Egorov Theorems and equidistribution of eigenfunctions for the quantized sawtooth and Baker maps.

S. De Bièvre
UFR de Mathématiques et
Laboratoire de Physique Théorique et Mathématique
Université Paris VII- Denis Diderot 75251 Paris
Cedex 05 France
e-mail: debievre@mathp7.jussieu.fr

M. Degli Esposti
Dipartimento di Matematica,
Università di Bologna.
Piazza di Porta S.Donato 5. 40127 Bologna, Italy
e-mail: desposti@dm.unibo.it

may, 1996

Abstract

We prove semi-classical Egorov estimate for the quantized Baker and sawthooth maps. Those are uniformly hyperbolic, but discontinuous, area preserving maps on the torus. Due to the discontinuities, the usual semi-classical Egorov Theorem breaks down. The estimates shown here are still strong enough to prove that the eigenfunctions of the quantized maps equidistribute in the classical limit.

1 Introduction.

When the classical limit of a quantum dynamical system is a Hamiltonian dynamical system which is ergodic with respect to the Liouville measure, it is expected that in the classical limit (most of) the eigenfunctions of the quantum system become equidistributed with respect to the Liouville measure . Made precise (see below) this is a statement about the diagonal matrix elements of quantized observables between eigenstates and is commonly referred to as the Schnirelman Theorem. It has been proven in many cases [Z1, CdV, HMR, GL, Z3, Z4, DIG, Sa, BD, ZZ]. If the system is in addition mixing, more can be inferred: in that case (most) off-diagonal matrix elements tend to zero [Z2, CR].

We will be interested here in the classical limit of quantized, discontinuous, ergodic or mixing symplectic transformations of the two-torus. Combining ideas of [Z4] and [ZZ] with the approach of [BD], we will show (Theorem 2) that, in this case, the equipartition result will follow provided one has a suitable version of the semi-classical Egorov Theorem (Definition 1). The problem is therefore reduced to proving this latter result.

The semi-classical Egorov Theorem states that quantization and evolution commute up to terms of order at most $\hbar$. It can take on several forms. In semi-classical analysis for systems where the underlying classical dynamics is smooth there are two versions of the result (see [R], Theorem 4.10 and Theorem 4.30). The deepest version states that the exact propagator differs by terms of order $\hbar$ from a suitable Fourier integral operator the phase of which is given in terms of the classical action. This implies an easier version which states that the quantization of an evolved classical observable differs from the quantum evolution of the quantized classical observable by terms of order $\hbar$. For our purposes, only the second version is needed. For billards, where the flow has singularities, a suitably adapted version of this result was proven in [C, F]. This is precisely the version used in [ZZ] to obtain the Schnirelman Theorem for ergodic billards. We will show Egorov estimates in the sense of Definition 1 below for the quantized sawtooth and Baker maps and derive equipartition results from it.

The sawtooth and Baker maps are prototypical discontinuous uniformly hyperbolic systems. The Baker is easily seen and well known to be a Bernoulli system (see e.g. [SV] and references therein). The dynamical properties of the sawtooth maps are much harder to derive and have been studied extensively in recent years. They are globally hyperbolic discontinuous systems and have been proven to be exponentially mixing and hence in particular ergodic [Ch, LW, Va, Li]. Their periodic orbits and various other dynamical properties have also been studied in detail (see [Va] for further references).

Those maps have attracted considerable attention in the context of "quantum chaos". The quantized sawtooth maps are analysed numerically in [La], and we refer to [SV] for a recent review and further references to the quantized Baker transformation. There are several reasons for such sharp interest in those maps.

First, there are very few explicit classical symplectic dynamical systems known to be hyperbolic, mixing, or even simply ergodic. In continuous time, we have the geodesic flows on (constant) negative curvature Riemannian manifolds, and ergodic billard flows. In discrete time, there are the hyperbolic automorphisms of the torus (and their perturbations), as well as the aforementioned discontinuous maps. Note that among those only the geodesic flows and the (smoothly perturbed) toral automorphisms are smooth. Their behaviour is often determined by special number theoretic properties and therefore not expected to be generic [HB, Ke1, Ke2, S]. This has been a major motivation for analyzing systems with singularities. Much attention has been paid to chaotic maps on the torus, even though they seem rather unrealistic as models for physical systems despite a recent attempt to realize the quantum Baker map as a physically realistic system through the use of an optical analogy [HKO]. It seems to be generally believed (or at least hoped) that the quantized sawtooth and Baker maps display "typical" behaviour of quantized chaotic systems [La, SV, OHT].

A second reason for the interest in those maps is that the discrete time variable and the finiteness of the quantum Hilbert spaces associated to the torus constitute a clear advantage for numerical – and to some extent theoretical – studies. Nevertheless, as has been pointed out elsewhere [SV], not much is known about them in terms of a rigorous semi-classical analysis.

Most efforts have concentrated on semi-classical expressions for the quantum propagator, its iterates, and matrix elements or traces thereof. The problems encountered there have three sources: the singularities in the maps, their chaotic character, and the fact that the finite dimensionality of the quantum Hilbert spaces also has a drawback since it replaces oscillating integrals by oscillating sums, for which a sufficiently complete and powerful stationary phase method is not available [SV]. With respect to this problem, it was shown in [BD] that the standard Weyl calculus is easily adapted to the torus and with it all "cinematic" semi-classical estimates, i.e. those that do not involve any particular dynamics. In addition, the simpler version of the Egorov Theorem for quantized smooth Hamiltonian flows and for quantized toral automorphisms (and their Hamiltonian perturbations) passes effortlessly to the torus [BD]. Recall that this is in turn sufficient for a proof of the Schnirelman Theorem which only depends on the ergodicity and not on more refined properties of the underlying dynamics. In fact, semi-classical estimates depending in a more refined way on the dynamics, such as trace formulas, do not come quite so easily. In this context, the use of Toeplitz quantization, advocated in [Z3, Z4], might prove more powerful in the future. It should be noted, however, that singularities in the classical maps introduce new problems making even the Egorov Theorem a non-trivial matter. We shall indeed prove that both for the sawtooth maps and the Baker transformation, the Egorov theorem does not hold in its usual form.

It seems therefore a good idea to start by establishing the Schnirelman theorem for discontinuous ergodic systems. Indeed, even if a suitable trace formula, on which most of the literature in the subject has concentrated, but which is very hard to establish rigorously, will imply this result, it can be proven much more simply provided one has a suitable Egorov Theorem at one's disposal. For ergodic billards, this is the strategy of the proof in [ZZ], using the Egorov Theorem of [C, F]. We will adapt the strategy of [ZZ] here for the quantized sawtooth and Baker maps, and therefore our main task is to establish a suitable version of the Egorov Theorem.

It is generally expected that standard semi-classical analysis breaks down in the presence of singularities due to diffraction effects (whether the system is chaotic or not). Some work has been done to analyse in which manner this breakdown occurs for the quantized Baker transformation [SV]. By comparing the matrix elements of the exact quantum propagator with those of its semi-classical approximation, the authors of [SV] show that – as expected – the breakdown occurs "close to" the singularities of the classical map. This is a way of saying that the Egorov Theorem breaks down at these sites. Presumably this phenomenon survives in other quantized discontinuous maps, although it was not observed in the quantized sawtooth maps [La].

Conversely, it seems reasonable to expect that, "away from" the singularities, the usual semi-classical analysis continues to hold. This should be all the more true in these particular models, since they are linear away from the singularities. It is this latter expectation that our work helps to confirm. Indeed, the Egorov Theorems we will prove roughly state that quantization and time evolution continue to commute up to terms of order $\hbar^n$, $\forall n \in \mathcal{N}$ provided two conditions are satisfied: both the classical observable and the quantum state should be supported away from the singularities of the classical map. This will be a sufficiently strong statement to still allow us to show the Schnirelman theorem.

Let us now be more precise. As we will recall in detail below, it is possible to associate to some symplectic maps T on the two-torus a corresponding quantum operator V_T. This operator acts on a suitable Hilbert space $\mathcal{H}_N$ of dimension $N \in \mathcal{N}$, where N is related to the Planck constant via $2\pi\hbar N = 1$.

We will not indicate the N dependence of V_T.

Given a function $f \in C^\infty(\mathcal{T}^2)$, we are interested in the behaviour of the following operator, $\forall k \geq 1$, and $N \to \infty$:

$$E_T^{(k)}(f) = V_T^{-k} Op_\hbar^W(f)\, V_T^k - Op_\hbar^W\left(f \circ T^k\right), \tag{1}$$

where $Op_\hbar^W(f)$ is the operator on $\mathcal{H}_N$ given by the (Weyl) quantization of f (see section 2) and where we have assumed that $f \circ T^k \in C^\infty(\mathcal{T}^2)$. The operators $E_T^{(k)}(f)$ measure the amount to which quantization and evolution do not commute. For future purposes, note that a simple induction procedure gives: $(f \circ T^\ell \in C^\infty(\mathcal{T}^2),\ 0 \leq \ell \leq k)$,

$$E_T^{(k)}(f) = V_T^{-1} E_T^{(k-1)}(f) V_T + E_T(f \circ T^{(k-1)}). \tag{2}$$

To put our results in a general setting, we give the following definition.

Definition 1 *Let T be a map on the torus and V_T its quantization. We say (T, V_T) satisfies an Egorov estimate up to time K if the following holds:*

1. *there exists a closed set Σ_K of measure zero so that, if $f \in C^\infty(\mathcal{T}^2)$ is supported away from Σ_K, then $f \circ T^\ell \in C^\infty(\mathcal{T}^2)$, $\forall \ell \leq K$;*
2. *for each family of orthonormal bases $\{\psi_j^{(N)}\}_{j=1,\ldots,N}$, there exists a family of index sets $\mathcal{E}^{(K)}(N) \subset \{1, 2, \ldots, N\}$ satisfying $\mathcal{E}^{(K)}(N)/N \to 1$ so that $\forall\, 0 \leq \ell \leq K$*

$$\sup_{j \in \mathcal{E}^{(K)}(N)} \parallel E_T^{(\ell)}(f)\psi_j^{(N)} \parallel \stackrel{N\to\infty}{\longrightarrow} 0. \tag{3}$$

The set Σ_K should be thought of as the union of the set of singularities for T with its image under $T, \ldots, T^K$. As $K \to \infty$, it tends to "fill" the torus, in the sense that it cuts the torus into disconnected pieces of increasingly small area, becoming eventually smaller than the elementary area $2\pi\hbar$. The first condition in the definition states that the classical observable must stay away from those. Since we need control for arbitrarily large K, this might look worrysome, since it seems to impose untenable restrictions on f. Fortunately, in the proof of the Schnirelman Theorem, one always takes $\hbar \to 0$ before taking $K \to \infty$, avoiding this difficulty. In particular, we have nothing to say on the existence or absence of the so-called $\log \hbar$ barrier [OHT]. The fact that (3) does not hold for all basis vectors is a reflection of the fact that they (meaning, roughly speaking, their Husimi or Wigner distributions) should also stay away from the singularities. This will become clearer in the proofs below, but has the following intuitive basis. Since the singularities are of zero measure, there is a subspace of the quantum Hilbert space of dimension $N - N^\epsilon$ (with ϵ as small as desired) consisting of vectors that essentially do not "touch" the singularities. For them, Egorov is expected to hold since they only "feel" the smooth part of the dynamics (up to time K). This is indeed what we prove in the various examples that we treat (see also the Remarks after Theorem 4).

