\rho(\mathbf{C}^{(0)}))$ is supercritical

In this hypothesis the energy decreases if we spin-flip all the (internal) sites that belong to $\bar\rho_1^{(\sigma)},\dots,\bar\rho_l^{(\sigma)}$ and that do not take part in the costruction of $C^{(0)}$ because they are subcritical. $C^{(0)}$ is supercritical, so it must exist a rectangle $R^*=R(l_b^*,l_h^*)$ that is contained in $\mathcal{N}(\rho(C^{(0)}))$.

• Let $C^{(0)}$ consist of $q(x)$ and of two quasi-rectangles ρ,ρ'. We know that ρ,ρ' are not interacting but they become interacting after the spin-flip at x; moreover we know that $q(x)$ must intersect both of the rectangular envelopes on two sides. Consider the circumscribed rectangles $R=R(\rho)$, $R'=R(\rho')$. Certainly we can choose R^* with non-empty intersections both with R and R'.

There are two cases, in which two clusters are not interacting and at a distance 1:
1) there exists a unique site x such that $q(x)$ intersects the two rectangles on two vertical sides; $x\in\mathcal{L}_2$.
2) x belongs to $\mathcal{L}_2$ and $q(x)$ intersects the two rectangles on two horizontal sides, but one of these rectangles is ephemere.

Consider case 1); the rectangles $R=R(\rho)$ and $R'=R(\rho')$ must start and end in rows belonging to $\mathcal{L}_2$, or they must have one row in common belonging to $\mathcal{L}_2$ and the others belonging to $\mathcal{L}_1$. Since R and R' are subcritical we further decrease the energy by removing their portions lying outside R^* After the intersection with R^* the two cases coincide; see Fig. (4.2 (a)).

Let l_1,l_2 be the sides of $R\cap R^*$, l_1',l_2' the ones of $R'\cap R^*$ and suppose that $l_1\ge l_1'$. We have:

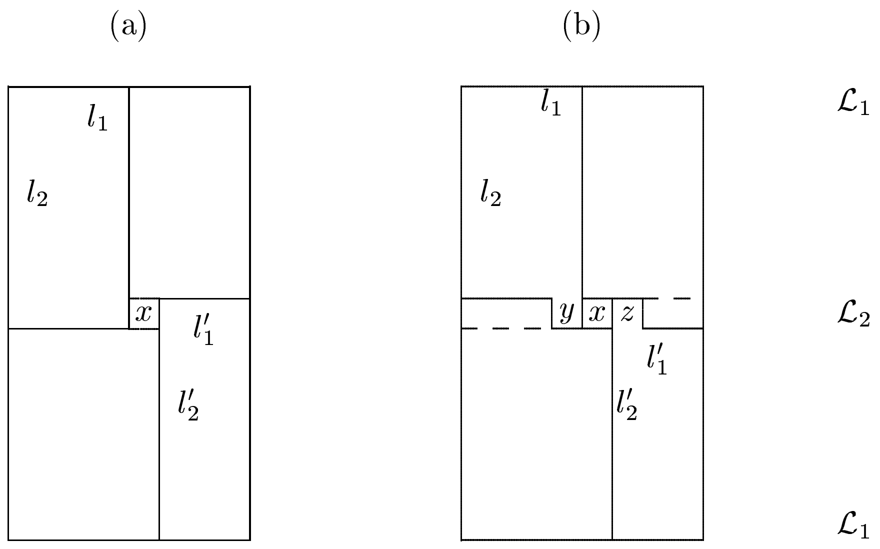
$$l_1\ge l_1' \qquad 1\le l_1\le l_b^*-2 \qquad 2\le l_2\le l_h^*-1=2\,(l_b^*-1)$$

Let

$$\Delta H' := H(\mathcal{P}_{\mathrm{I},a})-H(R^*)=h_1+\varepsilon(l_b^*-2),$$

$$\Delta H'' := H(\mathcal{P}_{\mathrm{II},a})-H(R^*)=\mu+\varepsilon(l_b^*-2),$$

$$\Delta H^* := H(\mathcal{P})-H(R^*)=\min\{\Delta H',\Delta H''\}$$



and let

$$\Delta H = H(C^{(0)}) - H(R^*).$$

This quantity is minimal if there exists a unique site y belonging to $\mathcal{L}_2 \cap R \cap R^*$ in the same row as x and a unique site z belonging to $\mathcal{L}_2 \cap R' \cap R^*$ in the same row as x (see Fig. 4.2 (b))

Thus we have:

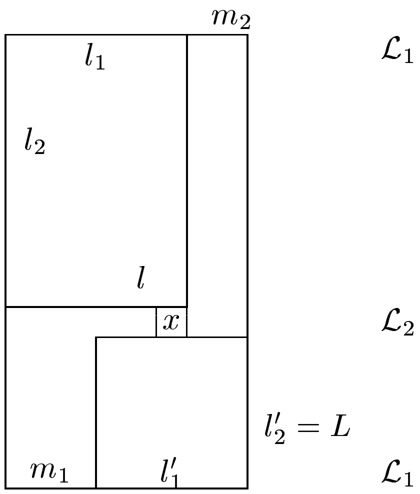
$$\Delta H \geq 4\,h_1 + \varepsilon(l_b^* - 3) + \frac{\varepsilon}{2}(l_2 - 2)(l_b^* - l_1) + \frac{\varepsilon}{2}(l_1 + 1)(l_h^* - (l_2 + 1)) \tag{4.18}$$

$$\Delta H - \Delta H' \geq 3h_1 - \varepsilon + \frac{\varepsilon}{2}(l_2 - 2)(l_b^* - l_1) + \frac{\varepsilon}{2}(l_1 + 1)(l_h^* - (l_2 + 1)) \geq 0 \qquad \text{always.}$$

Consider now case 2); either both rectangles R and R' start and end in rows belonging to $\mathcal{L}_1$, or they must have one row that intersects $q(x)$ belonging to $\mathcal{L}_1$ and the other belonging to $\mathcal{L}_2$. Also in this case, as before, after the intersection with R^*, the two cases coincide; they are as shown in Fig. 4.3 . In this case the situation with minimal energy corresponds to $\rho \cap R^* = R \cap R^*$, $\rho' \cap R^* = R' \cap R^*$ that we will then assume as hypothesis.

l_1, l'_1, l_2, l'_2 are defined as in case 1.

Let $m_2 = l_b^* - l_1 \qquad m_1 = l_b^* - l'_1 \qquad l = l_1 + l'_1 - l_b^*$, we denote by L the height of the rectangle that has the smaller horizontal side; therefore, since $l_1 \geq l_2$ we have $L = l'_2$ and $m_1 \geq m_2$ (see Fig. 4.3).



$$\Delta H = H(C^{(0)}) - H(R^*) = \mu(l-1) + \frac{\varepsilon}{2} m_2 (l_h^* - L) + \frac{\varepsilon}{2} m_1 (L+1) \tag{4.19}$$

$$\Delta H - \Delta H'' \geq \mu(l-2) + \varepsilon l_b^* (m_2 - 1) + \frac{\varepsilon}{2}(L+1)(m_1 - m_2) + 2\varepsilon \tag{4.20}$$

(i) If $l \geq 2 \quad m_2 \geq 1 \qquad\qquad \Delta H - \Delta H'' > 0$

(ii) If $l = 2 \quad m_2 = 0 \implies m_1 = l_b^* - 2 \quad \Delta H - \Delta H'' \geq -\varepsilon l_b^* + \varepsilon l_b^* - 2\varepsilon + 2\varepsilon = 0$ because $L \geq 1$

