Analytic regularity of solutions of Livsic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems

R. de la Llave
Department of Mathematics
The University of Texas at Austin
Austin, TX 78712-1802

Abstract

We study Livsic's problem of finding ϕ satisfying $X\phi = \eta$ where η is a given function and X is a given Anosov vector field.

We show that, if ϕ is a continuous solution and X, η are analytic, then ϕ is analytic.

We use the previous result to show that if two low-dimensional Anosov systems are topologically conjugate and the Lyapunov exponents at corresponding periodic points agree, the conjugacy is analytic.

Analogous results hold for diffeomorphisms.

1 Introduction and statement of results

In this paper, we extend the regularity theory for solutions of Livsic's cohomology equations to analyticity. We also discuss related results. We show that for two dimensional analytic Anosov diffeomorphisms and for three dimensional analytic Anosov flows, the set of eigenvalues at periodic orbits is a complete set of invariants for analytic conjugacy.

The crux of the proof of analytic regularity of solutions of Livsic equation is a regularity lemma that states that functions that are analytic restricted to the stable and unstable foliations are analytic. To prove that eigenvalues are a complete set of invariants, we need the regularity lemma established before as well as an argument that bootstraps the regularity along stable and unstable leaves from C^1 to analytic. (Recall that the fact that eigenvalues are a complete set of invariants for C^1 conjugacy was already established in [Ll], [LM])

More precisely, we prove

Theorem 1.1 *Let M be a compact manifold and X a C^ω Anosov vector field on M. Then, if ϕ is a continuous function so that $X\phi = \eta$ where $\eta \in C^\omega(M)$ then $\phi \in C^\omega(M)$.*

Notice that Theorem 1 does not assume transitivity of the flow.

If X were transitive, we could apply the fundamental Livsic's theorem [Li1] to obtain

Theorem 1.2 *Let M be a compact manifold and X a C^ω transitive Anosov vector field on M. Then, if η is a C^ω function on M with the property that*

$$\int_0^{T(\gamma)} \eta(\gamma(t))dt = 0$$

for all γ periodic orbit of X where $T(\gamma)$ is the period of γ.

Then, there exist a function ϕ in C^ω satisfying $X\phi = \eta$.

Moreover, ϕ is unique up to constants.

Analogous results hold for diffeomorphisms.

Theorem 1.3 *If f is an analytic Anosov diffeomorphism and ϕ is a continuous solution of $\phi \circ f - \phi = \eta$ with η analytic, then, Φ is analytic.*

Remarks. For a transitive system as above, the function ϕ solving $X\phi = \eta$ is unique up to constants not only among C^ω functions but also in much broader classes such as $L^1(\mu_+)$, $L^1(\mu_-)$ where μ_+ or μ_- are the Sinai-Ruelle-Bowen measures or C^0. This is because the SRB measures are ergodic. Of course, since there is a dense orbit, the solution is unique up to additive constants in C^0.

The most important shortcoming of these results with respect to the C^∞ analogues in [LMM] is that we do not know how to show analytic dependency on parameters.

This smooth dependence was essential in the discussion of conjugacy of Hamiltonian vector fields which are Anosov in each energy surface, since the energy was treated as an external parameter. Nevertheless, for the interesting case of geodesic flows, the change in energy is just a rescaling in time. Other results that do not involve dependence on parameters but that only use the regularity lemma. such as those of [MM] on cocycle vector equations or those of [HK] can be improved to analytic conjugacy.

Besides these applications, we were told by A. Katok that an analytic version of Livsic theorem could have applications for the theory of automorphic forms that a merely C^∞ one would not have.

Unfortunately Theorem 1.1 — or the lemmas used in its proof — do not seem to imply directly that the results of [Ll] [LM] about smooth conjugacy of low dimensional Anosov systems can be improved to analyticity.

In a third section we give the extra arguments needed to obtain such improvement. Namely we prove

Theorem 1.4 *Let f, g be two analytic Anosov diffeomorphisms of the two dimensional torus. If they are topologically conjugate and, moreover the Lyapunov exponents of corresponding periodic orbits are the same, then, the conjugating homeomorphism is analytic.*

The same methods of proof suffice to prove an analogous theorem for three-dimensional transitive Anosov flows. We can prove

Theorem 1.5 *Let X, Y be two transitive Anosov vector fields on a compact three dimensional manifold without boundary. If their flows are topologically conjugate and the Lyapunov exponents of corresponding periodic orbits are the same, then the conjugating homeomorphism is analytic.*

Notice that as an immediate corollary, if two flows as those considered in Theorem 1.5 are C^1 conjugate, the conjugacy is, then, analytic.

For geodesic flows in surfaces of negative curvature, such C^1 conjugacy is implied by equality of the lengths of closed geodesics in corresponding homotopy classes as shown in Feldman and Ornstein [FO].