We then have:

Theorem 2 *Suppose T is ergodic and (T, V_T) satisfies an Egorov estimate for all times K. Denote by $\varphi_j^{(N)}$ a normalized basis of eigenvectors of V_T and let $f \in C^\infty(\mathcal{T}^2)$. Then*

$$\frac{1}{N}\sum_{j=1}^{N} |< \varphi_j^{(N)}, Op_\hbar^W(f)\, \varphi_j^{(N)} > - \int f \; dq\, dp\; |^2 \to 0, \tag{4}$$

which is equivalent to the following statement.

There exists $\mathcal{E}(N) \subset \{1, \ldots, N\}$ with $\mathcal{E}(N)/N \to 1$ so that $\forall j_N \in \mathcal{E}(N)$, $\forall f \in C^\infty(\mathcal{T}^2)$

$$< \varphi_{j_N}^{(N)}, Op_\hbar^W(f)\, \varphi_{j_N}^{(N)} > \to \int f \; dq\, dp. \tag{5}$$

For any normalized $\psi \in \mathcal{H}_N$, one can construct a probability measure on $[0, 1]$ by defining, for all intervals $I \subset [0, 1]$:

$$\mu_\psi(I) = \sum_{\frac{k}{N} \in I} | < \mathbf{e}_k, \psi > |^2.$$

Here the $\mathbf{e}_k$, $k = 0, \ldots, N-1$, are the usual "position eigenvectors" (see Section 2 for the precise definition). Theorem 2 has now the following main corollary, which is the usual statement of the equidistribution of the eigenfunctions in the position variables [CdV, S, Z1].

Corollary 3 *Under the assumptions of Theorem 2:*

$$\mu_{\varphi_{j_N}}(I) = \sum_{\frac{k}{N} \in I} | < \mathbf{e}_k, \varphi_{j_N} > |^2 \longrightarrow |I|, \qquad \text{when } N \to \infty$$

The paper is organized as follows. In section 2 we briefly recall the basic elements of quantization on the torus. In section 3 we prove an Egorov estimate for quantized translations: this allows us to illustrate the strategy of the proof, used again in the following sections, in a simple setting .

The sawtooth maps are treated in section 4. The crucial estimate there is Proposition 15, which implies that the quantized sawtooth maps satisfy an Egorov estimate, so that Theorem 2 holds for them. In section 5 we deal with the Baker map B and its quantization V_B. In that case, we prove Theorem 2 for functions $f \in C^\infty(\mathcal{T}^2)$, depending on q alone. This is of course sufficient for Corollary 3 to hold as well. The key ingredient is the following Egorov estimate for such functions:

Theorem 4 *Let $k \in \mathcal{N}$, $0 < s < \frac{1}{2}$. Let $\left\{\psi_j^{(N)}\right\}_{j=1,\ldots,N}$ be a family of orthonormal bases for $\mathcal{H}_N$. Then there exists a family (in N) of subsets $\mathcal{E}^{(k)}(N) \subset \{1, 2, \ldots, N\}$ such that, for all $f \in C^\infty(\mathcal{T}^2)$ of the form $f(q) = \sum_{n\in\mathcal{Z}} f_n\, e^{2\pi i n q}$ there exists $C(k, s, f)$ so that: $\forall 0 \leq \ell \leq k$, $\forall j \in \mathcal{E}^{(k)}(N)$,*

$$\parallel \left[V_B^{-\ell} Op_\hbar^W(f)\, V_B^\ell - Op_\hbar^W\left(f \circ B^\ell\right)\right] \psi_j^{(N)} \parallel \leq \frac{C(k, s, f)}{N^s}. \tag{6}$$

Moreover

$$1 \geq \frac{\sharp \mathcal{E}^{(k)}(N)}{N} \geq 1 - 4 \cdot 2^k \cdot N^{s-\frac{1}{2}}. \tag{7}$$

The proof of Theorem 4 will show in addition that for $k \in \mathcal{N}$ fixed, for all $s > 0$ and $0 < \epsilon < 1$, there exists a "good" subspace of $\mathcal{H}_N$ of dimension at least $N - N^\epsilon$ so that for all ψ belonging to this subspace and for all $0 \leq \ell \leq k$,

$$\parallel \left[V_B^{-\ell} Op_\hbar^W(f)\, V_B^\ell - Op_\hbar^W\left(f \circ B^\ell\right)\right] \psi \parallel \leq \frac{C(k, s, f, \epsilon)}{N^s}. \tag{8}$$

This is a reflection of the fact that, away from the singularities, the Baker map is linear. Indeed, for linear maps, there is no error in the Egorov Theorem, whereas here the error can be made arbitrarily small on as large a subspace as one wishes. We also show that there exists a small subspace on which the Egorov estimate breaks down, and this despite the fact that smooth functions of q alone remain smooth under the classical evolution. For general smooth observables, the situation is technically considerably more complicated and we will not deal with it here.

Going back to (6) , note that the error there is at best of order $\sqrt{\hbar}$. In addition, as $s \to 1/2$ in (6), the estimate in (7) gets worse. This is due to the fact that we assume nothing about the projection of the basis vectors $\psi_j^{(N)}$ onto the "good" subspace. On the other hand, one can optimize the estimate in (7) by taking $s \to 0$, thereby sacrificing the estimate in (6).

Finally, section 6 contains the proof of Theorem 2.

Acknowledgments: The authors would like to thank Prof. M. Saraceno and Prof. S. Zelditch for most delightful and enlightening conversations during the semester "Chaos and Quantization" at the Institut H. Poincaré in the fall of 1995. Stephan De Bièvre has also benefited from helpful and stimulating conversations with Prof. S. Tomsovic. Finally, Mirko Degli Esposti would like to thank the *Laboratoire de Physique Théorique et Mathématique* (Université Paris VII): most of this work has been developed thanks to its hospitality and financial support.

2 Quantization: cinematics and dynamics.

The construction of the quantum Hilbert space associated to the two torus $\mathcal{T}^2 = \mathcal{R}^2/\mathcal{Z}^2$ and of the associated Weyl calculus is by now well known. We give a brief account of the results, mainly to set the notations. Further references to the original sources can be found in [BD, DE].

First of all, the compactness of the torus imposes a Bohr-Sommerfeld quantization condition:

$$2\pi\hbar N = 1, \tag{9}$$

for some $N \in \mathcal{N}$. For each choice of N, the quantum Hilbert spaces $\mathcal{H}_N(\theta)$ are indexed by $\theta = (\theta_1, \theta_2) \in [0,1[^2$. They are N-dimensional and carry each an irreducible unitary representation of the discrete Weyl-Heisenberg group with elements $(\frac{m}{N}, \frac{n}{N}, \phi)$, $n, m \in \mathcal{Z}$ and $\phi \in \mathcal{R}$.

There exists in each $\mathcal{H}_N(\theta)$ a suitable orthogonal basis $\mathbf{e}_j(\theta, N)$, $j = 0, \ldots, (N-1)$, representing a "state perfectly localized at $q_j(\theta, N) = \frac{j}{N} + \frac{\theta_2}{N}$". In this basis, the representation of the discrete Weyl-Heisenberg group can be written as follows:

$$U_\theta(\frac{m}{N}, \frac{n}{N}) = e^{\frac{i\pi}{N}nm} u_1^m u_2^n,$$

where

$$u_1\, \mathbf{e}_j(\theta, N) = \mathbf{e}_{j+1}(\theta, N), \qquad u_2\, \mathbf{e}_j(\theta, N) = e^{\frac{2i\pi}{N}(\theta_2 + j)}\, \mathbf{e}_j(\theta, N).$$

Here and in the following we use $\mathbf{e}_j(\theta, N)$, defined for all $j \in \mathcal{Z}$ via the quasi-periodicity condition

$$\mathbf{e}_{j+N}(\theta, N) = e^{-2\pi i \theta_1} \mathbf{e}_j(\theta, N).$$

The $\mathbf{e}_j(\theta,N)$ are the natural basis of the "position representation". One passes to the "momentum representation" via a discrete Fourier transform ($\forall k \in \{0,\ldots,N-1\}$):

$$\mathbf{f}_k(\theta,N) = \sum_{j=0}^{N-1} (\mathcal{F}_N)^{-1}_{jk}(\theta)\, \mathbf{e}_j(\theta,N) = \mathcal{F}_N^{-1}\, \mathbf{e}_j, \tag{10}$$

where

$$(\mathcal{F}_N)^{-1}_{jk}(\theta) = \frac{1}{\sqrt{N}} e^{2i\pi N q_j(\theta,N) p_k(\theta,N)} =< \mathbf{e}_j, \mathbf{f}_k >, \tag{11}$$

with

$$q_j(\theta,N) = \frac{1}{N}(j+\theta_2), \quad p_k(\theta,N) = \frac{1}{N}(k+\theta_1).$$

Note that

$$u_1\, \mathbf{f}_k(\theta,N) = e^{-\frac{i}{\hbar}\frac{1}{N}p_k(\theta,N)}\, \mathbf{f}_k(\theta,N), \qquad u_2\, \mathbf{f}_k(\theta,N) = \mathbf{f}_{k+1}(\theta,N).$$

with $\mathbf{f}_{k+N}(\theta,N) = e^{2i\pi\theta_2}\, \mathbf{f}_k(\theta,N)$. ¿From now on we will suppress the explicit (θ,N) dependence on the $\mathbf{e}_j, \mathbf{f}_k$ and on U. We are now able to define the Weyl-quantization $Op_\hbar^W(f)$ of a given function

$$f = \sum_{n,m\in\mathcal{Z}} f_{nm} e^{2\pi i(nq-mp)} \in C^\infty(\mathcal{T}^2). \tag{12}$$

as

$$Op_\hbar^W(f) = \sum_{n,m\in\mathcal{Z}} f_{nm} U(\frac{m}{N}, \frac{n}{N}). \tag{13}$$

We refer to [BD, DIG] for more details.

Turning to the dynamics, assume now that T is a symplectic map on the torus. In certain cases, it is possible to define a corresponding quantum operator V_T on $\mathcal{H}_N$. There is no general procedure avaible for doing this. The cases for which it has been done are:

1. When T is obtained by solving Hamilton's equations for some Hamiltonian $H \in C^\infty(\mathcal{T}^2)$, or as a product of such transformations as for kicked systems ($T = T_1 \circ T_2$). In that case, one simply quantizes H, using Weyl quantization and $V_T = e^{-\frac{i}{\hbar}Op_\hbar^W(H)}$ ($V_T = V_{T_1} \circ V_{T_2}$).