(iii) If $l > 2$ and $m_2 = 0$ $\Delta H - \Delta H'' = \mu(l-2) - \varepsilon l_b^* + \frac{\varepsilon}{2}(L+1) m_1 + 2\varepsilon$

using $\frac{\mu}{\varepsilon} < l_b^* \leq \frac{\mu}{\varepsilon} + 1$ we have $\Delta H - \Delta H'' \geq \mu(l-2) - \varepsilon - \mu + \frac{\varepsilon}{2}(L+1) m_1 + 2\varepsilon > 0$

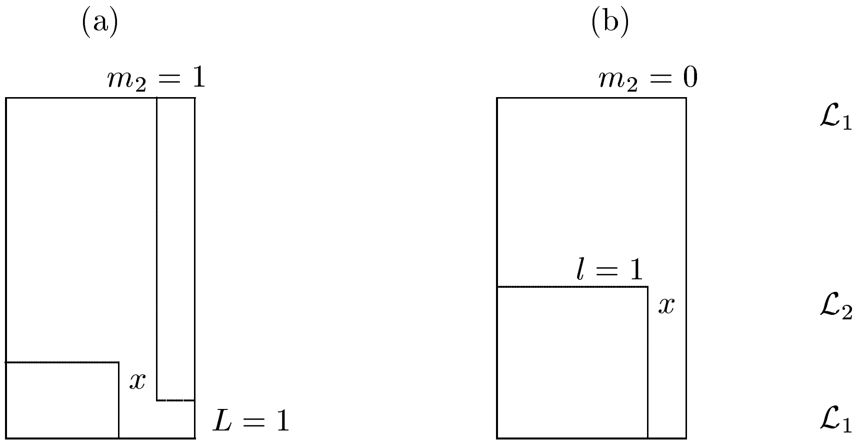
(iv) If $l = 1$ and $m_2 > 1$ $\qquad \Delta H - \Delta H'' > -\mu + \mu + \frac{\varepsilon}{2}(L+1)(m_1 - m_2) + 2\varepsilon > 0$

(v) If $m_2 = 1 = l_b^* - l_1$ and $l = 1 \implies l_1 = l_b^* - 1$ and $l'_1 = 2$ and $m_1 = l_b^* - 2$

$$\Delta H - \Delta H'' = -\mu + \frac{\varepsilon}{2}(L+1)(l_b^* - 3) + 2\varepsilon \tag{4.21}$$

(v_a) If $l = 1$ and $m_2 = 1$ and $L = 1$. This can be seen in Fig. 4.4 (a)

This configuration is not supercritical and then it does not belong to $\partial \mathcal{G}$. We have then to go back to the original configuration that we had before intersecting with R^*, taking



into account that it was consisting of two subcritical but stable rectangles. The cases of minimal energy are the ones in which the smaller rectangle in Fig. 4.4 is replaced by a 2×3 rectangle. We get a strictly positive value of $\Delta H - \Delta H''$ as it easily follows from a direct computation.

(v_b) If $l = 1$ and $m_2 = 1$ and $L \geq 3$

$$\Delta H - \Delta H'' \geq -\mu + 2\varepsilon\,(l_b^* - 3) + 2\varepsilon > +\mu - 4\varepsilon \geq 0 \quad \text{if} \quad \mu \geq 4\varepsilon \tag{4.22}$$

If $l_b^* = 4$ and $L = 3$ $\Delta H - \Delta H'' = -\mu + 4\varepsilon > 0$ if $\mu - 4\varepsilon < 0$

(vi) If $l = 1$ and $m_2 = 0 \quad \Longrightarrow \quad l_1' = 1$. Also this configuration, shown in Fig. 4.4(b), is subcritical.

• • If $C^{(0)}$ contains $q(x)$ and only one cluster ρ, let $R = R(\rho)$ be its rectangular envelope; there are three possibilities:

1) R is stable and $q(x)$ belongs to a strip s_a or s_b of R.
2) R is stable and $q(x)$ belongs to a column c_l or c_r of R.
3) R has one row in $\mathcal{L}_1$ and the other in $\mathcal{L}_2$.

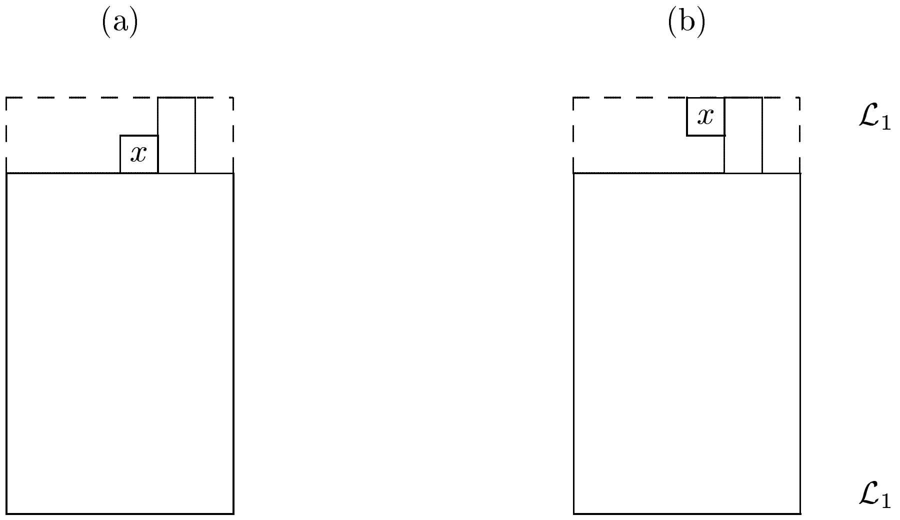
ρ is subcritical, but after a spin-flip at x it gets out of G.

In case 1) $R(\rho)$ must coincide with R^*; if for example x belongs to $s_t \cap \mathcal{L}_1$, then $\rho^{(x)}$ is what we called $\mathcal{P}_{I,b}$ (see Fig. 2.2 (b), 4.5 (a))

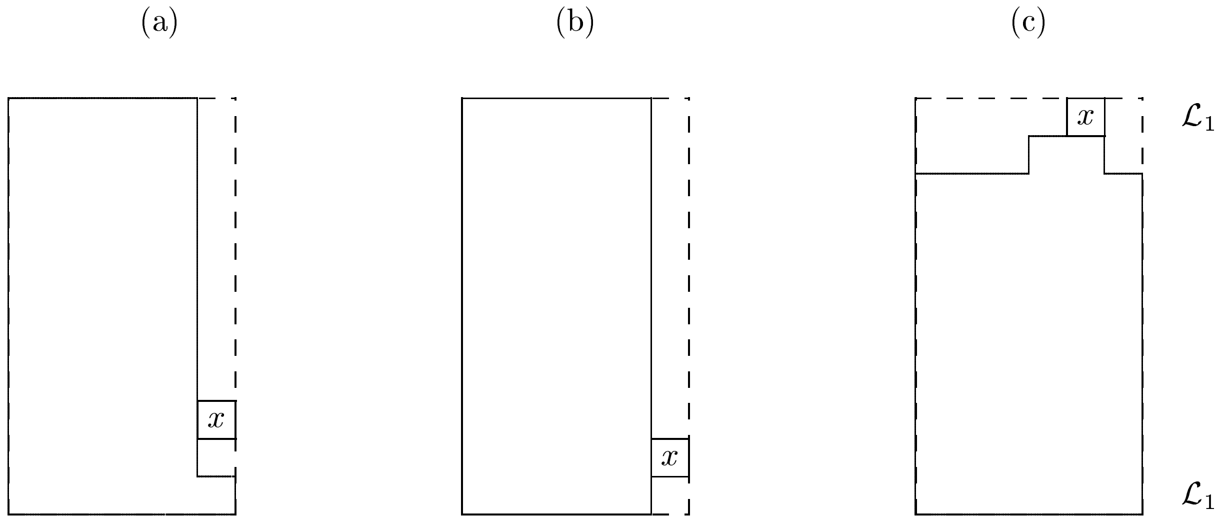
if $x \in \mathcal{L}_2 \qquad \Longrightarrow \qquad \rho^{(x)} = \mathcal{P}_{II,b}$ (see Fig. 2.3 (b) 4.5 (b)).