Remark. Transitivity does not appear in the hypothesis of Theorem 1.4 because it is automatic for Anosov diffeomorphisms in two dimensional manifolds [N]. For tree dimensional flows, it is not automatic as shown by the celebrated example [FW]. We also note that transitivity is not used in the proof that C^1 conjugated flows are analytically conjugate.

2 Analytic Livsic's theorem

The proof of Theorem 1 will be obtained by extending the methods of [LMM] and incorporating some ideas of [HK]. We will very often resort to the notation of [LMM]. So, it will be convenient for the reader to have a copy of that paper for reference.

The strategy of proof is very similar to that of [LMM]. There, p. 579 it was shown that, under the conditions of Theorem 1.1 or of Theorem 1.3, we have $\phi \in C^\omega_s \cap C^\omega_u$. (That is, the function ϕ is analytic when restricted to the leaves of the stable and unstable foliation.)

We refer to [LMM] for the full proof but we recall that the idea (explained here only for diffeomorphisms) is to notice that in the stable manifold of a fixed point p we have $\phi(x) = \phi(p)\sum_n^\infty \eta(f^n(x))$. Since f restricted to the stable manifold is contractive, the above sum converges in a complex neighborhood of the stable manifold. A density argument shows that we obtain uniform analyticity in all the leaves of the stable foliation. Similar formulas can be derived for flows.

Theorem 1.1 follows once we show that $C^\omega = C^\omega_s \cap C^\omega_u$.

Such a result was proved in [LMM] under the — rather restrictive — hypothesis that the stable and unstable foliations are analytic. In this paper we will establish the same result for all analytic Anosov diffeomorphisms. The technique we will use is a strengthening of the technique used in [HK] to prove $C^\infty = C^\infty_s \cap C^\infty_u$.

An important ingredient of this proof is the fact that the Jacobians of the stable and unstable foliations are smooth when restricted to the leaves. We will use the same argument as in [LMM] — in turn this is related to the argument in [A] for absolute continuity — to show that the Jacobian of the stable foliation is in C^ω_s (and that of the unstable foliation in C^ω_u).

Another important ingredient will be a cut-off method adapted to analytic functions. In [LMM] one such method is used (after the publication of the paper, we learned from Alan Sokal that the same method appears in [AR] and even there it is referred back to [B]). This method turns out not to be convenient for the problems here — even if it is possible to use it — but a slight improvement suffices.

We now start to adapt some definitions from [LMM] to analytic regularity. For the definitions we use without any mention, we refer to that paper.

We will take as standing convention that the dimension of M is n, s is the dimension of the stable foliation, u the dimension of the unstable direction. Hence, for flows, $n = s + u + 1$.

The notation for flows is always a little bit contorted because of the neutral direction which we can choose to lump with the stable, the unstable or on a class of its own.

Putting a notation to cover all the cases is sort of painful, particularly so because, for our purposes, this direction does not matter. (Regularity along the direction of the flow follows just by integrating.)

What [LMM] did was to include regularity along the flow — as well as in the properly stable directions — in the definition of C^j_s. For the purposes of this paper, this would complicate the considerations in Lemma 2.5. So we have decided to explicitly say whether the flow direction is included or not. Therefore, many of the objects that [LMM] were denoted by s or u subscripts will have now an overscore "$\bar{\ }$" denoting that the direction of the flow is included.

Nevertheless, all these considerations, except for the desire of having a perfectly consistent notation, can be safely ignored since the flow direction is trivial to deal with anyway.

The following is an adaptation of Definition 2.1 of [LMM] for analytic regularity. Of course part 1 goes back to [A].

Definition 2.1 *We say that $\Lambda_s : U \times V \to M$ (where $U \subset R^s$, $V \subset R^{u+1}$ are balls) is a local parameterization of the stable foliation*

1. *Λ is a homeomorphism from $U \times V$ to an open set of M, (which we will assume is contained in an analytic chart, so we can consider $\Lambda_s(x, y)$ as a mapping from a ball in R^{s+u+1} to itself).*

2. *For each $y \in V, \Lambda_{s,y} : U \to M$ given by $\Lambda_{s,y}(x) = \Lambda_s(x, y)$ is an analytic immersion whose image is an open set of a stable leaf.*

3. *there is a $\xi > 0$ such that all The $\{\Lambda_{s,y}\}_{y \in V}$ extend to a domain*

$$D_\xi = \{(z_1, \dots, z_s) \in C^s / |\mathrm{Im}\ z_i| \le \xi,\ (\mathrm{Re}\, z_1 \dots \mathrm{Re}\, z_s) \in V\}$$

4. *If we give analytic functions the topology of the sup in D_ξ, the mapping that to each y associates $\Lambda_{s,y}$ is continuous and*

$$\sup_{\substack{y \in \overline{V} \\ \underline{z} \in D_\xi}} |\Lambda_{s,y}(z)| \le K < \infty.$$

There is an analogous definition for unstable foliations and center stable and center unstable. The local parameterizations of center-stable we will denote by $\overline{\Lambda}_s$ and those of the center unstable by $\overline{\Lambda}_u$.