2. For $T \in SL(2,\mathcal{Z})$ [HB, DE] and for Hamiltonian perturbations thereof [BD, BO].

3. For general contact transformations [Z4].

4. When T is a translation on the torus [BD, DDG].

5. For certain piecewise affine maps on the torus, such as the baker transformation [BV, Sa], the sawtooth map [DDG, La] and the D-map [SVa].

In cases (1), (2) and (3) the Egorov Theorem is easy to establish [BD, Z4]. We will deal here with (4)-(5), proving a sufficiently strong version of the Egorov Theorem (see Definition 1) to allow us to prove an equipartition result.

3 Ergodic translations.

In this section we show that the translations on the torus satisfy an Egorov estimate in the sense of Definition 1. A weaker result was already obtained in [BD]. The proof we give here already contains, in a simple setting, the main ideas of the proofs in Sections 4 and 5. Given $\alpha, \beta \in [0,1[$, we denote by $T_{(\alpha,\beta)}$ the corresponding translation

$$T_{(\alpha,\beta)} : (q,p) \longrightarrow (q+\alpha, p+\beta) \in \mathcal{T}^2.$$

By considering $T_{(\alpha,\beta)}$ as a composition of two commuting translations,

$$T_{(\alpha,\beta)} = T_{(\alpha,0)}\, T_{(0,\beta)},$$

we are lead to define the following unitary operators [BD, DDG] $(2\pi\hbar N = 1)$

$$M_2(\beta)\, \mathbf{e}_j = e^{\frac{i}{\hbar}(\frac{j}{N})\beta}\, \mathbf{e}_j = e^{2\pi i \beta\, j}\, \mathbf{e}_j,$$

for $j = 0, \ldots, (N-1)$. For simplicity, we will always assume $\theta = 0$ in this section although the results hold for general θ. Similarly, the quantization $M_1(\alpha)$ of $T_{(\alpha,0)}$ is diagonal in the momentum representation and hence its matrix in the position representation is given by

$$M_1(\alpha) = \mathcal{F}_N^{-1} \tilde{M}_1(\alpha) \mathcal{F}_N,$$

where

$$\tilde{M}_1(\alpha) = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{-\frac{i}{\hbar}(\frac{N-1}{N})\alpha} \end{pmatrix}.$$

Finally, the quantization $M_{(\alpha,\beta)}$ of $T_{(\alpha,\beta)}$ is defined to be (see [BD])

$$M = M_{(\alpha,\beta)} = e^{i\pi N\alpha\beta}\, M_1(\alpha) \circ M_2(\beta).$$

The choice of the phase ensures that $M_{(\frac{n}{N},\frac{m}{N})} = U(\frac{n}{N},\frac{m}{N})$. Note that the $M_{(\alpha,\beta)}$ depend continuously on (α,β).

This quantization, while reasonable, is not completely natural, as explained also in [DDG]. Indeed, it suffices to note that

$$M_2(\beta)\mathbf{e}_N = M_2(\beta)\mathbf{e}_0 = \mathbf{e}_0 = \mathbf{e}_N \neq e^{2i\pi\beta N}\,\mathbf{e}_N,$$

unless $\beta = k/N$. In other words, the quantization treats the "left" edge ($\mathbf{e}_0$) and the "right" edge ($\mathbf{e}_N$) of the torus differently. In this sense the smoothness of the classical map is broken at the quantum level and a kind of "quantum discontinuity" is introduced. A similar phenomenon occurs in the quantization of the sawtooth, where the "quantum discontinuity" superimposes itself on the classical one.

We now turn to estimates, showing that the Egorov theorem in its usual form fails in this case.

Lemma 5 *Let* $\alpha,\beta \in [0,1[$ *and* $(n,m) \in \mathcal{Z}\times\mathcal{Z}$, $(n,m) \neq (0,0)$. *Then, for* N *large enough:*

$$\| \, M_2(-\beta)\, U(\frac{n}{N},\frac{m}{N})\, M_2(\beta) \; - \; e^{-2\pi i\beta n}\, U(\frac{n}{N},\frac{m}{N}) \, \|_{\mathcal{H}_N} = 2 \mid \sin\pi N\beta \mid \tag{14}$$

and

$$\| \, M_1(-\alpha)\, U(\frac{n}{N},\frac{m}{N})\, M_1(\alpha) \; - \; e^{2\pi i\alpha m}\, U(\frac{n}{N},\frac{m}{N}) \, \|_{\mathcal{H}_N} = 2 \mid \sin\pi N\alpha \mid . \tag{15}$$

Proof. Note that

$$M_2(-\beta)\, U(\frac{n}{N},\frac{m}{N})\, M_2(\beta) = e^{i\pi\frac{nm}{N}}\, M_2(-\beta)\, u_1^n\, M_2(\beta)\, u_2^m,$$

so that we only have to control the terms

$$E = M_2(-\beta)\, u_1^n\, M_2(\beta) - e^{-2\pi i\beta n} u_1^n.$$

For that purpose, we can compute its matrix elements $(0 \leq k,l \leq N-1)$

$$E_{k,\ell} = \left[e^{2\pi i\beta(\ell-k)} \; - e^{-2\pi i\beta n}\right] \, (\mathbf{e}_k,\, u_1^n\mathbf{e}_\ell). \tag{16}$$

The element $(\mathbf{e}_k,\, u_1^n \mathbf{e}_\ell)$ is non vanishing if and only if $\ell + n = k \mod N$, namely $k - \ell = n + s_n(k,\ell)\cdot N$, where $s_n(k,\ell)$ takes only the values $0, 1$ and -1, provided N is large enough. Hence,

$$E_{k,\ell} = e^{-2\pi i\beta n}\left[e^{-2\pi i N\beta s_n(k,\ell)} - 1\right]\cdot \delta^k_{\ell+n}, \tag{17}$$

where $\delta^b_a = 1$ if $a = b \mod N$ and 0 otherwise. Using $\| E \|_{\mathcal{H}_N} = \| E^* E \|^{\frac{1}{2}}_{\mathcal{H}_N}$, it is now easy to estimate the operator norm of E. We have:

$$\begin{aligned}
(E^*E)_{k,\ell} &= \sum_{r=0}^{N-1} E^*_{kr}E_{r\ell} = \sum_{r=0}^{N-1} \bar{E}_{rk}E_{r\ell} \\
&= \sum_r \left[e^{2\pi i N\beta s_n(r,k)} - 1\right]\left[e^{-2\pi i N\beta s_n(r,\ell)} - 1\right]\delta^{n+k}_r\,\delta^{n+\ell}_r \\
&= \left(4\sin^2 \pi N\beta s_n\right)\cdot \delta^\ell_k.
\end{aligned}$$

Here $s_n = 0$ if $0 \le k + n \le N - 1$, and $| s_n |= 1$ otherwise. As a result, E^*E is diagonal and since it has at least one non zero matrix element for each $n \neq (0,0) \in \mathcal{Z}^2$, the proof is complete. The second statement regarding $M_1(\alpha)$ follows now in an analogous way, thanks to the unitarity of the Fourier transform. $\square$

Note that, if $\beta = \frac{t}{N}$, $t \in \mathcal{N}$, then (14) shows that the Egorov theorem holds exactly, without error terms, as expected. On the other hand, if β is a fixed N-independent number, then $\limsup_{N\to\infty} | \sin 2\pi N\beta | \neq 0$ and the usual Egorov theorem does not hold in this case for $M_2(\beta)$, quantization of $T_{(0,\beta)}$. We now turn to the solution of this problem (Theorem 7).

First, for future purposes, let us define $\forall n, m \in \mathcal{Z}$ (see (1))

$$E^{(k)}_T(n,m) = E^{(k)}_T(e^{2\pi i(mq-np)}). \tag{18}$$

Note furthermore the following simple relation: $\forall f \in C^\infty(\mathcal{T}^2)$,

$$E_{T_{(\alpha,\beta)}}(f) = V^{-1}_{T_{(\alpha,0)}} E_{T_{(0,\beta)}}(f) V_{T_{(\alpha,0)}} + E_{T_{(\alpha,0)}}(f \circ T_{(0,\beta)}) \tag{19}$$

Putting together this formula with the last lemma, we get

Proposition 6 *Let $\alpha, \beta \in [0,1[$ and $(n,m) \in \mathcal{Z}\times\mathcal{Z}$. Then, for N large enough*

$$\| E_{T_{(\alpha,\beta)}}(n,m) \|_{\mathcal{H}_N} \le 2\left(| \sin \pi N\alpha | + | \sin \pi N\beta |\right) \tag{20}$$

¿From (20) one sees, as in [BD], that for (α, β) in a G_δ dense set there exists a sequence $N_k \to \infty$ so that $\| E_{T_{(\alpha,\beta)}}(n,m) \|_{\mathcal{H}_{N_k}} \to 0$, $\forall m, n \in \mathcal{Z}$. This is a way to circumvent the above problem with Egorov. It is not very satisfactory (see [BD] for further comments) and Theorem 7 below gives a much better result.

Let us go back to (17) and remark that for n fixed and N much larger than $|n|$, the condition $0 \le \ell + n \le (N-1)$ is satisfied for many ℓ: hence Egorov is actually exact on a large subspace of $\mathcal{H}_N$. It only breaks down close to the "edges", i.e. when ℓ is close to zero or N. This is a reflection of the quantum discontinuity we mentioned earlier. We will now show how to use this remark to prove the following Egorov type result, sufficient for our purposes.

Theorem 7 *Let $\alpha, \beta \in [0,1[$, $k \in \mathcal{N}$ and $0 < s < \frac{1}{2}$. Let for each $N \in \mathcal{N}$ an orthonormal basis $\psi_1, \ldots, \psi_N$ of $\mathcal{H}_N$ be given. Then there exists a family (in N) of subsets $\mathcal{E}^{(k)}(N) \subset \{1, 2, \ldots, N\}$ such that, for all $f \in C^\infty(\mathcal{T}^2)$ there exists $C(s,k,f)$ so that: $\forall 0 \le \ell \le k$, $\forall N \in \mathcal{N}$ and $\forall j \in \mathcal{E}^{(k)}(N)$,*

$$\| \left[M_{(\alpha,\beta)}^{-\ell} Op_\hbar^W(f) \, M_{(\alpha,\beta)}^{\ell} - Op_\hbar^W\left(f \circ T_{(\alpha,\beta)}^{\ell}\right)\right] \psi_j \| \le \frac{C(s,k,f)}{N^s}. \tag{21}$$

Moreover

$$1 \ge \frac{\sharp \mathcal{E}^{(k)}(N)}{N} \ge 1 - 4kN^{s-\frac{1}{2}}. \tag{22}$$

To prove this, we first introduce "good" subspaces of $\mathcal{H}_N$, where Egorov is exact on trigonometric polynomials.