In case 2), $R(\rho)$ can coincide with R^*; then, if $x \in \mathcal{L}_1$, we obtain $\rho^{(x)} = \mathcal{P}_{II,a}$. (see Fig. 2.3 (a), 4.6 (a))

On the other hand, if $x \in \mathcal{L}_2$, we obtain $\rho^{(x)} = \mathcal{P}_{I,a}$ (see Fig. 2.2 (a)).



If, instead, $R(\rho) = R(l_b^* - 1, l_h^*)$ x must belong to $\mathcal{L}_2$ (otherwise $\rho^{(x)}$ would not be supercritical), we have a configuration in ∂G with energy larger than $H(\mathcal{P}_{I,a})$ and so it can be disregarded. (see Fig. 4.6 (b)).



$$\Delta H = H(\rho^{(x)}) - H(R^*) = \varepsilon\,(l_b^* - 2) + 2h_1 > \Delta H'$$

In case 3), $R(\rho)$ must coincide with $R(l_b^*, l_h^* - 1)$ and ρ must contain a 2×1 rectangular protuberance in s_t; the minimal energy, in this case is reached when $\rho^{(x)} = \mathcal{P}_{I,b}$. Moreover

x must belong to $\mathcal{L}_1$, so we obtain a configuration as the one shown in Fig. 4.6 (c).

2. $\mathcal{N}(\rho(C^{(0)}))$is subcritical

Let $\bar\rho_1^{(\sigma)},\dots,\bar\rho_l^{(\sigma)}$ be the clusters not involved in the construction of $C^{(0)}$. From $C^{(0)}$, $\bar\rho_1^{(\sigma)},\dots,\bar\rho_l^{(\sigma)}$ we contruct by iteration the set of $\mathcal{C}_j^{(r)}$ of r-th generation. For every generatione r we get a chain say $\mathcal{C}_1^{(r)}$ containing $q(x)$, and certain subset of $\bar\rho_1^{(\sigma)},\dots,\bar\rho_l^{(\sigma)}$. The other chains $\mathcal{C}_j^{(r)}$, $j>1$ contain each just one quasi-rectangle from those among $\bar\rho_1^{(\sigma)},\dots,\bar\rho_l^{(\sigma)}$ that have not appeared in $\mathcal{C}_1^{(p)}$, $p\le r$ in the preceding steps. Clearly at the final step we have $\mathcal{C}_1^{(f)}=\bar\rho_j^{(\zeta)}=\tilde\zeta$ namely the unique supercritical quasi-rectangle in $\tilde\zeta$. Let p be the order such that $\rho(\mathcal{C}_1^{(p)})$ is subcritical but $\rho(\mathcal{C}_1^{(p+1)})$ is supercritical.

Now, starting from $\mathcal{C}_1^{(p)}$, we construct $\mathcal{C}_1^{(p+1)}$ using a different procedure. Let $\xi_1,\dots,\xi_q$ be the quasi-rectangles that we unite to $\rho\big(\mathcal{C}_1^{(p)}\big)$ to obtain $\mathcal{C}_1^{(p+1)}$; we join them in the lexicografic order of the top left corner. We consider $\rho\Big(\rho\big(\mathcal{C}_1^{(p)}\big)\cup\xi_1\Big)$; if this is subcritical we continue this procedure, otherwise we stop it. However there exists an index j for which the procedure ends i.e. such that

$$\rho\Big(\rho\big(\mathcal{C}_1^{(p)}\big)\cup\xi_1\cup\ldots\cup\xi_j\Big)\qquad\text{is subcritical, but}$$

$$\rho\Big(\rho\big(\mathcal{C}_1^{(p)}\big)\cup\xi_1\cup\ldots\cup\xi_{j+1}\Big)\qquad\text{is supercritical.}$$

We call $\tilde\xi_1=\rho\Big(\rho\big(\mathcal{C}_1^{(p)}\big)\cup\xi_1\cup\ldots\cup\xi_j\Big)$ and $\tilde\xi_2=\xi_{j+1}$, therefore $\mathcal{N}(\rho(\tilde\xi_1\cup\tilde\xi_2))$ is supercritical and so it must contain a critical rectangle R^*. This R^* can be chosen so that if we set:

$$\begin{cases}\rho_1=\tilde\xi_1\cap R^*\\ \rho_2=\tilde\xi_2\cap R^*\end{cases}\qquad\text{we have}\qquad\rho(\rho_1\cup\rho_2)=R^*. \tag{4.23}$$

Since the $\tilde\xi_i$'s are subcritical we certaily decrease the energy by substituting the ρ_i's to the $\tilde\xi_i$'s. If ρ_1 and ρ_2 are still interacting rectangles we can forget about the portion of $\tilde\xi_i$ lying outside R^* and replace $\tilde\xi_i$ with ρ_i still remaining in $\partial\mathcal{G}$.

We know that $\tilde\xi_1$ and $\tilde\xi_2$ are two interacting clusters; so one of the following four situations can take place:

1) the rectangular envelopes intersect each other.
2) There exists y belonging to $\mathcal{L}_1$ such that $q(y)$ intersects the rectangles $R(\tilde\xi_1)$ and $R(\tilde\xi_2)$ on two vertical sides.
3) There exists y in $\mathcal{L}_1$ such that $q(y)\cap\big(R(\tilde\xi_1)\cup R(\tilde\xi_2)\big)$ is given by two horizontal sides.

4) There exists y in $\mathcal{L}_2$ such that $q(y)\cap\big(R(\tilde{\xi_1})\cup R(\tilde{\xi_2})\big)$is given by two horizontal segments; moreover supposing that $R(\tilde{\xi_1})$ lies over $R(\tilde{\xi_2})$ we have:

$$\rho(\tilde{\xi_1})\cap R(\tilde{\xi_1}) = s_b\big(R(\tilde{\xi_1})\big) \quad \text{and} \quad \rho(\tilde{\xi_2})\cap R(\tilde{\xi_2}) = s_t\big(R(\tilde{\xi_2})\big)$$

where both $R(\tilde{\xi_1})$ and $R(\tilde{\xi_2})$ cannot be ephemere (and so $\mathcal{N}(\tilde{\xi_1})$ and $\mathcal{N}(\tilde{\xi_2})$ must contain $R(l_1^*, l_2^*)$).