It is a consequence of the standard regularity theory for invariant foliations [HPS] (see also [HP], [S]) that the stable foliations of analytic Anosov vector fields admit local parameterizations in the sense of Definition 2.1 around every point.

One just has to modify their proof making the contractive operators they consider act in appropriate spaces of analytic functions endowed with the supremum norm. All the papers mentioned above present detailed sketches, so we will not repeat them here.

It also follows from this proof that the mapping that to each y associates $\Lambda_{s,y}$ is Hölder. This later property will not be used, but see the remarks after the statement of Lemma 2.5.

Definition 2.2 *If we fix a finite set of local parameterizations covering M, we say that a function $\phi : M \to R$ is in C_s^ω when $\phi(\Lambda_s(\cdot, y))$ are uniformly analytic (that is to say they extend to a domain D_ξ as above with ξ independent of y and there is a bound for the absolute value of this extension in D_ξ) and, moreover $y \to \phi(\Lambda_s(\cdot, y))$ is a continuous mapping to the appropriate of analytic functions in D_ξ.*

Given a fixed set of local parameterization and a number $\xi > 0$ we give a Banach space structure to $C_s^{\omega,\xi} \subset C_s^\omega$ by the sup of the absolute value of all the complex extensions of the functions $\Phi(\Lambda(\cdot, y))$.

We, moreover, say that a family of functions ϕ_ε is uniformly in C_s^ω when there exist a $\xi > 0$ so that all $\phi_\varepsilon \in C_s^{\omega,\xi}$, and are uniformly bounded and, the mapping that to ε, y associates $\phi_\varepsilon(\Lambda_s(\cdot, y))$ is uniformly continuous when the range is given the sup norm on D_ξ.

Even if these definitions have been given in terms of a fixed covering set of local parameterizations it is easy to check that any two such covering sets will give rise to the same concept of C_s^ω and uniformly in C_s^ω — even if $C_s^{\omega,\xi}$ change —

Analogous definitions hold for the unstable, center stable, center unstable foliations and we will call them $C_u^\omega C_{\overline{s}}^\omega$, $C_{\overline{u}}^\omega$. If we denote by C_X^ω the functions analytic when restricted to the integral lines of the flow. It will turn out that

$$C_{\overline{u}}^\omega = C_u^\omega \cap C_X^\omega, \quad C_{\overline{s}}^\omega = C_s^\omega \cap C_X^\omega.$$

and this will follow from our main regularity lemma. Analogous definitions of C^ω regularity hold for geometric objects which can be reduced locally to functions, e.g., vector fields, forms, etc. The reason why this result is easy is because the

Lemma 2.3 *For an Anosov diffeomorphism or the time t map of an Anosov flow, it is possible to choose a local parameterization of the stable foliation so that the Jacobian of the stable foliation agrees with a C^ω_s function.*

Proof. We follow the argument in [LMM] p. 582 and following. Roughly, we follow the strategy of [A] to prove absolute continuity, but we pay considerable attention to the way that the limits are reached.

To the stable foliation of an Anosov system we can associate a $(n-s)$ form ψ such that ψ contracted with vectors tangent to the stable foliation is zero.

Moreover, there is a continuous l-form α, whose contraction with the flow or the vectors tangent to the unstable foliation vanishes, such that

$$d\psi = \alpha \Lambda \psi$$

where the derivative is understood in a weak sense.

The key for the proof of the regularity of the Jacobians in [LMM] is the following prescription to compute α : Fix any C^∞ metric. If we split $TM = E^s \oplus \widehat{\overline{E^u}}$ where $\widehat{\overline{E^u}}$ is a C^∞ bundle transverse to E^s (approximately $\overline{E^u}$ — the center unstable bundle). Then, for a fixed t, we have that Φ_t — the time t map — has a derivative of the form

$$\begin{pmatrix} A_t(x) & B_t(x) \\ 0 & D_t(x) \end{pmatrix}$$

Call $\mathcal{D}_t(x) = \log\ \det(D_t(x))$.

Given a C^∞_s function h define $d_s h$ as the continuous l-form that vanishes on vectors tangent to the center unstable bundle and moreover $(d_s h)|_{W^s_x} = d(h|_{W^s_x})$ (where the last d is on the sense of the exterior derivative of W^s_x).