Definition 8 *Let $\alpha, \beta \in [0,1[$ and $0 < \eta_N < N$. Then, $\forall k \ge 1$*

$$\mathcal{G}_{\eta_N}^{(k)}(\alpha,\beta) = \{\psi \in \mathcal{H}_N : \quad E_{T_{(\alpha,\beta)}}^{(\ell)}(n,m)\psi = 0, \ \forall |(n,m)| \le \eta_N, 0 \le \ell \le k\}. \tag{23}$$

We then immediately have $(\mathcal{G}_{\eta_N}(\alpha,\beta) = \mathcal{G}_{\eta_N}^{(1)}(\alpha,\beta))$

Lemma 9 *Let $\alpha, \beta \in [0,1[$ and $0 < \eta_N < N$ as before. Then, $\forall k \ge 1$*

1. $\dim \mathcal{G}_{\eta_N}(0,\beta) \ge N - 2\eta_N$; $\quad \dim \mathcal{G}_{\eta_N}(\alpha,0) \ge N - 2\eta_N$;
2. $\dim \mathcal{G}_{\eta_N}(\alpha,\beta) \ge N - 4\eta_N$;
3. $\dim \mathcal{G}_{\eta_N}^{(k)}(\alpha,\beta) \ge N - 4k\eta_N$;

Proof of Lemma 9. The first statement follows immediately from (16) and the comments following it. Now write, as in (19)

$$E_{T_{(\alpha,\beta)}}(n,m) = M_2(-\beta)E_{T_{(\alpha,0)}}(n,m)M_2(\beta) + e^{2\pi i m\alpha}\, E_{T_{(0,\beta)}}(n,m).$$

It follows immediately that $M_2(-\beta)\,[\mathcal{G}_{\eta_N}(\alpha,0)] \bigcap \mathcal{G}_{\eta_N}(0,\beta) \subseteq \mathcal{G}_{\eta_N}(\alpha,\beta)$. The codimension of this intersection can not be bigger than $4\eta_N$, because of the unitarity of $M_2(\beta)$. It is now easy to prove the last statement inductively, using the invariance of the characters under translations. Indeed, since

$$\exp 2\pi i\,[m(q+\alpha) - n(p+\beta)] = \exp 2\pi i(m\alpha - n\beta)\ \exp 2\pi i(mq - np),$$

we have,

$$\begin{aligned} E^{(k)}_{T_{(\alpha,\beta)}}(n,m)\psi &= M^{-(k-1)}_{(\alpha,\beta)}\left[E_{T_{(\alpha,\beta)}}(n,m)\psi\right] M^{(k-1)}_{(\alpha,\beta)}\psi \\ &\quad + e^{2\pi i(m\alpha - n\beta)} E^{(k-1)}_{T_{(\alpha,\beta)}}(n,m)\psi. \end{aligned}$$

So if

$$\psi \in \mathcal{G}^{(k-1)}_{\eta_N}(\alpha,\beta) \cap M^{-(k-1)}_{(\alpha,\beta)}\mathcal{G}_{\eta_N}(\alpha,\beta),$$

then $E^{(k)}_{T_{(\alpha,\beta)}}(n,m)\psi = 0$ and hence $\psi \in \mathcal{G}^{(k)}_{\eta_N}(\alpha,\beta)$, which proves the result. $\square$

We can now decompose $\mathcal{H}_N$ into a "good" and a "bad" subspace as follows:

$$\mathcal{H}_N = \mathcal{G}^{(k)}_{\eta_N}(\alpha,\beta) \oplus \mathcal{B}^{(k)}_{\eta_N}(\alpha,\beta), \qquad \forall k \geq 1.$$

We write $\pi_{\mathcal{G}^{(k)}_{\eta_N}}$ and $\pi_{\mathcal{B}^{(k)}_{\eta_N}}$ for the corresponding projections. The last ingredient needed for the proof of Theorem 7 is this:

Proposition 10 *Let $\alpha,\beta \in [0,1[$, $\psi_1,\ldots,\psi_N \in \mathcal{H}_N$ as in the statement of Theorem 7. Let $0 < \eta_N < N$, $\delta > 0$ and introduce $\forall k \geq 1$*

$$\mathcal{E}^{(k)}_N(\delta) = \left\{ j \mid \parallel \pi_{\mathcal{B}^{(k)}_{\eta_N}}\psi_j \parallel \leq \delta \cdot \parallel \psi_j \parallel \right\}. \tag{24}$$

Then

$$\sharp\mathcal{E}^{(k)}_N(\delta) \geq N - \frac{4\eta_N k}{\delta^2} \tag{25}$$

Proof of Proposition 10. First of all, note that

$$\mathrm{Tr}\,\pi_{\mathcal{B}^{(k)}_{\eta_N}} = \sum_{j=0}^{N-1} < \pi_{\mathcal{B}^{(k)}_{\eta_N}}\psi_j,\, \pi_{\mathcal{B}^{(k)}_{\eta_N}}\psi_j > \geq \left(N - \sharp\mathcal{E}^{(k)}_N(\delta)\right)\delta^2.$$

¿From this we get, using Lemma 9

$$\left(N - \sharp\mathcal{E}^{(k)}_N(\delta)\right)\cdot\delta^2 \leq \sum_{j=0}^{N-1} \parallel \pi_{\mathcal{B}^{(k)}_{\eta_N}}\psi_j \parallel^2 = \mathrm{Tr}\,\pi_{\mathcal{B}^{(k)}_{\eta_N}} \leq 4\eta_N k,$$

which concludes the proof. □

Proof of Theorem 7. Choose sequences $\eta_N = N^{1/2-s}$, $\delta_N = N^{-s}$, $N \in \mathcal{N}$. Define (see (24)) $\mathcal{E}^{(k)}(N) = \mathcal{E}^{(k)}_N(\delta_N)$, then (25) implies (22). To prove (21), write first

$$\begin{aligned} f(q,p) &= \sum_{n,m\in\mathcal{Z}} f_{mn}\exp 2\pi i\,[mq - np] \\ &= \left(\sum_{|(n,m)|\leq\eta_N} + \sum_{|(n,m)|>\eta_N}\right) f_{mn}\exp 2\pi i\,[mq-np] \\ &= f_{\eta_N} + \mathcal{R}_{\eta_N}(f). \end{aligned}$$

Hence, for $j \in \mathcal{E}^{(k)}(N)$, $\forall 1 \leq \ell \leq k$

$$\parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(f)\psi_j \parallel \leq \parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(f_{\eta_N})\psi_j \parallel + \parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(\mathcal{R}_{\eta_N}(f))\psi_j \parallel \tag{26}$$

Since $\left(f\circ T^\ell_{(\alpha,\beta)}\right)_{\eta_N} = f_{\eta_N}\circ T^\ell_{(\alpha,\beta)}$ and hence $\mathcal{R}_{\eta_N}(f\circ T^\ell_{(\alpha,\beta)}) = \mathcal{R}_{\eta_N}(f)\circ T^\ell_{(\alpha,\beta)}$, the fast decrease of the f_{mn} guarantees that $\forall 0 \leq \ell \leq k$ and for all $s' \in \mathcal{R}^+$, there exists $C^k_{s'}(f)$ so that

$$\parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(\mathcal{R}_{\eta_N}(f)) \parallel \leq \frac{C^k_{s'}(f)}{\eta_N^{s'}}. \tag{27}$$

On the other hand

$$\begin{aligned} \parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(f_{\eta_N})\psi_j \parallel \quad &\leq \textstyle\sum_{|(n,m)|\leq\eta_N} \mid f_{mn}\mid \cdot \parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(n,m)\pi_{\mathcal{G}^{(\ell)}_{\eta_N}}\psi_j \parallel \\ &+ \textstyle\sum_{|(n,m)|\leq\eta_N} \mid f_{mn}\mid \cdot \parallel E^{(\ell)}_{T_{(\alpha,\beta)}}(n,m)\pi_{\mathcal{B}^{(\ell)}_{\eta_N}}\psi_j \parallel, \end{aligned} \tag{28}$$

so that (23) and (24) imply

$$\| E^{(\ell)}_{T_{(\alpha,\beta)}}(f_{\eta_N})\psi_j \| \leq \left(\sum_{|(n,m)|\leq \eta_N} | f_{mn} | \right) 2\delta_N. \tag{29}$$

Inserting (29) and (27) into (26), (21) follows upon choosing $s' = s(1/2 - s)^{-1}$. □

Remark 11 *To prove the equivalent of (8) note that, for* $\psi \in \mathcal{G}^{(k)}_{\eta_N}(\alpha,\beta)$, *the r.h.s. of (28) is identically zero. Hence (26)-(27) imply,* $\forall 0 \leq \ell \leq k$, $\forall \psi \in \mathcal{G}^{(k)}_{\eta_N}(\alpha,\beta)$,

$$\| E^{(\ell)}_{T_{(\alpha,\beta)}}(f)\psi \| \leq \frac{C(s,k,f,\epsilon)}{N^s}, \qquad \dim \mathcal{G}^{(k)}_{\eta_N}(\alpha,\beta) \geq N - N^{\epsilon}.$$

Theorem 7 shows that $(T_{(\alpha,\beta)}, M_{(\alpha,\beta)})$ satisfies an Egorov estimate in the sense of Definition 1. Hence, if $T_{(\alpha,\beta)}$ is ergodic ($\alpha,\beta \in \mathcal{R}\setminus\mathcal{Q}$, $\alpha/\beta \in \mathcal{R}\setminus\mathcal{Q}$), Theorem 2 applies. Note that in this case the "singular set" Σ_K can be taken to be empty for all K. Finally, observe that the analog of Theorem 7 for the Baker map is Theorem 4, proven in section 6 and that the remarks following Theorem 4 apply here as well.

4 The sawtooth map.

Given $a,b \in \mathcal{R}$, we consider the following discontinuous maps (for $a,b \in \mathcal{R}\setminus\mathcal{Z}$) A_1, A_2 on the torus:

$$A_1 : (q,p) \longrightarrow (q+ap, p),$$
$$A_2 : (q,p) \longrightarrow (q, p+bq).$$

and we set $A = A(a,b) = A_1 \circ A_2$.

For the classical properties of this maps, we refer to [Ch, LW, Li] and [Va]. As showed in [DDG], we have the corresponding quantum operators V_1, V_2 in the position representation:

$$V_1 = \mathcal{F}_N^{-1} \circ D_1 \circ \mathcal{F}_N, \qquad V_2 = D_2,$$

where

$$D_1 = \begin{pmatrix} e^{-i\pi Nap_0^2} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{-i\pi Nap_{N-1}^2} \end{pmatrix},$$

and,

$$D_2 = \begin{pmatrix} e^{i\pi N b q_0^2} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{i\pi N b q_{N-1}^2} \end{pmatrix}.$$

Finally, we set

$$V_A = V_1 \circ V_2.$$

We take from now on $\theta = (0,0)$, but all the proofs and the results still hold in the general case.