If the rectangles have non-empty intersection, (case 1))we can make odd and even horizontal translations and (only) even vertical translations until they intersect each others, but without having any common site. After the translations we can obtain two different types of clusters, that intersect only along two sides, and this intersection is of length one or two. In both cases we obtain:

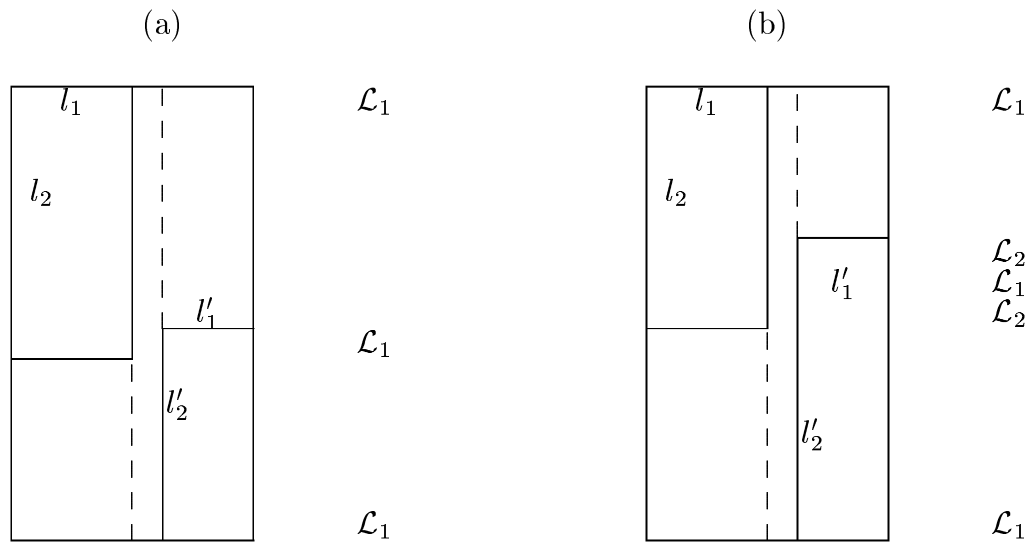
$$\Delta H = H(\rho_1) + H(\rho_2) - H(R^*) \geq 2J \geq \Delta H'. \tag{4.24}$$

It is easily seen that we have only the following two possibilities:

i) For all R^* satisfying (4.23): ρ_1 and ρ_2 start and end in rows in $\mathcal{L}_1$.

ii) For all R^* satisfying (4.23): ρ_1 starts on a row in $\mathcal{L}_1$ and ends on a row in $\mathcal{L}_2$ and conversely for ρ_2.

We will see that the most delicate case is 4), i) where some detailed analysis will be needed; in all the other cases very crude estimates will be sufficient.



In case 2), i) we decrease the energy still remaining in $\partial\mathcal{G}$ by supposing ρ_1 and ρ_2 to be rectangles. Let $l = l_2 + l_2' - l_h^*$. The energy is minimal for $l = 1$, but in any case the column between ρ_1 and ρ_2 is empty (see Fig. 4.7 (a)).

$$\Delta H \geq h_1 + \frac{\varepsilon}{2}(l_h^* - 1) = h_1 + \varepsilon\,(l_b^* - 1) > \Delta H'. \tag{4.25}$$

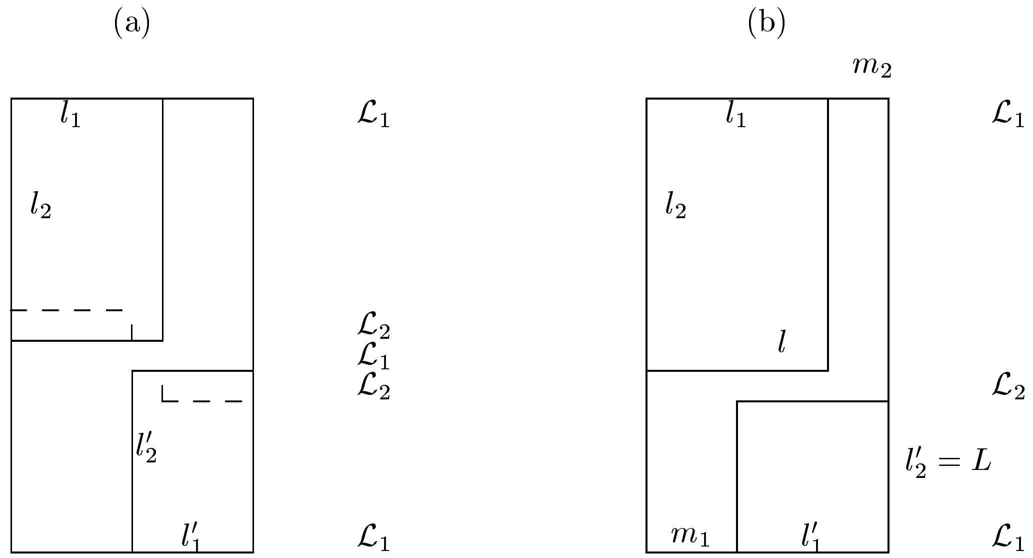
In case 2), ii) there exists at least one site y, belonging to $\mathcal{L}_1$, such that $q(y)$ intersects both R(ρ_1) and $R(\rho_2)$. So if we define $l = l_2 + l_2' - l_h^*$, then l has to be ≥ 3. If $l = 3$ (see Fig. 4.7 (b)) the configuration is of minimal energy.

In any case the column between the two rectangles is empty. We get:

$$\Delta H \geq h_1 + \frac{\varepsilon}{2}(l_h^* - 3) + 2\mu > \ \Delta H'.$$

In case 3), ii), since the rectangles are interacting along two strips belonging to $\mathcal{L}_2$, the energy is minimal if $\rho_1 \cap s_b(R(\rho_1))$ contains only one site z_1 and, similarly $\rho_2 \cap s_b(R(\rho_2))$ contains only one site z_2. So it is enough to consider this last case.

(see Fig. 4.8 (a))



$$\Delta H > h_1 + \varepsilon(l_b^* - 1) > \ \Delta H'.$$

In case 4),i), we can find an R^* satisfying (4.23) in such a way that the corresponding ρ_1 and ρ_2 are interacting if and only if each of them contains a rectangle $R(l_1^*, l_2^*)$ namely:

$$2l_2^* + 1 \leq l_h^* \qquad \Longleftrightarrow \qquad 2l_1^* \leq \ l_b^* \tag{4.26}$$

This condition is indeed a condition on the parameters h_1, h_2.

We say that our model for given values of parameters h_1, h_2 is *reducible* if for those values of h_1, h_2 condition (4.26) (condition of reducibility) is satisfied.

In any case we further decrease the energy by supposing that ρ_1 and ρ_2 are stable rectangles.

Now we are going to partition region I∪II of the space of parameters into several regions mainly using the criterion of reducibility.