Then [LMM] proves (eq. 2.18) $\alpha = \lim_{t\to\infty} d_s \log \mathcal{D}_t$ and that the convergence is uniform along with that of all the stable derivatives, (and, of course, that the limit is independent of the choices of bundle and of metric since it has an geometric meaning that is independent of the choice of coordinates). □

Here we prove.

Proposition 2.4 *With the notations above:*

$$\alpha \in C^\omega_s$$

Proof of the proposition. The proof consists essentially in walking through the proof in [LMM] but checking that we have uniform convergence in a complex extension of the stable manifold. Since we only have to estimate suprema of functions, the proofs are even technically easier than those in [LMM] for finitely many derivatives.

First we claim that we can choose our approximation to the center-unstable bundle to be analytic. This follows from the fact, mentioned above that subbundles of the tangent bundle can be identified with forms and forms can be approximated by analytic ones. We refer to [Gr] for a discussion of what objects in an analytic manifold can be approximated by analytic ones. (Forms certainly can because we can for example use an analytic version or Whitney's embedding theorem — this is what is proved in [Gr] — , then, extend the forms using Whitney extension theorem, then use the customary smoothing by convolving with approximations of identity and restricting.) We emphasize that we only need to approximate the center-stable bundle by an analytic one and not the center-stable foliation. Approximating foliations is very delicate even with finite differentiability. (See [EHP] for examples of foliations that cannot be approximated by smoother ones)

Hence, we have that $\mathcal{D}_t \in D_s^\omega$ uniformly for $t \in [0,1]$.

By the chain rule we have (See [LMM] (2.17))

$$\log \mathcal{D}_{t+u} = \log \mathcal{D}_u + \log \mathcal{D}_t \circ \Phi_u$$

If we take the coordinates along the stable manifolds as done in [LMM] p. 573 we get calling

$$\widetilde{\mathcal{D}}_{x,t} = \mathcal{D}_t \circ w_x^s$$

where the w_x^s is a coordinate system that maps a ball around 0 in R^s into neighborhood of x into its center stable manifold.

After we take coordinates in this fashion, as pointed out in [LMM] we can think as the x's only as parameters and the arguments of the functions are just vectors in R^s

$$\begin{aligned} d_s(\log\ \widetilde{\mathcal{D}}_{x,t}) = \quad & \textstyle\sum_{j=0}^{N-1} [d_s[\log \widetilde{\mathcal{D}}_{x_j,1}] \circ \overline{\Phi}^s_{x,j}] \cdot (\Phi^s_{x,j})' \\ & +[d_s \log\ \mathcal{D}_{x_N,(t-N)} \circ \overline{\Phi}^s_{x,N}] \cdot (\Phi^s_{x,N})' \end{aligned}$$

where $N-1$ is the integer part of t, x_j is $\Phi_j x$ and $\overline{\Phi}^s_t$ is the version in coordinates of Φ^s_t, the flow restricted to the stable bundle. (In the corresponding formula of [LMM] there is a misprint, the last N in the first line should be a j.)

In order to prove that this is an l-form in C_s^ω we just observe that all the functions involved extend to a domain of the form $D = \{(z_1 \dots z_s) \in C^s \mid |z_i| \le \xi\}$ where $\xi > 0$ can be picked uniformly for $x \in M$, $t \in [0,1]$. Moreover, in this complex extension we have that all the functions are uniformly bounded and Φ' satisfies (2.7) of [LMM] which tells it decreases exponentially fast in N, so that the sums converge uniformly by Weierstrass M test. This finishes the proof of the proposition.

Once that we have that α is in C_s^ω we can invoke a functional equation for the Jacobian ρ derived in [LMM] p. 587 following Anosov [A] p. 134-135.

$$\frac{d\rho_{\nu,v}^{(w)}}{d\nu} =< \alpha(\Lambda(\tau v, t, w)) > \rho_{\nu,v}$$

out of which we can obtain that the Jacobian is indeed in C_s^ω. This finishes the proof of Lemma 2.3.

The corresponding result for diffeomorphisms can be proved by invoking the usual suspension trick.

This concludes the proof of Proposition 2.4 and of Lemma 2.3. □

Remark. We believe that this result can also be obtained from the methods of [PS] who consider the holonomy maps of the transversals, but have not checked all the details nor written a coherent notation for it.

Analogous proofs hold, of course, for the unstable and center stable and center unstable foliation. The unstable is the stable for $-X$ and the introduction of the extra direction of the flow is trivial since the vector field is C^ω by assumption.

Now we finish the proof that $C^\omega = C_s^\omega \cap C_{\overline{u}}^\omega$.