We first consider the case of V_2. The proof and even the formulation of the appropriate Egorov theorem are complicated by the following problem, which adds on to the difficulties encountered already for the translations. When $f \in C^\infty(\mathcal{T}^2)$, one does not in general have $f \circ A_2 \in C^\infty(\mathcal{T}^2)$. This means that $Op_\hbar^W(f \circ A_2)$ might not make sense, since typically, $f \circ A_2$ will not even be continuous so that nothing guarantees the convergence of the series $\sum_{m,n} (f \circ A_2)_{nm} U(\frac{m}{N}, \frac{n}{N})$ in (13). One could try to avoid this problem as follows. For $f \in C^\infty(\mathcal{T}^2)$, define

$$Op_\hbar^W(f \circ A_2) = \sum_{n,m} f_{nm}\, Op_\hbar^W(\chi(n,m) \circ A_2)\,. \tag{30}$$

But now $\chi(n,m) \circ A_2 = e^{-2i\pi m b q}\chi(n,m)$, which is not continuous if $b \notin \mathcal{Z}$. As a result, the Fourier series

$$e^{-2i\pi m b q}\, \chi(n,m) = \sum_k c_k(b,m)\, \chi(n+k,m), \quad q \in]0,1[, \tag{31}$$

with

$$c_k(b,m) = \frac{e^{-i\pi m(b+k)} \sin[\pi(mb+k)]}{\pi(k+mb)} \tag{32}$$

converges conditionally and not absolutely. Moreover the convergence is not uniform on $]0,1[$. In addition the equality in (31) holds only for $q \in \mathcal{R} \setminus \mathcal{Z}$, not for $q \in \mathcal{Z}$. Nevertheless, in this case, if we try to define, following (13),

$$Op_\hbar^W(\chi(n,m) \circ A_2) = \sum_k c_k(b,m)\, U(\frac{m}{N}, \frac{n+k}{N}), \tag{33}$$

then the series on the right hand side does actually converge, as is readily verified. So this would give a meaning to (30), but a problem still remains. Suppose indeed $f \in C^\infty(\mathcal{T}^2)$ is such that $f \circ A_2 \in C^\infty(\mathcal{T}^2)$ as well. Then we should prove that the usual definition of $Op_{\hbar}^{W}(f\circ A_2)$ from (13) coincides with (30). That is clear from a formal manipulation, involving a change of summation order in a conditionally convergent series. We will carefully deal with this problem, avoiding the use of (30) and of (33), but we point out that the results in this section can be obtained from obvious formal manipulations using (33). In order to prove Proposition 13 below, we will use the following lemma, the proof of which we will postpone until after the proof of the proposition.

Lemma 12 *Let $g \in C^{\infty}([0,1])$ and denote by g also its extension to a 1 periodic function on $\mathcal{R}$. Then, there exists a constant C so that $\forall\epsilon>0$ and $\forall M_1, M_2 \in \mathcal{N}$, with $\min\{M_1,M_2\} > \frac{11}{\epsilon}$:*

$$\begin{aligned} &\sup_{x\in[\epsilon,1-\epsilon]}\left|g(x)-\sum_{n=-M_2}^{M1} g_n\, e^{2i\pi nx}\right| < \\ <\; & C[\left(\| g\|_{\infty}\,\epsilon^{-1}+\| g'\|_{\infty}+\| g''\|_{\infty}\right)\cdot\left(\min\{M_1,M_2\}\right)^{-1} \\ & +(\| g\|_{\infty}+\| g'\|_{\infty})\,|\ln M_1/M2|\,], \end{aligned}$$

where $g_n=\int_0^1 g(x)e^{-2i\pi nx}\,dx$.

As the proof will show, the term in ϵ^{-1} is absent if $g\in C^{\infty}(S^1)$. We will apply the lemma to $g(x)=\exp 2i\pi mbx$, where this is of course not the case ($b\notin\mathcal{Z}$). We then have,

Proposition 13 *Let $f \in C^{\infty}(\mathcal{T}^2)$, with $f\circ A_2, f\circ A_1 \in C^{\infty}(\mathcal{T}^2)$. Let $\beta_N = N^{1-\gamma}$, $0<\gamma<1$. Define, for all $0\le j<N$, $|\, j\,|_N=\min\{j,N-1-j\}$.*

Then, for all $|\, j\,|_N>\beta_N$ and for all $s\in\mathcal{R}^+$, there exists $C_s(f,\gamma)>0$ such that

$$\|\left[V_2^{-1}Op_{\hbar}^{W}(f)\,V_2-Op_{\hbar}^{W}(f\circ A_2)\right]\mathbf{e}_j\,\|_{\mathcal{H}_N}\le\frac{C_s(f,\gamma)}{N^s}. \tag{34}$$

Similarly, for all $|\, j\,|_N>\beta_N$ and for all $s\in\mathcal{R}^+$, there exists $\tilde{C}_s(f,\gamma)>0$ such that

$$\|\left[V_1^{-1}Op_{\hbar}^{W}(f)\,V_1-Op_{\hbar}^{W}(f\circ A_1)\right]\mathbf{f}_j\,\|_{\mathcal{H}_N}\le\frac{\tilde{C}_s(f,\gamma)}{N^s}. \tag{35}$$

Remark 14 *(34) and (35) make a remark from the introduction clear. The singularities of A_2, for example, are the lines $q=0$ and $q=1$; (34) states that an Egorov estimate holds with arbitrary precision, provided the "position eigenvectors" $\mathbf{e}_j$ stays away from the edges ($|j|_N>\beta_N$).*

Proof. We only prove (34), the proof of (35) is similar. It is easy to see that

$$(f \circ A_2)_{\ell m} = \sum_{n \in \mathcal{Z}} c_{\ell - n}(b,m)\, f_{nm}, \tag{36}$$

with $c_k(b,m)$ defined in (31-32). Note that this sum is absolutely convergent, uniformly in m and ℓ. Thanks to the assumptions on $f \circ A_2$, for any divergent sequences $\gamma_N, \eta_N \in \mathcal{N}$, $(N \in \mathcal{N})$ to be chosen later, we have

$$Op_\hbar^W(f \circ A_2) = \sum_{|\ell| \le \gamma_N, |m| \le \eta_N} (f \circ A_2)_{\ell m}\, U(\frac{m}{N}, \frac{\ell}{N}) + O((\gamma_N + \eta_N)^{-\infty}). \tag{37}$$

Here and below the notation $O((\gamma_N + \eta_N)^{-\infty})$, for $\gamma_N, \eta_N \to +\infty$, means as usual that $\forall s > 0$, there exists C_s so that

$$\| Op_\hbar^W(f \circ A_2) - \sum_{|\ell| \le \gamma_N, |m| \le \eta_N} (f \circ A_2)_{\ell m}\, U(\frac{m}{N}, \frac{\ell}{N}) \| \le C_s \cdot (\gamma_N + \eta_N)^{-s}.$$

Now introduce (36) into (37) and split the sum in n using a divergent sequence $\xi_N \in \mathcal{N}$ (to be chosen later) as follows:

$$\begin{aligned} & Op_\hbar^W(f \circ A_2) = \\ & \sum_{|\ell| \le \gamma_N, |m| \le \eta_N} \left[\left(\sum_{|n| < \xi_N} + \sum_{|n| \ge \xi_N} \right) f_{nm}\, c_{\ell - n}(b,m)\, U(\frac{m}{N}, \frac{\ell}{N}) \right] \\ & +O((\gamma_N + \eta_N)^{-\infty}) \\ = \;& I_1 + I_2 + O((\gamma_N + \eta_N)^{-\infty}). \end{aligned} \tag{38}$$

For the second term I_2 in (38) we write, after changing the summation order and remarking from (32) that $| c_k(b,m) | < 1$,

$$\| I_2 \| \le \sum_{|m| \le \eta_N} \sum_{|n| \ge \xi_N} |f_{nm}| (2\gamma_N + 1) \le \gamma_N \cdot O(\xi_N^{-\infty}). \tag{39}$$

We now turn to the estimate of I_1. We will assume from now on that $\eta_N \le \frac{1}{2}\beta_N$ so that, if $| j |_N > \beta_N$ and $\epsilon_N = \frac{3\beta_N}{4N}$, we have

$$\epsilon_N \le \frac{1}{N}\left(j + \frac{m}{2} \right) \le 1 - \epsilon_N. \tag{40}$$

Now compute:

$$I_1 \mathbf{e}_j = \sum_{|m| < \eta_N} \sum_{|n| < \xi_N} f_{nm} \left[\sum_{k = -n - \gamma_N}^{-n + \gamma_N} c_k(b,m) e^{\frac{2i\pi}{N} k (j + \frac{m}{2})} \right] e^{\frac{i\pi}{N} nm} e^{\frac{2i\pi}{N} nj} u_1^m \mathbf{e}_j$$

We will now apply Lemma 12 to the expression in the square bracket with $M_1 = -n + \gamma_N$, $M_2 = n + \gamma_N$, $g(x) = e^{-2i\pi bmx}$ and $\epsilon = \epsilon_N$. This will yield the following bound, provided $\epsilon_N(\gamma_N - \xi_N) > 11$. We take henceforth $\xi_N = O(\gamma_N)$ so that:

$$
\begin{aligned}
&\| I_1 \, \mathbf{e}_j - \sum_{|m|<\eta_N} \sum_{|n|<\xi_N} f_{nm} e^{-\frac{2i\pi}{N} bm(j+\frac{m}{2})} U(\frac{m}{N}, \frac{n}{N}) \mathbf{e}_j \| \leq \\
\leq \;& C[\frac{1}{\epsilon_N \gamma_N} \left(\sum_{n,m} | f_{nm} | \right) + \frac{1}{\gamma_N} \left(\sum_{m,n} m^2 \, | f_{nm} | \right) \\
&+ \left(\sum_{m,n} | m | \cdot \left(\frac{| n |}{\gamma_N} \right) \cdot | f_{nm} | \right)]
\end{aligned}
\tag{41}
$$

Inserting (41) and (39) into (38) yields

$$
\begin{aligned}
&\| Op_\hbar^W (f \circ A_2) \, \mathbf{e}_j - \sum_{n,m \in \mathcal{Z}} f_{nm} e^{-\frac{2i\pi}{N} bm(j+\frac{m}{2})} U(\frac{m}{N}, \frac{n}{N}) \mathbf{e}_j \| \leq \\
&C_f \left[\frac{1}{\epsilon_N \gamma_N} + \frac{1}{\gamma_N} + \gamma_N O(\xi_N^{-\infty}) + O((\gamma_N + \eta_N)^{-\infty}) + O((\xi_N + \eta_N)^{-\infty}) \right],
\end{aligned}
$$

where we used the fast decrease of the f_{nm}. For β_N given, we can choose $\gamma_N = \frac{N}{\beta_N} N^s$, $\xi_N \sim \sqrt{\gamma_N}$ and, $\eta_N \leq \frac{1}{2}\beta_N$, which yields the result. □