We will write:

$$\mathrm{I} \cup \mathrm{II} = \mathrm{I}^{'} \cup \mathrm{I}^{''} \cup \mathrm{I}^{'''} \cup \mathrm{II}_a^{'} \cup \mathrm{II}_a^{''} \cup \mathrm{II}_a^{'''} \cup \mathrm{II}_b^{'} \cup \mathrm{II}_b^{''} \cup \mathrm{II}_b^{'''} \tag{4.27}$$

The various regions appearing in (4.27) are defined in the sequel.

$$\mathrm{I}^{'} := \mu > \varepsilon,\ h_2 < \mu - \varepsilon, \quad 2l_1^* \le l_b^* \ \text{ namely } l_b^* \ge 4$$

$$\mathrm{I}^{''} := \mu > \varepsilon,\ h_2 < \mu - \varepsilon, \quad 2l_1^* > l_b^* > l_1^* \ \text{ namely } l_b^* = 3$$

$$\mathrm{I}^{'''} := \mu > \varepsilon,\ h_2 < \mu - \varepsilon, \quad l_1^* \ge l_b^* \ \text{ namely } l_b^* = 2$$

$$\mathrm{II}_a^{'} := \mu > \varepsilon,\ \mu - \varepsilon h_2 < \mu, \quad 2l_1^* \le l_b^* \text{ namely } l_b^* \ge 4$$

$$\mathrm{II}_a^{''} := \mu > \varepsilon,\ \mu - \varepsilon h_2 < \mu, \quad 2l_1^* > l_b^* > l_1^* \text{ namely } l_b^* = 3$$

$$\mathrm{II}_a^{'''} := \mu > \varepsilon,\ \mu - \varepsilon h_2 < \mu, \quad l_1^* \ge l_b^* \text{ namely } l_b^* = 2$$

$$\mathrm{II}_b^{'} := \mu > \varepsilon, \mu < h_2, \quad 2l_1^* \le l_b^* \text{ namely } h_2 \le \frac{3}{2}\mu - 2\varepsilon - (r - \frac{a}{2})\varepsilon$$

where

$$a = 1 - \left\{\frac{\mu}{\varepsilon}\right\} \quad \text{and} \quad r = 1 - \left\{\frac{h_2 - \mu}{\varepsilon}\right\} \tag{4.28}$$

$$\mathrm{II}_b^{''} := \mu > \varepsilon, \mu < h_2, \quad 2l_1^* > l_b^*,\ l_1^* < l_b^* \text{ namely}$$

$$h_2 \geq \frac{3}{2}\mu - \frac{3}{2}\varepsilon - (2r-a)\frac{\varepsilon}{2} \tag{4.29}$$

$$\mathrm{II}_b^{'''} := \mu > \varepsilon, h_2 < \mu - \varepsilon, \quad 2l_1^* > l_b^*,\ l_1^* \geq l_b^*$$

Let

$$\zeta^* = \rho_1 \cup \rho_2$$

and denote by $M_{sub}^{(1)}$ the set of V_1-stable subcritical minima (sets of non-interacting subcritical rectangles). Since there exists an almost down-hill path

from our original configuration ζ to ζ^* and from ζ to $M_{sub}^{(1)}$, we have:

$$H(\zeta) \geq H(S(\zeta^*, M_{sub}^{(1)})).$$

On the other hand from Proposition 3.3 we get:

$$H(S(\zeta^*, M_{sub}^{(1)})) \geq V_2 + H(\zeta^*). \tag{4.30}$$

Where

$$V_2 := \min\{h_1, \mu\}$$

So to exclude ζ from $U(\partial\mathcal{G})$ it will suffice to show that

$$H(\zeta^*) + V_2 > H(\mathcal{P}) \tag{4.31}$$

Now we are going to prove that in case 4) i) we always have that condition (4.31) is satisfied . We will use different methods in different regions of the space of parameters corresponding to the partition given by (4.27).

We first remark that region $\mathrm{II}_b^{'''}$ (as well as $\mathrm{I}^{'''}$, $\mathrm{II}_a^{'''}$) can be neglected. Indeed, there, we have $l_1^* \geq l_b^*$ namely there do not exist subcritical non-ephemere rectangles but this is exactly the kind of pairs of rectangles that we wish to consider in case 4) i).

It will turn out that in regions $\mathrm{I}^{'}, \mathrm{II}_a^{'}$ one can easily show that $H(\zeta^*) > H(\mathcal{P})$ so that only in regions $\mathrm{II}_b^{'}$,$\mathrm{II}_b^{''}$ we really need the more refined argument leading to the sufficient condition (4.31) to show that $H(\zeta) > H(\mathcal{P})$. Finally the only irreducible cases in which condition (4.26) is not satisfied are $\mathrm{I}^{''}$, $\mathrm{II}_a^{''}$ and $\mathrm{II}_b^{''}$ which then requires a different argument w.r.t. all the other cases: 1), 2), 3) and 4) i), $(\mathrm{I}^{'} \cup \mathrm{II}_a^{'} \cup \mathrm{II}_b^{'})$.

We first state an easy result useful in our computations:

4.3 Proposition

Suppose $l_1^* < l_b^*$; in case 4), i) among all possible pairs of rectangles ρ_1 and ρ_2 the ones with minimal energy are those in which:

$$l = 1 \qquad m_2 = l_1^* - 1 \qquad m_1 = l_b^* - l_1^* \qquad L = l_2^*. \tag{4.32}$$

Proof

The proof follows from a direct computation. See [N] for more details. ∎

4.4 Proposition

The minimal saddles $H(S(\zeta^*, M_{sub}^{(1)}))$ are obtained from ζ^*: in region I by flipping a plus spin in an angle belonging to $\zeta^* \cap \mathcal{L}_1$; in region II by flipping a plus spin in an angle belonging to $\zeta^* \cap \mathcal{L}_2$ and we always have:

$$\Delta H_1 := H(S(\zeta^*, M_{sub}^{(1)})) - H(R^*) = -h_2 + \varepsilon l_1^* l_b^* + \varepsilon l_1^*(l_b^* - 2l_1^* + 1) + V_2 \tag{4.33}$$

Proof

The proof follows from Proposition 4.1. ∎

Now we make a sistematic comparison between

ΔH_1 and $\Delta H^* = H(\mathcal{P}) - H(R^*)$ in the different regions (4.27).

In region I' (where $l_1^* = 2$) we have:

$$\Delta H_1 - \Delta H' = -h_2 + 2\varepsilon l_b^* + 2\varepsilon(l_b^* - 3) - \varepsilon(l_b^* - 2) > -h_2 + \mu + 2\varepsilon(l_b^* - 2) > 0 \tag{4.34}$$

In region II we have:

$$\Delta H_1 - \Delta H'' = -h_2 + \varepsilon l_1^* l_b^* + \varepsilon l_1^*(l_b^* - 2l_1^* + 1) - \varepsilon(l_b^* - 2) \tag{4.35}$$

It becomes in II_a' where $l_1^* = 2,\ l_b^* \geq 4$ and $\mu > h_2$

$$= -h_2 + \varepsilon l_b^* + 2\varepsilon(l_b^* - 2) > -\mu + 3\varepsilon l_b^* - 4\varepsilon > 0.$$

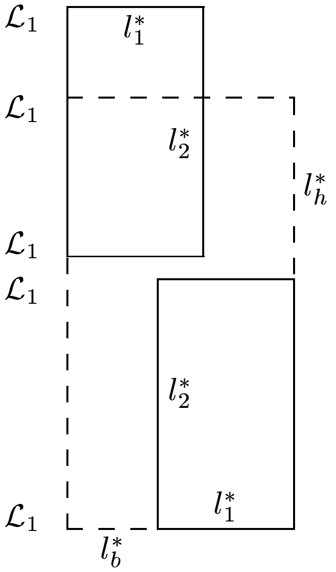
In region II_b', using $h_2 \leq \frac{3}{2}\mu - \frac{3}{2}\varepsilon$ (4.35) becomes

$$\geq -\frac{3}{2}\mu + \frac{3}{2}\varepsilon + \varepsilon(l_1^* - 1)l_b^* + \varepsilon(l_b^* - 2l_1^* + 1)l_1^* > \frac{1}{2}\mu + \varepsilon l_1^*(l_b^* - 2l_1^* + 1) > 0$$

because $l_1^* \geq 3$ and $l_b^* \geq 2l_1^*$.