Lemma 2.5 *If the stable and center unstable foliations admit local parameterizations in the sense of Definition 2.1 and, if their Jacobians are in C_s^ω, $C_{\overline{u}}^\omega$ respectively, then $C^\omega = C_s^\omega \cap C_{\overline{u}}^\omega$.*

Lemma 2.5 is really a regularity result independent of the theory of Anosov systems. It should – perhaps – have been stated as saying that given two transverse absolutely continuous foliations with uniformly analytic leaves such that the Jacobians are also uniformly analytic restricted to the leaves have the property that functions that uniformly analytic restricted to the leaves are analytic.

We also point out that the restriction to two foliations is not essential.

Similar regularity questions have been considered in harmonic analysis, specially in the theory of Λ_α spaces and it is related to regularity of integral transforms such as Riesz transform that play a fundamental role in elliptic regularity theory. (See [Ste] [Kr].) The C^ω regularity relates to "edge of the wedge theorems" [AR]. In the analytic category this theorem is somewhat reminiscent of the Hartoggs theorem that establishes that separate analyticity implies analyticity. The hardest part of Hartoggs theorem is to recover the a-priori bounds that we are already given. Under the hypothesis of uniform analyticity, this result was established in [B]. We also call attention to the fact that similar questions when regularity is assumed only on sets of full measure have been considered in [Ll2] for finite regularity and motivated by non-uniformly hyperbolic systems and in the analytic case in [UY] Section 6.

With C^∞ regularity it was proved by [LMM], [Jo1],[Jo2], [HK]. Some improvements on the method of [Jo1], [Jo2] was introduced in [Ll2].

The proofs of [LMM] and [HK] use the regularity of the Jacobian but only continuity of the foliation. [Jo1], on the other hand, uses Hölder continuity of the foliation but makes no assumptions on the Jacobian. The paper [Jo2] only needs to assume that the foliations are continuous.

In this paper we will pattern our proof along the lines of [HK] since it seems to be the shortest. This will have the inconvenient that the estimates that we obtain will not be tame. This problem is related to the fact that the proof in [HK] relies only on the size of the Fourier coefficients. The proof of [LMM], bases on elliptic regularity theory makes a more efficient use of the coefficients. Since elliptic regularity is somewhat cumbersome for the case of analyticity, we thought it best to try the faster [HK]. The proofs of [Jo1], [Jo2] seem to run into problems with analyticity.

Note that as a simple corollary of 2.5 we have 2. If we restrict ourselves to a leaf of the stable foliation, the strong stable foliation and the direction of the flow have are smooth along the leaves of the corresponding foliations.

Proof of Lemma 2.5. Taking analytic coordinates we can always reduce to proving the theorem in an open set in R^n.

Given $x_0 \in R^n$, $\{v_i\}_{i=1}^n$ a basis in $R^n \delta \in R^+$ we will construct a mapping $\tau : T^n \to R^n$ (T^n is the standard torus) and a function $\Omega : T^n \to R$ by

$$\tau(\theta_1 \dots \theta_n) = x_0 + \delta \sum_{i=1}^n v_i \sin\theta_1;\ \Omega(\theta_1 \dots \theta_n) = (\cos\theta_1 \cdot \dots \cdot \cos\theta_n)^n \ .$$

We will prove that, if $\phi \in C_s^\omega \cap C_{\overline{u}}^\omega$ then, $(\Phi \circ \tau)\Omega$ will be analytic provided that x_0, δ, $\{v_i\}$ satisfy some mild conditions. Clearly, if $(\phi \circ \tau)\Omega$

is analytic then Φ itself is analytic in a subset of Range (τ). The conditions of smallness will be such that we can cover the whole of M with a finite number of those subsets, hence establishing the theorem.

We will pick any arbitrary point x_0 and $v_{-1} \dots v_{-s}, v_{-s+1}$ spanning the tangent space to the center stable manifold through x_0. The vectors $v_{-s+2} \dots v_{-n}$ will span the tangent space to the center unstable manifold.

We study the Fourier coefficients

$$\hat{\phi}_{\underline{k}} = \int_{T^n} e^{i\underline{k}\boldsymbol{\theta}} \phi \circ \tau(\underline{\theta}) \Omega(\theta) d\theta_1 \dots d\theta_n$$

we can rewrite those coefficients as

$$\hat{\phi}_{\underline{k}} = \int_{T^n} e^{i\underline{k}\boldsymbol{\theta}} \phi \circ \Lambda_s \circ \Lambda_s^{-1} \tau(\theta) \Omega(\theta) d\theta.$$

Now, if $\underline{\psi}$ is a system of coordinates in the range of $\Lambda_s^{-1}\tau(\theta)$ in such a way that $\psi_1 \dots \psi_s$ are the coordinates corresponding to the stable directions.