Proof of Lemma 12. We first consider the case where $M_1 = M_2 = M$ and since it is enough to get the estimate for the real and imaginary part of g separately, we restrict to the case of a real function g. Then

$$
\left| g(x) - \sum_{n=-M}^{M} g_n \, e^{2i\pi nx} \right| = \left| \int_0^1 [g(x-y) - g(x)] \, D_M(y) \, dy \right|, \tag{42}
$$

where the Dirichlet kernel $D_M(y)$ is given by

$$
D_M(y) = \frac{\sin{(2M+1)\pi y}}{\sin \pi y}.
$$

Introducing, for $x \in [\epsilon, 1-\epsilon]$, ($\epsilon > \frac{11}{M}$)

$$
h(x,y) = \frac{g(x-y) - g(x)}{\sin \pi y} \, e^{i\pi y},
$$

equation (42) can be rewritten as

$$\left| g(x) - \sum_{n=-M}^{M} g_n \, e^{2i\pi nx} \right| \le \left| \int_0^1 h(x,y) e^{2i\pi My} \, dy \right| . \qquad (43)$$

For $x \in [\epsilon, 1-\epsilon]$, $h(x,y)$ is a C^∞, 1-periodic function of y, except at most for a single finite jump discontinuity at $y_* = x$, present only if $g(0) \neq g(1)$. It is then easy to see that ([Ed], pag.36):

$$\begin{aligned} \left| \int_0^1 h(x,y) e^{2i\pi My} \, dy \right| &< \frac{1}{2} \int_0^1 \left| h(x, y + \frac{1}{2M}) - h(x,y) \right| \, dy \qquad (44) \\ < \ \frac{1}{2} &\left[\left(\int_0^{x-\frac{1}{M}} + \int_{x+\frac{1}{M}}^1 + \int_{x-\frac{1}{M}}^{x+\frac{1}{M}} \right) \left| h(x, y + \frac{1}{2M}) - h(x,y) \right| \, dy \right] \end{aligned}$$

Clearly, ($I_1 = \left[x - \frac{1}{M}, x + \frac{1}{M} \right], I_2 = \left[0, x - \frac{1}{M} \right] \cup \left[x + \frac{1}{M}, 1 \right]$)

$$\begin{aligned} \int_{I_1} \left| h(x, y + \frac{1}{2M}) - h(x,y) \right| \, dy \ &< \ \frac{2}{M} \sup_{y \in [x - \frac{1}{M}, x + \frac{3}{2M}]} 2 \left| h(x,y) \right| \\ &< \ \frac{4}{M} \left(2 \parallel g \parallel_\infty \right) \frac{1}{\epsilon} C \qquad (45) \end{aligned}$$

For the other terms of (44), we have the following simple estimate:

$$\sup_{x \in [\epsilon, 1-\epsilon]} \sup_{y \in I_2} \left| h(x, y + \frac{1}{2M}) - h(x,y) \right| \le \frac{C}{M} \left(\parallel g' \parallel_\infty + \parallel g'' \parallel_\infty \right) . \qquad (46)$$

Inserting (45) and (46) into (44) yields

$$\sup_{x \in [\epsilon, 1-\epsilon]} \left| g(x) - \sum_{n=-M}^{M} g_n e^{2i\pi nx} \right| \le \frac{C}{M} \left[\frac{1}{\epsilon} \parallel g \parallel_\infty + \parallel g' \parallel_\infty + \parallel g'' \parallel_\infty \right] .$$

Now, if $M_1 > M_2$ we have

$$\begin{aligned} &\sup_{x \in [\epsilon, 1-\epsilon]} \left| g(x) - \sum_{n=-M_2}^{M_1} g_n \, e^{2i\pi nx} \right| \le \\ \le \ &\sup_{x \in [\epsilon, 1-\epsilon]} \left| g(x) - \sum_{n=-M_2}^{M_2} g_n \, e^{2i\pi nx} \right| + \left| \sum_{n=M_2}^{M_1} g_n \, e^{2i\pi nx} \right| \\ \le \ &C_1 \left(\parallel g \parallel_\infty \epsilon^{-1} + \parallel g' \parallel_\infty + \parallel g'' \parallel_\infty \right) M_2^{-1} \\ &+ C_2 \left(\parallel g \parallel_\infty + \parallel g' \parallel_\infty \right) \ln \frac{M_1}{(M_2 - 1)} , \end{aligned}$$

since $| g_n |< 2\,(\| g \|_\infty + \| g' \|_\infty) \mid n \mid^{-1}$. $\square$

We can now state the following,

Proposition 15 *Let $f \circ A^j \circ A_1, f \circ A^j \in C^\infty(\mathcal{T}^2)$, for all $0 \le j \le k$, $k \in \mathcal{N}$. Let $s > 0$ and $\beta_N = N^{1-\gamma}$, $0 < \gamma < 1$. Then there exists $C(f,s,\gamma,k) > 0$ and an $N - 4k\beta_N$ dimensional subspace $\mathcal{G}_N^{(k)}(A)$ of $\mathcal{H}_N$ so that*

$$\| E_A^{(j)}(f)\psi \| \le \frac{C(f,s,\gamma,k)}{N^s} \| \psi \| \tag{47}$$

for all $0 \le j \le k$, $\forall \psi \in \mathcal{G}_N^{(k)}(A)$.

Proof. We first prove this for $k = 1$.

Let $\psi \in \mathcal{H}_N$ so that $< \psi, \mathbf{e}_j >=< \mathbf{f}_k, V_2\psi >= 0$, for all $| j |_N, | k |_N < \beta_N$. Those ψ form an at least $(N - 4\beta_N)$-dimensional vector space. Now compute, for such ψ:

$$\| E_A(f)\psi \| \le \| E_{A_2}(f \circ A_1)\psi \| + \| E_{A_1}(f)V_2\psi \| \tag{48}$$

The result for $k = 1$ then follows immediately from Proposition 13. For $k > 1$, we then proceed by induction:

$$E_A^{(k+1)}(f)\psi = V_A^{-1} E_A^{(k)}(f) V_A \psi + E_A(f \circ A^k)\psi. \tag{49}$$

So if $\psi \in \mathcal{G}_N(A) \cap V_A^{-1}\mathcal{G}_N^{(k)}(A)$, then (47) follows. Since

$$\dim \left(\mathcal{G}_N(A) \cap V_A^{-1}\mathcal{G}_N^{(k)}(A) \right) \ge N - 4\beta_N k - 4\beta_N,$$

the proof is completed. $\square$

It is now easy to prove analogs of Proposition 10 and of Theorem 7, with f as in Proposition 15. This implies that (A, V_A) satisfies an Egorov estimate as in Definition 1, and hence Theorem 2 and Corollary 3 apply.

5 An Egorov Theorem for the Baker transformation.

In this section we prove Theorem 4. We start by introducing the model and fixing some notations. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{50}$$

The Baker map is a discontinuous map B on the torus, defined as follows: $(x = (q, p) \in \mathcal{T}^2)$

$$B(q,p) = \begin{cases} Ax, & q \in [0,1/2[, \\ (T_{(-1,1/2)} \circ A)x, & q \in]1/2,1]. \end{cases} \tag{51}$$

We refer to [AA] for the ergodic and topological properties of the map B. For what concerns the action of the classical maps on the characters $\chi(n,m) = e^{2\pi i[nq-mp]}$, it follows from the definition that $(\forall n, r \in \mathcal{Z})$

$$\chi(n,2r) \circ B \quad = \quad \chi(2n,r), \tag{52}$$

$$\chi(n,2r+1) \circ B \quad = \quad \begin{cases} e^{-i\pi p}\chi(2n,r) \text{ if } q \in [0,1/2[\\ -e^{-i\pi p}\chi(2n,r) \text{ if } q \in]1/2,1]. \end{cases} \tag{53}$$

With the *usual* assumption $N = 0 \bmod 2$, the quantization of the Baker map B is then given by the following unitary operator in the basis $\mathbf{e}_k$, $k = 0, \ldots, N-1$ [BV, DDG, Sa, SV]

$$V_B(\theta) = \mathcal{F}_N^{-1}(\theta) \circ \begin{pmatrix} \mathcal{F}_{N/2}(\theta) & 0 \\ 0 & \mathcal{F}_{N/2}(\theta) \end{pmatrix} = \mathcal{F}_N^{-1}(\theta) \circ Q_B(\theta) \tag{54}$$

$V_B(\theta)$ acts on $\mathcal{H}_N(\theta) = \mathcal{H}_{N,L}(\theta) \oplus \mathcal{H}_{N,R}(\theta)$. Here, with $\mathcal{Z}_N = \{0, \ldots, \frac{N}{2} - 1\} \cup \{\frac{N}{2}, \ldots, N-1\} = L \cup R$:

$$\mathcal{H}_{N,L} = \ \mathrm{span}_{\mathcal{C}} \{\mathbf{e}_\ell\}_{\ell \in L}, \qquad \mathcal{H}_{N,R} = \ \mathrm{span}_{\mathcal{C}} \{\mathbf{e}_\ell\}_{\ell \in R}. \tag{55}$$

We will now assume $\theta = (0,0)$, which is the case considered in [BV]. All the results, in particular the asymptotic ones, can easily be generalized to the more natural case $\theta = (1/2, 1/2)$ [Sa]. With this convention, writing $\mathcal{F}_N(0) = \mathcal{F}_N$ and $\mathcal{F}_{N/2}(0) = \mathcal{F}_{N/2}$:

$$\left(\mathcal{F}_{N/2}\right)_{k,\ell} = \sqrt{\frac{2}{N}}\, e^{-\frac{2\pi i}{\frac{N}{2}} k\ell} \quad k, \ell \in \{0, \ldots, \frac{N}{2} - 1\}. \tag{56}$$

Given $\ell \in \mathcal{Z}_N$, let the "strip" S_ℓ of ℓ be defined by $S_\ell = L$ ($S_\ell = R$) if $\ell \in L$ ($\ell \in R$). Then $\forall \ell, k \in \mathcal{Z}_N$

$$< \mathbf{e}_k,\, Q_B \mathbf{e}_\ell >= \begin{cases} \sqrt{\frac{2}{N}} e^{-\frac{4i\pi}{N}\ell k} & \text{if} \quad S_\ell = S_k \\ 0 & \text{if} \quad S_\ell \neq S_k \end{cases} \tag{57}$$

It is now easy to prove the following formula: $\forall \ell \in \mathcal{Z}_N$ and $\forall m, n \in \mathcal{Z}$

$$V_B^{-1} U(\frac{m}{N}, \frac{n}{N}) V_B \mathbf{e}_\ell = \frac{2}{N} \sum_{j \in S_\ell} \sum_{k \in S_{j+n}} e^{-\frac{4i\pi}{N}\ell j} e^{-\frac{i\pi}{N}mn} e^{-\frac{2i\pi}{N}mj} e^{\frac{4i\pi}{N}k(j+n)} \mathbf{e}_k. \tag{58}$$