Let us now start to analyse the irreducible regions I'', II_a'' and II_b''

It is immediately seen that, due to subcriticality of the clusters $\tilde{\xi}_1$ and $\tilde{\xi}_2$, we certainly decrease the energy if we pass to the configuration where $\tilde{\xi}_1$, $\tilde{\xi}_2$ are replaced by two interacting rectangles $R(l_1^*, l_2^*)$. This configuration is still in $\partial\mathcal{G}$ so to proceed it will suffice to compare its energy with $H(\mathcal{P})$. Moreover, since the energy does not change if we make odd or even horizontal translation and (only) even vertical translations and since $l_b^* + 1 \leq 2l_1^*$ we can translate these rectangles to include their bases inside a rectangle $R^* \equiv R(l_b^*, l_h^*)$ in such a way that only one of them is vertically outside R^* as it is shown in Fig. 4.9.



$$2l_1^* - l = l_b^*$$

$$1 \leq l \leq l_b^* - 2$$

The difference between the height of the two rectangles $\mathrm{R}(l_1^*, l_2^*)$ and R^* is $2l$. Indeed knowing that $l_2^* = 2l_1^* - 1$ and that $l_h^* = 2l_b^* - 1$ we have:

$$2l_2^* + 1 - l_h^* = 2(2l_1^* - 1) + 1 - 2l_b^* + 1 = 2(2l_1^* - l_b^*) = 2l. \tag{4.36}$$

From this it follows that

$$2\mathrm{H}\left(\mathrm{R}\left(l_1^*, l_2^*\right)\right) - \mathrm{H}\left(\mathrm{R}^*\right) = 4Jl - \varepsilon l l_1^* - h_2 + \mu(l-1) + \varepsilon(l_b^* - l_1^*)(l_2^* + 1 - l) \tag{4.37}$$

Using $l_2^* = 2l_1^* - 1$ we get:

$$\text{r.h.s.of (4.37)} = 4Jl - \varepsilon l l_1^* + \mu l - 2J + \varepsilon(l_b^* - l_1^*)(2l_1^* - l) \tag{4.38}$$

So we finally obtain:

$$H(S(\zeta^*, M_{sub}^{(1)})) - H(\mathrm{R}^*) \geq 4Jl - \varepsilon l l_1^* + \mu l - 2J + \varepsilon(l_b^* - l_1^*)(2l_1^* - l) + V_2 \tag{4.39}$$

$$H(S(\zeta^*, M_{sub}^{(1)})) - H(\mathcal{P}_{II}) \geq 4Jl - 2J - \varepsilon l l_1^* + \mu l - \varepsilon(l_b^* - 2) + \varepsilon(l_b^* - l_1^*)(2l_1^* - l) \quad (4.40)$$

Using $l_b^* - l_1^* = l_1^* - l$ we get

$$\text{r.h.s.of } (4.40) = 4Jl - 2J + \mu l + \varepsilon(2{l_1^*}^2 + l^2 - 4l_1^* l - l_b^* + 2)$$

In regions I'' and II_a'' the expression in r.h.s. of (4.37) becomes particularly simple and in both cases we get

$$H(S(\zeta^*, M_{sub}^{(1)})) - H(\mathcal{P}) > 0$$

In region II_b'' $h_2 \geq \frac{3}{2}\mu - \frac{5}{2}\varepsilon$ so the function is always growing, then it is enough to make the computation for $l = 1 \Longrightarrow 2l_1^* = l_b^* + 1$:

$$H(S(\zeta^*, M_{sub}^{(1)})) - H(\mathcal{P}_{II}) = 2J + \frac{\varepsilon}{2}(l_b^* - 3)(l_b^* - 1) > 0$$

■

Section 5. The tube of exit.

In this Section we briefly describe the set of typical paths followed by our process during its first excursion from $-\underline{1}$ to $+\underline{1}$.

In [OS1], in the general context of reversible Markov chains with transition probabilities small in a large parameter β, the authors give a complete description of the typical features of the first excursion outside a generic cycle A (possibly containing many local equilibria) as well as of the first fall to its bottom $F(A)$. These two phenomena are, in an appropriate sense, related by time reversal because of the reversibility of the dynamics. Let us underline that the peculiar characteristic of these phenomena, when the concerned cycle A contains many local minima, is that generally they involve suitable time fluctuations given by stochastic "resistance times", typically exponentially long in β, in suitable "permanence sets". In other words, for instance, contrary to the case in which A contains only one local equilibrium coinciding with its bottom $F(A)$ in the general case the first fall to $F(A)$ does not take place along a rapid downhill path but rather an efficient mechanism involves long waiting times inside suitable subcycles of A.

The results in [OS1] about first exit from a cycle and first fall to the bottom of a cycle can be easily combined to give rise to a complete description of our "tunnelling" between $-\underline{1}$ and $+\underline{1}$.

Let us first consider for our stochastic Ising model the first fall to the bottom of a cycle $A \subset \mathcal{S}_\Lambda$. It is shown in [OS1] that, with probability tending to one as β tends to infinity, the trajectory of our process starting from a point (configuration) $\sigma_0 \in A$ during the first excursion to the bottom $F(A)$ of A belongs to one of the possible *standard cascades* emerging from σ_0 . A standard cascade emerging from σ_0 is a set $\omega_0, Q_1, \omega_1, Q_2, \dots, \omega_{n-1}, Q_n, \omega_n$, where, for $i = 1, \dots, n-1$, ω_i are downhill paths emerging from η_i and ending inside Q_{i+1}. $\eta_0 = \sigma_0$ and ω_0 is a downhill path emerging from σ_0. If μ_1 is its ending point, Q_1 is the cycle given by the maximal connected set in $\mathcal{S}_\Lambda$ containing μ_1 with energy smaller than $H(S(\mu_1, F(A))$ whereas $\eta_1 \in \partial Q_1 \cap S(\mu_1, F(A))$. Similarly ω_1 is a downhill path emerging from η_1; if μ_2 is its ending point then Q_2 is the cycle given by the maximal connected set in $\mathcal{S}_\Lambda$ containing μ_2 with energy smaller than $H(S(\mu_2, F(A))$ and so on, up to Q_n. Finally ω_n is a downhill path ending in $F(A)$.

Notice that it may (and it will in our present case) happen that $\omega_i \cap (\mathcal{S}_\Lambda \setminus Q_{i+1}) \equiv \omega_i \cap \partial Q_{i+1} \equiv \eta_i$.

In [OS1] it is shown that given any $\delta > 0$, with probability approaching one as $\beta \to \infty$ during its first fall to the bottom of a cycle A starting from $\sigma_0 \in A$ the system follows a path belonging to a standard cascade $\omega_0, Q_1, \omega_1, Q_2, \dots, \omega_{n-1}, Q_n, \omega_n$ emerging from σ_0 spending inside each cycle Q_i a (random) time T_i satisfying:

$$\exp([H(\eta_i) - H(F(Q_i) - \delta)]\beta) \le T_i \le \exp([H(\eta_i) - H(F(Q_i) + \delta)]\beta) \tag{5.1}$$

The set of all the above specified paths (belonging to some standard cascade and satisfying (5.1)) is called the (typical) δ-*tube of fall* from σ_0 to $F(A)$. It is denoted by $\mathcal{T}^f_{\delta,\sigma_0}$. Moreover it is shown in [FW], [OS1] that, with a probability $\to 1$ as $\beta \to \infty$, the typical point of first exit from a cycle A belongs to the set $U(A)$. Then, using reversibility it is shown that the first excursion from the bottom $F(A)$ of a cycle A to ∂A belongs to the (typical) δ-tube of escape $\mathcal{T}^e_\delta$; where a path $\omega = \sigma_1, \dots \sigma_m$ in $\mathcal{T}^e_\delta$ is the time reversal of a typical path of fall from ∂A to $F(A)$. More precisely let $\mathcal{R}$ be the time-reversal operator acting on paths :

$$\tilde\omega = \tilde\sigma_1, \dots \tilde\sigma_m = \mathcal{R}\omega \Leftrightarrow \ \tilde\sigma_i = \sigma_{m-i}$$

we have $\omega = \sigma_1, \dots \sigma_m \in \mathcal{T}^e_\delta$ iff $\omega = \mathcal{R}\tilde\omega$ and $\tilde\sigma_1 = \eta^*, \tilde\sigma_2 = \sigma^*,\ (\tilde\sigma_2, \dots, \tilde\sigma_m) \in \mathcal{T}^f_{\delta,\sigma^*}$ with $\eta^* \in \partial A$, η^* downhill connected to $\sigma^* \in A$.