Since the multiplicity of the mapping τ is 4^n except in a set of zero measure we have

$$\hat{\phi}_{\underline{k}} = 4^n \int_{\text{Range of } \Lambda_s^{-1} \circ \tau} e^{i\underline{k}\boldsymbol{\theta}(\psi)} \phi \circ \Lambda_s(\underline{\psi}) \underline{\Omega}(\underline{\theta}) \frac{\partial(\underline{\theta})}{\partial(\underline{\psi})} d\psi.$$

Notice that, by assumption, the Jacobian $\frac{\partial(\psi)}{\partial(\theta)}$ exists and its only zeros are those of the Jacobian of τ, which are less than n. Hence, $\Omega(\theta)\frac{\partial(\theta)}{\partial(\psi)}$ is an analytic function.

Moreover, we have that θ, $\phi \circ \Lambda_s$ are analytic functions of $\psi_1 \dots \psi_s$ and, by making δ small enough we can assume $\theta_1 \dots \theta_s$ are very close to $\psi_1 \dots \psi_s$ respectively.

All these statements are uniform in the remaining variables.

In order to perform the integration in $\psi_1 \dots \psi_s$ we deform the contour of integration and, using the fact that, for sufficiently small δ, $\underline{\theta}(\psi_1) \approx \psi_i \;\; i \leq s+1$ in an analytic extension of the domain. We can get:

$$|\hat{\phi}_{\underline{k}}| \leq M e^{-\xi\left(|k_1|+\ldots+|k_s|\right)+\varepsilon(\delta)|\underline{k}|}. \tag{1}$$

Where ξ is the width of the strip of analyticity, M, a uniform bound of the functions in this complex strip — times 4^n — and ε a number which can be made as small as desired by taking δ small enough.

An analogous argument using the unstable foliation will show that

$$|\hat{\phi}_{\underline{k}}| \leq Me^{-\xi\left(|k_{s+1}|+\ldots+|k_n|\right)+\varepsilon(\delta)|k|}. \tag{2}$$

Hence $|\hat{\phi}_k| \leq \sqrt{|\hat{\phi}_{\underline{k}}|^2} \leq Me^{-\left(\xi/2-\varepsilon(\delta)\right)|\underline{k}|}$ and hence, the lemma is established. □

Remark. Note that the proof is not restricted to considering just two foliations which are transversal. If we had several foliations – with uniformly analytic leaves and Jacobians analytic along the leaves – such that the tangent spaces of the leaves span the whole tangent space, we could have decomposed the tangent space in several pieces and obtain a finite number of bounds analogues to (1), (2) which would also lead to the conclusion of exponentially fast decay.

Remark. Unfortunately, the bounds we obtained are not tame in the sense of [H] [Ze1],[Ze2]. (Note that the parameter ξ that measures the analyticity domain for the solution is roughly half of what it was for the data.)

It is possible to choose other norms (for example $\| \quad \|_{C^\omega_s} + \| \quad \|_{C^\omega_u} + \| \quad \|_{C^\omega_X}$) in which the estimates are indeed tame, but with the choices we could come up with, analytic differential operators do not act nicely except in the case that the foliations are smooth.

This is unfortunate, because one of the original motivations to study regularity of Livsic's equation was its application in Nash-Moser type theorems. Except in the case where the foliations are analytic, already covered in [LMM], we do not know how to do that. Using the strategy of [LMM], proving smoothness in parameters would allow to avoid hard implicit function theorems.

Remark. We learned the idea of using two different changes of coordinates to estimate the Fourier coefficients from [HK] where it is attributed to C. Toll (In [LMM] only one was used, which required the extra assumption that the foliations were analytic.)

Remark. Composing the functions with a map of the torus is used in [B] (see [AR]) much before its use in [LMM]. The further introduction of the Ω function done here, simplifies matters but it is not quite essential.

3 Analytic conjugacy of low dimensional Anosov systems

In view of the results of the previous section, to prove Theorem 1.4 it suffices to show that the conjugating homeomorphism is in $C_s^\omega \cap C_u^\omega$.

In order to show it belongs to C_s^ω it suffices to show that the restriction to the stable manifold of a fixed point is uniformly analytic because a closed graph argument will conclude that, since it is possible to extend continuously to all stable leaves, the extension will be uniformly analytic in the stable direction (every stable leaf is dense because of transitivity [Ka]).

In order to prove that it is uniformly analytic on this stable manifold we will show that the restriction of the conjugacy to the stable manifold satisfies a certain ordinary differential equation that, under the hypothesis of equality of the Lyapunov exponents, has uniformly analytic coefficients.