This follows from a simple calculation using

$$\mathcal{F}_N U(\frac{m}{N}, \frac{n}{N}) \mathcal{F}_N^{-1} = U(\frac{n}{N}, \frac{-m}{N}), \tag{59}$$

where $\mathcal{F}_N$ is defined as in (10)-(11):

$$\begin{aligned} & V_B^{-1} U(\frac{m}{N}, \frac{n}{N}) V_B \mathbf{e}_\ell \\ = & \sum_{j \in \mathcal{Z}_N} < \mathbf{e}_j, Q_B \mathbf{e}_\ell > Q_B^{-1} U(\frac{n}{N}, \frac{-m}{N}) \mathbf{e}_j \\ = & \sqrt{\frac{2}{N}} \sum_{j \in S_\ell} e^{-\frac{4i\pi}{N}lj} Q_B^{-1} U(\frac{n}{N}, \frac{-m}{N}) \mathbf{e}_j \\ = & \sqrt{\frac{2}{N}} \sum_{j \in S_\ell} \sum_{k \in \mathcal{Z}_N} e^{-\frac{4i\pi}{N}lj} e^{-\frac{i\pi}{N}nm} e^{-\frac{2i\pi}{N}jm} < \mathbf{e}_k, Q_B^{-1} \mathbf{e}_{j+n} > \mathbf{e}_k \\ = & \frac{2}{N} \sum_{j \in S_\ell} \sum_{k \in S_{j+n}} e^{-\frac{4i\pi}{N}\ell j} e^{-\frac{i\pi}{N}nm} e^{-\frac{2i\pi}{N}jm} e^{\frac{4i\pi}{N}k(j+n)} \mathbf{e}_k \end{aligned}$$

We are now turning to the proof of a suitable Egorov estimate (Theorem 4). Suppose that $f \in C^\infty(\mathcal{T}^2)$ does only depend on the q-coordinate, that is $f(q,p) = \sum_{n \in \mathcal{Z}} f_n \, \chi(n, 0)$. As in the previous sections, we want to study the operator

$$E_B^{(k)}(f) = V_B^{-k} Op_\hbar^W(f) \, V_B^k - Op_\hbar^W\left(f \circ B^k\right) \tag{60}$$

We consider $k = 1$ first, and prove the following crucial formula:

$$E_B(\chi(n, 0)) = \frac{2}{N} \sigma \Gamma_n \sigma, \tag{61}$$

where

$$\gamma_{k\ell} =< \mathbf{e}_k, \Gamma_n \mathbf{e}_\ell >= \sum_{j \in S''_\ell} e^{-i\frac{4\pi}{N}j(\ell - k)} \, e^{i\frac{4\pi}{N}kn},$$

and

$$< \mathbf{e}_k, \sigma \mathbf{e}_\ell >= \sigma_\ell \, \delta_{k\ell},$$

with $\sigma_\ell = i$ if $\ell \in L$ and $\sigma_\ell = -i$ if $\ell \in R$. To prove (61), we split for each $\ell \in \mathcal{Z}_N$ the strip S_ℓ in two (n-dependent) disjoint pieces:

$$\begin{aligned} S_\ell^{''} &= \{j \in S_\ell, \quad S_{j+n} = S_{N-\ell}\} = S_\ell \cap \mathcal{S}_n, \\ S_\ell^{'} &= \{j \in S_\ell, \quad S_{j+n} = S_\ell\} = S_\ell \setminus S_\ell^{''}. \end{aligned} \tag{62}$$

Here

$$\mathcal{S}_n = \{j \in \mathcal{Z}_N \mid S_{j+n} \neq S_j\}, \qquad \sharp\mathcal{S}_n = 2 \mid n \mid .$$

Then, (58) implies

$$\begin{aligned}
& V_B^{-1} Op_\hbar^W\left(\chi(n,0)\right) V_B \, \mathbf{e}_\ell = \frac{2}{N} \sum_{j \in S_\ell} \sum_{k \in S_{j+n}} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k \\
&= \frac{2}{N} \sum_{j \in S_\ell^{'}} \left[\sum_{k \in S_\ell} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k \right] + \frac{2}{N} \sum_{j \in S_\ell^{''}} \left[\sum_{k \in S_{N-\ell}} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k \right] \\
&= \frac{2}{N} \sum_{k \in S_\ell} \sum_{j \in S_\ell} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k + \frac{2}{N} \sum_{k \in S_{N-\ell}} \sum_{j \in S_\ell^{''}} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k \\
&\quad -\frac{2}{N} \sum_{k \in S_\ell} \sum_{j \in S_\ell^{''}} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k \\
&= \frac{2}{N} \sum_{k \in S_\ell} \sum_{j \in S_\ell} e^{-i\frac{4\pi}{N}\ell j} \, e^{i\frac{4\pi}{N}k(j+n)} \, \mathbf{e}_k + \frac{2}{N} \sum_{k=0}^{N-1} \left(\gamma_{k\ell}\sigma_k\sigma_\ell\right) \mathbf{e}_k \\
&= e^{i\frac{4\pi}{N}\ell n} \, \mathbf{e}_\ell + \frac{2}{N} \sum_{k=0}^{N-1} \left(\gamma_{k\ell}\sigma_k\sigma_\ell\right) \mathbf{e}_k,
\end{aligned} \tag{63}$$

and this proves (61).

Equation (61) implies that the usual Egorov Theorem does not hold in this case. Indeed, taking $n = 1$ it is easy to check that $\| E_B(\chi(1,0))\mathbf{e}_{N-1} \|= \sqrt{2}$, $\forall N \in \mathcal{N}$.

The following result is then the basic ingredient for the proof of equidistribution of smooth observables in configuration space.

Proposition 16 *Let $\eta_N < N/2$ be given, then*

1. *If $|n| < \eta_N$ then $\forall \ell \in \{0, 1, \ldots, N-1\}$*

$$\| \left[V_B^{-1} Op_\hbar^W\left(\chi(n,0)\right) V_B - Op_\hbar^W\left(\chi(2n,0)\right) \right] \mathbf{e}_\ell \| \leq \frac{\eta_N}{\sqrt{N}}. \tag{64}$$

2. *There exists a subspace $\mathcal{G}_{\eta_N} \subset \mathcal{H}_N$, such that* $\dim \mathcal{G}_{\eta_N} \geq N - 2\eta_N$ *and* $\forall \psi \in \mathcal{G}_{\eta_N}$*:*

$$E_B(\chi(n,0))\,\psi = 0, \qquad \forall \mid n \mid < \eta_N \tag{65}$$

Proof. Because of the assumptions, $\sharp S'_\ell \geq N/2 - \eta_N$ and $\sharp S''_\ell \leq \eta_N$ (see (62)), that is $|\gamma_{k\ell}| \leq \eta_N$. This immediately implies

$$\parallel \left[V_B^{-1} Op_\hbar^W\left(\chi(n,0)\right) V_B - Op_\hbar^W\left(\chi(2n,0)\right) \right] \mathbf{e}_\ell \parallel^2 \leq \frac{\eta_N^2}{N}, \tag{66}$$

which proves the first assertion. To prove the second result, we introduce the space:

$$\mathcal{G}_{\eta_N} = \left\{ \psi \in \mathcal{H}_N \mid \sigma\psi \in \operatorname{Ker}\Gamma_n;\, \forall |n| < \eta_N \right\}.$$

In view of (61), (65) holds on $\mathcal{G}_{\eta_N}$. We will now show that the dimension of this space is in fact bigger than $N - 2\eta_N$. To see this we compute the kernel of Γ_n ($\forall |n| < \eta_N$). It is easy to see that

$$\gamma_{k\ell} = \frac{1}{2} \sum_{j \in \mathcal{S}_n} e^{-i\frac{4\pi}{N} j(\ell - k)}\, e^{i\frac{4\pi}{N} kn}.$$

For $\varphi = \sum_{\ell=0}^{N-1} c_\ell\, \mathbf{e}_\ell = \sum_{k=0}^{N-1} d_k\, \mathbf{f}_k$, we have

$$\begin{aligned} < \mathbf{e}_k, \Gamma_n \varphi > &= \sum_{\ell=0}^{N-1} c_\ell\, \gamma_{k\ell} = \frac{1}{2} \sum_{j \in \mathcal{S}_n} e^{\frac{4i\pi}{N} kn} \sum_{\ell=0}^{N-1} c_\ell\, e^{-\frac{4i\pi}{N} j(\ell - k)} \\ &= \frac{\sqrt{N}}{2} \sum_{j \in \mathcal{S}_n} e^{\frac{4i\pi}{N} k(n+j)}\, d_{2j}. \end{aligned} \tag{67}$$

Hence the kernel of Γ_n is $(N - 2|n|)$-dimensional and the result easily follows. □

As an immediate consequence of this proposition, we can state the following

Corollary 17 *Let $\eta_N < N/2$ and $k > 0$ be given, such that $2^k \eta_N < N$. Then there exists a subspace $\mathcal{G}_{\eta_N}^{(k)} \subset \mathcal{H}_N$ such that*

1. $\dim \mathcal{G}_{\eta_N}^{(k)} \geq N - 2k\eta_N$.

2. $\forall \psi \in \mathcal{G}_{\eta_N}^{(k)}$*:*

$$E_B^{(j)}(\chi(n,0))\psi = 0, \qquad \forall \mid n \mid < \eta_N \text{ and } \forall 0 \leq j \leq k.$$

Using this corollary and a natural adaptation of Proposition 10, one now readily proves Theorem 4, along the lines of the proof of Theorem 7.