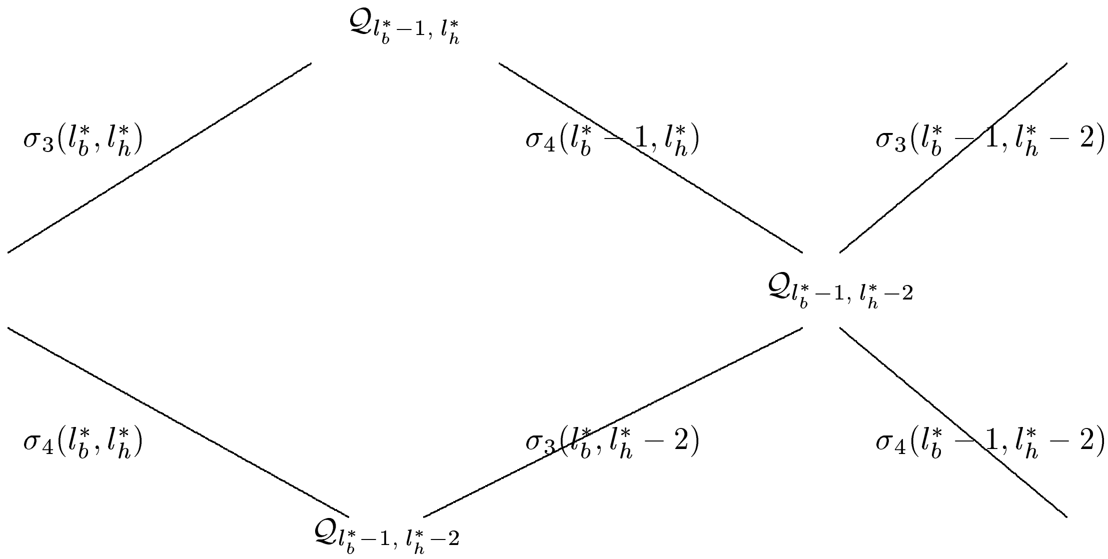
Now let $\bar{\mathcal{G}}$ be the cycle given by the maximal connected set of configurations containing $-\underline{1}$ with energy smaller than $H(\mathcal{P})$ and let $\widetilde{\mathcal{G}}$ be the cycle given by the maximal connected set containing $+\underline{1}$ with energy smaller than $H(\mathcal{P})$. With a probability approaching one as $\beta \to \infty$, the first excursion from $-\underline{1}$ to $+\underline{1}$ follows an "up-down" path consisting in an " ascent path" from $-\underline{1}$ to a configuration η from $\mathcal{P}$ and a "descent path" going from η to $+\underline{1}$. These ascent and descent paths are, respectively typical (in the above specified sense) for the first excursion outside $\bar{\mathcal{G}}$ and first fall to the bottom of $\widetilde{\mathcal{G}}$ given by $F(\widetilde{\mathcal{G}}) \equiv -\underline{1}$. The above statement has been given in a precise way for the first time in [S1] for the standard Ising model and in [CO] (see Appendix A therein) in a more general setup. All the arguments developed in [CO] can be applied to our present case as well. We refer the reader to [CO] for more details.

Consider now h_1, h_2 belonging to region I. We want first to concentrate our attention to the description of the ascent path from $-\underline{1}$ to $\mathcal{P}$. It is clear from the above discussion that all we need is to solve the sequence of variational problems consisting in finding the sequence of saddles $\eta_1, \eta_2, \ldots$ characterizing a generic standard cascade emerging from a generic $\sigma^* \in \bar{\mathcal{G}}$ to which $\mathcal{P}$ is downhill connected. ¿From the arguments developed in Section 3 we easily deduce the result. Given a stable rectangle $R(l_1, l_2)$ let

$$\mathcal{Q}_{l_1,l_2} = \mathcal{B}(R(l_1, l_2)) \tag{5.2}$$

(see notation introduced after (3.14)).

Consider the "fundamental graph" $\mathcal{V}$ depicted in Fig. 5.1



where $\sigma_3(l_1,l_2)$, $\sigma_4(l_1,l_2)$ are depicted in Fig. 3.1

It is not difficult to deduce from the arguments of proof in Section 3 that the configurations in $\bar{\mathcal{G}}$ to which $\mathcal{P}$ is downhill connected are $\sigma_3(l_b^*,l_h^*)$, $\sigma_4(l_b^*,l_h^*)$; moreover a generic standard cascade emerging from $\sigma_3(l_b^*,l_h^*)$ $(\sigma_4(l_b^*,l_h^*))$ is given by a one-dimensional oriented graph (chain), obtained as a subgraph of the fundamental graph $\mathcal{V}$, starting from $\sigma_3(l_b^*,l_h^*)$ $(\sigma_4(l_b^*,l_h^*))$, passing through $R(2,3)$ and then following a downhill path made of six steps up to $-\underline{1}$. For instance a possible path is:

$$\sigma_3(l_b^*,l_h^*), \mathcal{Q}_{l_b^*-1,l_h^*}, \sigma_4(l_b^*-1,l_h^*), \mathcal{Q}_{l_b^*-1,l_h^*-2}, \dots, R(2,3), \dots, -\underline{1} \tag{5.3}$$

Notice that the possible paths follow canonical or almost canonical shapes : $l_2 = 2l_1 - 1$. If $h_1, h_2 \in$ II we get a very similar result. It suffices to replace $\bar{\sigma}_3, \bar{\sigma}_4$ to σ_3, σ_4 (see Fif. 3.2).

It also follows from the arguments of Section 3 that the supecritical growth is described by the generalized standard cascade:

$$\mathcal{P},\ \mathcal{Q}_{l_b^*,l_h^*},\ \sigma_1(l_b^*,l_h^*),\ \mathcal{Q}_{l_b^*+1,l_h^*},\ \sigma_1(l_b^*+1,l_h^*),\dots,\mathcal{Q}_{N,l_h^*},\ \sigma_2(N,l_h^*),\ \mathcal{Q}_{N,l_h^*+2},\dots$$
$$\dots,\sigma_2(N,N-2),+\underline{1} \qquad (5.4).$$

where the part from $\sigma_1(l_b^*,l_h^*)$ up to $+\underline{1}$ in (5.4) is nothing but a standard cascade which is typical of the first fall to the bottom $+\underline{1}$ of $\widetilde{\mathcal{G}}$ starting from $\sigma_1(l_b^*,l_h^*) \in \widetilde{\mathcal{G}}$. This mechanism of supercritical growth is very similar to the one which has been found for the anisotropic Ising model (see [KO1]).