Lemma 3.1 *Let f, g be contractive diffeomorphisms of class $C^k k \geq 2$ of the real line $f(0) = g(0) = 0$; $f'(0) = g'(0) = \lambda \quad 0 < |\lambda| < 1 \quad \lambda$ and h a C^ℓ,$\ell > 1$ diffeomorphism satisfying $f \circ h = h \circ g$. Then,*

$$\Gamma(x, y) = \lim_{n\to\infty} \frac{g'(x)g'(g(x))\dots g'(g^n(x))}{f'(y)f'(f(y))\dots f'(f^n(y))}$$

exists and it is a C^{k-1} function. Moreover, h satisfies

$$h'(x) = \Gamma(x, h(x))h'(0).$$

Proof. Once we know that the limit exists the fact that $h'(x)$ satisfies the differential equation is easy because out of the equation $f^n \circ h = h \circ g^n$ we get by taking derivatives

$$h'(x) = \frac{g'(x)g'(g(x))\dots g'(g^n(x))h(g^{n+1}(x))}{f'(h(x))f'(fh(x))\dots f'(f^n h(x))}$$

To prove the existence of the limit it is easier to study the logarithms (we will assume for the sake of notation that f, g preserve orientation). We can also assume that x, y are small.

To prove that the limit exists, it suffices to show

$$\lim_{n\to\infty} \sum^n (\log g'(g^i(x)) - \log f'(f^i(y)))$$

converges and is differentiable.

Convergence follows from the fact that $\log g'(0) = \log f'(0)$ and that we have

$$\log g'(g^i(x)) = \log g'(0) + k|g'(0)|^n x + \circ(|g'(0)|^n)|x|$$

and likewise for f. Then, the series converges absolutely and uniformly in a compact interval around zero because it can be dominated by a geometric series.

To prove existence of higher derivatives we just have to take formal derivatives and check they are uniformly bounded by numerical series that converge absolutely.

When we take derivatives with respect to x we get factors $(g^i)'$ in each term involving g and the ones containing y drop. So we are left again with a geometric series.

Getting higher derivatives proceeds along the same lines but we have to cope with the extra terms coming from higher derivatives of g. Since we only have to study those derivatives in the interval $[0, x]$ they are uniformly bounded and the argument of [LMM] p. 574 shows these extra factors can be dominated by a geometric series times a polynomial. Notice that all mixed derivatives are zero.

Notice also that, even when f and g are uniformly close over the whole real line, it is not clear that $\Gamma(x, y)$ would be uniformly bounded on the whole real line. In fact, for the diffeomorphisms of the line obtained by restricting two dimensional Anosov diffeomorphism to a stable manifold, the proof of smoothness of the conjugacy we will present, amounts to showing that $\Gamma(x, y)$ is indeed uniformly bounded. So, any proof of uniform boundedness for $\Gamma(x, y)$ will require extra hypothesis; in our case, they will be implied by equality of Lyapunov exponents of the two-dimensional diffeomorphisms.

In order to complete the proof of Theorem 1.4 we will resort to the notation of [Ll] which is better adapted to this technique rather than that of [LM] and later on we will explain which modifications are necessary to prove the result for flows on three dimensional manifolds.

We will henceforth call f, g the two Anosov diffeomorphisms satisfying the hypothesis of theorem 2 and h the conjugacy between then $f \circ h = h \circ g$.

First of all we observe that it suffices to prove the theorem under the assumption that the conjugating homeomorphism is C^k-close to the identity $k = 1, \ldots, \infty$.

In effect by [LM] we know h is a C^∞ diffeomorphism and hence we can approximate it in C^{k+1} by an analytic diffeomorphism h^*. It suffices to prove the theorem for f and $g^* = h^* g h^{*-1}$.

Now, following [Ll] we observe that it suffices to study what happens in the stable manifolds of the fixed point, which we will assume is at 0.

We will take coordinates in these stable manifolds and call $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$, the expression in coordinates of f, g and h respectively.

We will require that the systems of coordinates satisfy:

1) Derivatives with respect to the coordinates extend to $C^{\omega}_{s(f)}$ $C^{\omega}_{s(g)}$ vector fields respectively.

2) $\sup_{t\in R} |\tilde{h}(t) - t| < \infty$.

The construction in [Ll] works without modification, but there is a more elegant construction due to Moriyón [MM], which we will now describe.

For one of the stable manifolds, we choose arc-length as our parameterization. For the second manifold, we say that the coordinate of a point in the first manifold closest to the given point. (Later on, we will see yet another construction.)

Notice that due to assumption 1), and standard facts about the stable manifolds the derivatives $\tilde{f}'$, $\tilde{g}'$, $\tilde{h}'$, extend to functions in the manifold which are $C^{\omega}_{s(f)}$, $C^{\omega}_{s(g)}$, $C^{\omega}_{s(h)}$ respectively. Moreover, there is a complex extension to a domain $D_\xi = \{z| \ |\mathrm{Im}\ z| \le \xi\}$ in such a way that $\tilde{f}$, $\tilde{g}$ are defined there and are uniform contractions.

Notice that the function in parenthesis extends to the whole manifold and that, the sum over periodic orbits is zero because of the assumption on Lyapunov exponents. We can, then, apply Livsic's theorem [Ll] to conclude that there is a bounded function ψ in such a way that

$$[\log \tilde{g}' - \log \tilde{f}' \circ \tilde{h}](x) = \psi(x) - \psi(\tilde{g}(x))$$

so that all the sums telescope. This concludes the proof when $y = \tilde{h}(x)$, $x \in R$.

If $x, y \in D_\xi$ we proceed as follows

$$\begin{aligned}\log \Gamma_n(x,y) \quad &= \quad \log \Gamma_n(x,y) = \log \Gamma_n(Re(x), \tilde{h}(Re(x)) \\ &\quad -[\log \Gamma_n(Re(x), \tilde{h}(Re(x))) - \log \Gamma_n(x, \tilde{h}(Re(x))] \\ &\quad -[\log \Gamma_n(x, \tilde{h}(Re(x))) - \log \Gamma_n(x,y)]\end{aligned}$$

and we can bound each one of the terms — using the uniform contractivity estimates — independently of n. (This is done in [Ll] in more detail.)

Clearly the Γ_n are analytic and since they are uniformly bounded on a complex domain, they converge pointwise on the real line. Hence it follows that they converge uniformly in the domain D_ξ.

It then, follows that $\Gamma(x, y)$ is not only defined in the real line but also extends to a complex neighborhood.

Since $\tilde{h}'(x) = \Gamma(x, \tilde{h}(x))$ in the real line, it follows that $\tilde{h}$ extends to a domain $D_{\xi'}$ and in that domain $\tilde{h}(t) - t$ is uniformly bounded. To prove $h \in C^\omega_s$, we just observe that any stable leaf is dense, so we can find segments of the stable leaf of zero approaching any leaf and that $\tilde{h}'$ restricted to these segments is uniformly analytic and converges in the real line. It follows it converges in the complex extension.

Since there is an identical argument to show $h \in C^\omega_u$, an invocation to our Lemma 2.5 finishes the proof of Theorem 1.4 as stated.

The proof of the Theorem 1.5 for three dimensional flows goes along similar lines. Hence, we just sketch the differences.

First, we can assume, by the same argument as before, that the two vector fields, X, Y are C^k close and the conjugating homeomorphism Φ is C^k close to the identity.

We now pick a periodic orbit γ for X of period T — which we will assume to be 1 — $W^{s(X)}_\gamma$ can be mapped analytically to $S^1 \times R$, in such a way that the angle coordinate represents the phase of the orbit and the real coordinate is a parameterization of the stable manifold. (In other words, the lines given by "angle = cte" are stable manifolds.)

In $W^{s(y)}_{\Phi(\gamma)}$ we can introduce a similar system of coordinates. We can either use Moriyón's device or we can realize that the smoothed out homeomorphism of [LM] induces a C^∞ coordinate system. We can further smooth out the restriction to $S^1 \times R$ to obtain analyticity.

The same properties as before about the possibility of extending the derivatives in coordinates hold. If we now look to the time T map in these coordinates, we see that it preserves the vertical directions. We just restrict to one stable leaf and obtain a pair of one dimensional maps to which we can apply the same trick with the differential equation and the $\Gamma(x, y)$ function.

The reason why the functions $\Gamma_n(x, y)$ corresponding to the restrictions to a one dimensional stable manifold are bounded is the following: In [LM] equation (2) gives a formula for the derivative of such a mapping as the exponential of the integral along an orbit of a function in the manifold. $\Gamma_n(x, y)$ can then be expressed as the exponential of an integral along an piece of an orbit running from 0 to n. The integrand (called η in [LM], has zero integral along any periodic orbit, if the Lyapunov exponents are to be the same along corresponding periodic orbits. η is also a Hölder function. Hence, by Livsic's theorem, it is the derivative along the flow of a Hölder and function ψ hence $\Gamma_n(x, y)$ is the exponential of the difference of the function ψ at two different points, hence bounded independently of n. The rest of the proof is as the previous one. $\square$

Remark. The use of the function Γ to improve smoothness of the conjugacy shows that in one dimension, when the conjugacy is C^1 is actually as smooth as the function. This should be compared with the results of [St] — where one derivative is lost — and with his Example 9 which shows that the assumption $h \in C^1$ in this remark cannot be removed. (See also the discussion in [Ne].)

This approach also can be used to prove the results of [Pr], which were an inspiration for us.

We also point out that after this paper was written, there appeared two papers that clarify the role of Γ. In [Po] it was shown that another way of considering the bootstrap of regularity is noticing that the SRB measures have to be transported into each other. Hence the function Γ has the dynamical interpretation of quotient of densities. This method was also used in [Ll2]. We point out that the approach described here, even if less geometric. makes sense even outside of the support of the SRB measure but in the basin.

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