Section 6. The other regions of the h_1, h_2 plane.

In this section we want to rapidly discuss the other regions of parameters namely III, IV and V (see Fig. 2.1).

First of all we remark that in V $-\underline{1}$ is no more a local minimum so it is only of interest to analyse the transition from $\pm\underline{1}$ to $+\underline{1}$; this will be done also for regions IV, V at the end of this Section.

It is not difficult to realize that, in regions III, IV, to describe the first excursion from $-\underline{1}$ to $+\underline{1}$ it is sufficient to observe that:

$$\text{in III} \qquad V_1 = h_2 \ \le\ 4J - h_1 \tag{6.1}$$

whereas

$$\text{in IV} \qquad V_1 = 4J - h_1 \ \le \ h_2 \tag{6.2}$$

In both regions the protocritical droplet for the transition from $-\underline{1}$ to $+\underline{1}$ is just a unit square ($l_b^* = l_h^* = 1$) belonging to $\mathcal{L}_1$ for which it is still true that $l_h^* = 2l_b^* - 1$ namely it is still of " canonical shape". We then have, in regions III, IV a result similar to the statement of Theorem 1; but now the global saddle $\mathcal{P}$ is just a unit square on $\mathcal{L}_1$ and $H(\mathcal{P}) - H(-\underline{1}) = 4J - h_1$. We deduce from (6.1), (6.2) that in region IV during the transition from $-\underline{1}$ to $+\underline{1}$ we typically visit the configuration $\pm\underline{1}$ (we fill the whole $\mathcal{L}_1$ of pluses before we see any plus in $\mathcal{L}_2$) whereas in region III we typically do not visit $\pm\underline{1}$ before $+\underline{1}$ (a minus spin in $\mathcal{L}_2$ with two plus spins in its n.n. sites in $\mathcal{L}_1$ is likely to flip before an isolated minus in $\mathcal{L}_1$ flips). We get:

$$\text{in III} \qquad \frac{1}{\beta}\log\tau_{+\underline{1}} \ \sim \ 4J - h_1,$$

$$\text{in IV} \qquad \frac{1}{\beta}\log\tau_{+\underline{1}} \ \sim \ h_2$$

It also follows from the previous discussion that in regions III, IV and V we have:

$$S(\pm\underline{1}, +\underline{1}) \ \equiv \widetilde{\mathcal{P}}$$

where $\widetilde{\mathcal{P}}$ is the configuration obtained from $\pm\underline{1}$ by flipping a minus spin in $\mathcal{L}_2$. In other words the critical droplet is just a unit square in $\mathcal{L}_2$ (microscopic size) and the tipical value for the first hitting time to $+\underline{1}$ is independent of $\varepsilon = h_1 - h_2$ and is given by

$$\frac{1}{\beta}\log\tau_{+\underline{1}} \ \sim \ h_2$$

This corresponds to "absence of metastability" in the sense that the lifetime of the metastable equilibrium does not diverge as we approach the coexistence line. This is the dynamical counterpart of the expected fact that the concerned coexistence line disappears at positive temperature.

We want to notice that from the above discussion and from the results of previous Sections it follows that a very peculiar behaviour takes place in the vicinities of the triple point $\widetilde{P}$. Now it is clear what we announced in the introduction. Given a halfline r_α emerging from $\widetilde{P}$ lying in the interior of II$\cup$III$\cup$IV forming an angle α with the horizontal line $h_2 = 2J$ let λ be the distance from $\widetilde{P}$ of the generic point P of the line r_α. We have that in the whole region III$\cup$IV the protocritical droplet for the transition from $-\underline{1}$ to $+\underline{1}$ is a unit square in $\mathcal{L}_1$; given a line r_α in region III ($0 < \alpha < \frac{\pi}{2}$) the protocritical droplet depends only on α and not on λ. Its horizontal size l_b^* runs from 2 to ∞ as α runs from $\frac{\pi}{2}$ to $\frac{3}{4}\pi$ (corresponding to the line $h_1 = h_2$) and stays always independent of λ. Correspondingly the typical value of $\frac{1}{\beta} \log \tau_{+\underline{1}}$ runs from $\min\{4J - h_1, h_2\}$ to ∞.

Aknowledgements

We want to aknowledge L. Chayes, R.Kotecký, E. Scoppola, M. Zahradník for interesting discussions and in particular J. Fruttero for his contribution to the early stage of this work.

This work has been partially supported by the grant CHRX-CT93-0411 and CIPA-CT92-4016 of the Commission at European Communities.

References.

[C] O. Catoni, " Sharp large deviation estimates for simulated annealing.", Ann. Inst. Henri Poincaré, **27**,291,384, (1991)

[CGOV] M.Cassandro, A.Galves, E.Olivieri, M.E.Vares, " Metastable behaviour of stochastic dynamics: A pathwise approach", Journ.Stat.Phys. **35**, 603-634 (1984)

[CO] E. Cirillo, E.Olivieri "Metastability and nucleation for the Blume-Capel model : different mechanisms of transition ", CARR Preprint,1995, in press on Journ. Stat. Phys.

[FW] M. I. Freidlin, A. D. Wentzell, "Random Perturbations of Dynamical Systems", Springer-Verlag (1984).

[KO1] R. Kotecky, E. Olivieri, "Droplet dynamics for asymmetric Ising model", Journ. Stat. Phys. **70**, 1121-1148 (1993).

[KO2] R. Kotecky, E. Olivieri, "Shapes of growing droplets - a model of escape from a matastable phase", Journ. Stat. Phys. **75**, 409-507 (1994).

[MOS] F. Martinelli, E. Olivieri, E. Scoppola, "Metastability and exponential approach to equilibrium for low temperature stochastic Ising models", Journ. Stat. Phys. **61**, N. 5/6 1105 (1990).

[N] F.R. Nardi, "Laurea" thesis, Math. Dept. II University of Rome, (1995)

[NS1] E. J. Neves, R. H. Schonmann, "Critical Droplets and Metastability for a Glauber Dynamics at Very Low Temperatures", Comm. Math. Phys. **137**, 209 (1991).

[NS2] E. J. Neves, R. H. Schonmann, "Behaviour of droplets for a class of Glauber dynamics at very low temperatures", Prob. Theor. Rel. Fields **91**, 331 (1992).

[OS1] E. Olivieri, E. Scoppola, "Markov chains with exponentially small transition probabilities: First exit problem from a general domain - I. The reversible case",Journ. Stat. Phys.**79**, 613-647 (1995)

[OS2] E. Olivieri, E. Scoppola, "Markov chains with exponentially small transition probabilities: First exit problem from a general domain - II. The general case". CARR. Preprint, Roma, mp-arc n. 95-423,(1995)

[PL1] O. Penrose, J. L. Lebowitz, "Towards a rigorous molecular theory of metastability." In Fluctuation Phenomena (second edition). E. W. Montroll, J. L. Lebowitz, editors. North-Holland Physics Publishing, (1987).

[PL2] O. Penrose, J. L. Lebowitz, "Molecular theory of metastability: an update." Appendix to the reprinted edition of the article "Towards a rigorous molecular theory of metastability"; by the same authors. In Fluctuation Phenomena (second edition). E. W. Montroll, J. L. Lebowitz, editors. North-Holland Physics Publishing, (1987).

[S] R. H. Schonmann, "The pattern of escape from metastability of a stochastic Ising model", Comm. Math. Phys. **147**, 231-240 (1992).