6 Proof of Theorem 2

For $f \in C^\infty(\mathcal{T}^2)$, we write $\bar{f} = \int_{\mathcal{T}^2} f \; dq\, dp$. We will denote the time-average of f and of $Op_\hbar^W(f)$ by

$$\{f\}_K = \frac{1}{K}\sum_{\ell=0}^{K-1}\left(f \circ T^\ell\right), \qquad \left\{Op_\hbar^W(f)\right\}_K = \frac{1}{K}\sum_{\ell=0}^{K-1} V_T^{-\ell} Op_\hbar^W(f)\, V_T^\ell,$$

and write

$$Z_N(f) = \frac{1}{N}\sum_{j=1}^{N}\left| < \varphi_j^{(N)},\, Op_\hbar^W(f-\bar{f})\, \varphi_j^{(N)} > \right|^2, \tag{68}$$

for any $f \in C^\infty(\mathcal{T}^2)$ (see [ZZ]). For any fixed K and ϵ to be chosen later, introduce a smooth characteristic function $\chi_{\epsilon,K}$ of Σ_K, with the property that $\mu\,(\mathrm{Supp}\,\chi_{\epsilon,K}) \leq \epsilon$, where μ denotes the Lebesgue measure. Then, for any $f \in C^\infty(\mathcal{T}^2)$

$$f = f\,\chi_{\epsilon,K} + f(1-\chi_{\epsilon,K}). \tag{69}$$

We shall write $f_{\epsilon,K} \equiv f(1-\chi_{\epsilon,K})$,

$$\gamma_1 = \int_{\mathcal{T}^2} f_{\epsilon,K}\, d\mu, \qquad \gamma_2 = \int_{\mathcal{T}^2} f\,\chi_{\epsilon,K}\, d\mu.$$

Clearly (see 68)

$$Z_N(f) \leq 2\left[Z_N(f_{\epsilon,K}) + Z_N(f\chi_{\epsilon,K})\right]. \tag{70}$$

We control $Z_N(f_{\epsilon,K})$ first. Note that, since (T, V_T) satisfies an Egorov estimate up to time K, $\left(f_{\epsilon,K}\circ T^\ell\right) \in C^\infty(\mathcal{T}^2)$ for all $\ell \leq K$ and hence $Op_\hbar^W\left(f_{\epsilon,K}\circ T^\ell\right)$ is well defined. Since the φ_j are eigenfunctions of V_T (we drop the N on the $\varphi_j^{(N)}$),

$$Z_N(f_{\epsilon,K}) = \frac{1}{N}\sum_{j=1}^{N}\left| < \varphi_j, \left\{Op_\hbar^W(f_{\epsilon,K}-\gamma_1)\right\}_K \varphi_j > \right|^2. \tag{71}$$

Noting that $|< \varphi_j\,, A\varphi_j >|^2 \leq\, < \varphi_j,\, A^* A\varphi_j >$ for any $A \in \mathcal{L}(\mathcal{H}_N)$, we get

$$Z_N(f_{\epsilon,K}) \leq \frac{1}{N}\sum_{j=1}^{N} < \varphi_j, \left\{Op_\hbar^W(f_{\epsilon,K}-\gamma_1)\right\}_K^* \left\{Op_\hbar^W(f_{\epsilon,K}-\gamma_1)\right\}_K \varphi_j > . \tag{72}$$

With an eye towards using the Egorov estimate (3) in Definition 1, we rewrite this as follows:

$$
\begin{aligned}
Z_N(f_{\epsilon,K}) &\leq \frac{1}{N}\sum_{j=1}^{N} < \varphi_j, \left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K^* \left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K \varphi_j > \\
&= \frac{1}{N}\sum_{j\notin\mathcal{E}^{(K)}(N)} < \varphi_j, \left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K^* \left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K \varphi_j > \\
&+ \frac{1}{N}\sum_{j\in\mathcal{E}^{(K)}(N)} < \varphi_j, Op_\hbar^W\left(\{f_{\epsilon,K}-\gamma_1\}_K\right)^* Op_\hbar^W\left(\{f_{\epsilon,K}-\gamma_1\}_K\right)\varphi_j > \\
&+ \frac{1}{N}\sum_{j\in\mathcal{E}^{(K)}(N)} < \varphi_j, \left[\left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K - Op_\hbar^W\left(\{f_{\epsilon,K}-\gamma_1\}_K\right)\right]^* \\
&\qquad\qquad \left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K \varphi_j > \\
&+ \frac{1}{N}\sum_{j\in\mathcal{E}^{(K)}(N)} < \varphi_j, \left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K^* \\
&\qquad\qquad \left[\left\{Op_\hbar^W\left(f_{\epsilon,K}-\gamma_1\right)\right\}_K - Op_\hbar^W\left(\{f_{\epsilon,K}-\gamma_1\}_K\right)\right]\varphi_j > .
\end{aligned}
$$

Now we use that $\forall f,g\in C^\infty(\mathcal{T}^2)$

$$
\| \, Op_\hbar^W(f)\, Op_\hbar^W(g) - Op_\hbar^W(fg)\, \|_{\mathcal{H}_N} \leq \frac{C(f,g)}{N} \tag{73}
$$

to conclude that there exists a positive constant $C_{\epsilon,K}(f)$, such that the following estimate holds

$$
\begin{aligned}
Z_N(f_{\epsilon,K}) &\leq \frac{N-\sharp\mathcal{E}^{(K)}(N)}{N} C_{\epsilon,K}(f) \\
&+ \frac{1}{N}\sum_{j\in\mathcal{E}^{(K)}(N)} < \varphi_j, Op_\hbar^W\left(|\,\{f_{\epsilon,K}-\gamma_1\}_K\,|^2\right)\varphi_j > + \frac{C_{\epsilon,K}(f)}{N} \\
&+ C_{\epsilon,K}(f)\frac{\sharp\mathcal{E}^{(K)}(N)}{N} \sup_{j\in\mathcal{E}^{(K)}(N)} \sup_{k\in\{0,\ldots,K-1\}} \| \, E_T^{(k)}(f_{\epsilon,K}-\gamma_1)\varphi_j \, \|_{\mathcal{H}_N} .
\end{aligned}
\tag{74}
$$

Using that $\left|\frac{1}{N}\mathrm{Tr}\, Op_\hbar^W(g) - \int_{\mathcal{T}^2} g\, d\mu\right| \leq \frac{C(g)}{N}$, for any $g\in C^\infty(\mathcal{T}^2)$, (74) yields, modulo yet another easily controlled error term,

$$
Z_N(f_{\epsilon,K}) \leq \int_{\mathcal{T}^2} \left|\{f_{\epsilon,K}-\gamma_1\}_K\right|^2 d\mu
$$

$$
\begin{aligned}
&+ \quad C_{\epsilon,K}(f)[\left(1-\frac{\sharp\mathcal{E}^{(K)}(N)}{N}\right)+\frac{1}{N} \\
&+ \quad \frac{\sharp\mathcal{E}^{(K)}(N)}{N}\sup_{j\in\mathcal{E}^{(K)}(N)}\ \sup_{k\in\{0,\ldots,K-1\}} \parallel E_T^{(k)}(f_{\epsilon,K}-\gamma_1)\varphi_j \parallel].
\end{aligned} \tag{75}
$$

Now, choose $\delta > 0$ fixed. Then, since T is ergodic, we have for all K large enough

$$
\int_{\mathcal{T}^2} \left|\{f\}_K - \bar{f}\right|^2 \, d\mu < \frac{\delta}{11}. \tag{76}
$$

Note that, for such K, $\lim_{\epsilon\to 0}\{f_{\epsilon,K}\}_K = \{f\}_K$ and $\lim_{\epsilon\to 0}\gamma_1 = \bar{f}$, so that the dominated convergence theorem yields that, for ϵ sufficiently small

$$
\int_{\mathcal{T}^2} \left|\{f_{\epsilon,K}\}_K - \gamma_1\right|^2 \, d\mu \leq \frac{\delta}{4}. \tag{77}
$$

Taking K and ϵ as above, and inserting (77) in (75), the Egorov estimate (3) implies that, for N sufficiently large

$$
Z_N(f_{\epsilon,K}) \leq \frac{\delta}{2}. \tag{78}
$$

It remains to control the term $Z_N(f\chi_{\epsilon,K})$. Note that, by using the previous observation regarding the trace of $Op_\hbar^W(g)$, we can write

$$
\left|\frac{1}{N}\mathrm{Tr}\, Op_\hbar^W\left(\mid f\chi_{\epsilon,K}-\gamma_2\mid^2\right) - \int_{\mathcal{T}^2} \mid f\chi_{\epsilon,K}-\gamma_2\mid^2 \, d\mu\right| \leq \frac{1}{N}C_{\epsilon,K}(f). \tag{79}
$$

In particular, given $\epsilon > 0$ and K, we can choose N sufficiently large such that (see also (73))

$$
\begin{aligned}
Z_N(f\chi_{\epsilon,K}) \quad &\leq \quad \frac{1}{N}\sum_{j=1}^{N} < \varphi_j,\, Op_\hbar^W\left(f\chi_{\epsilon,K}-\gamma_2\right)^* Op_\hbar^W\left(f\chi_{\epsilon,K}-\gamma_2\right)\varphi_j > \\
&\leq \quad \frac{1}{N}\mathrm{Tr}\, Op_\hbar^W\left(\mid f\chi_{\epsilon,K}-\gamma_2\mid^2\right) + \frac{C'(\epsilon,K)}{N} \\
&\leq \quad \int_{\mathcal{T}^2} \mid f\chi_{\epsilon,K}-\gamma_2\mid^2 \, d\mu + \frac{C''(\epsilon,K)}{N} \leq \frac{\delta}{2}.
\end{aligned} \tag{80}
$$

The first equation in Theorem 2 now follows from (70), (78), and (80). The rest follows from standard diagonalization and density arguments which we omit [CdV, Z1].

References

[AA] V.I. Arnold, A. Avez, *Ergodic Problems in Classical Mechanics*, Benjamin, New York (1968)

[BV] N.L. Balazs, A. Voros, *The Quantized Baker's Transformation*, Annals of Physics **190** (1989), 1-31.

[BD] A.Bouzouina, S. De Bièvre, *Equipartition of the eigenfunctions of quantized ergodic maps on the torus*, to appear in Comm.Math.Phys., (1996).

[BO] M. Basilio de Matos and A.M. Ozerio de Almeida, *Quantization of Anosov maps*, (preprint 1995).

[C] J. Chazarain, *Construction de la paramétrix du problème mixte hyperbolique pour l'équation d'ondes*, C.R. Acad. Sci. Paris, **276** (1973), 1213-1215.

[Ch] N. Chernoff, *Ergodic and statistical properties of piecewise linear hyperbolic automorphisms of the two-torus*, J.Stat.Phys. **69** (1992), 111-134.

[CdV] Y. Colin de Verdière, *Ergodicité et functions propres du Laplacien*, Commun. Math. Phys. **102** (1985), 497-502.

[CR] M. Combescure, D. Robert, *Distribution of matrix elements and level spacings for classical chaotic systems*, Ann. Inst. H. Poincaré **61**, 4 (1994), 443-483.

[DDG] S. De Bièvre, M. Degli Esposti and R. Giachetti,*Quantization of a class of piecewise affine transformations on the torus*, Comm. Math. Phys. **176**, (1995), 73-94.

[DE] M. Degli Esposti, *Quantization of the orientation preserving automorphisms of the torus*, Ann. Inst. Henri Poincaré **58** (1993), 323-341.

[DIG] M. Degli Esposti, S. Isola and S. Graffi, *Classical limit of the quantized hyperbolic toral automorphisms*, Comm.Math.Phys. **167** (1995), 471-507.

[Ed] R.E. Edwards, *Fourier Series. A modern Introduction. Volume 1*, Graduate Text in Mathematics **64** (1979).

[F] M. Farris, *Egorov's theorem on a manifold with diffractive boundary*, Comm. P.D.E. **6**, 6 (1981), 651-687.

[GL] P. Gerard, E. Leichtnam, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. Jour. **71** (1993), 559-607.

[HB] J.H. Hannay, M.V. Berry, *Quantization of linear maps on a torus - Fresnel diffraction by a periodic grating*, Physica D **1** (1980), 267-291.

[HKO] J.H. Hannay, J.P. Keating, A.M. Ozorio de Almeida, *Optical realization of the baker's transformation*, Nonlinearity **7** (1994), 1327-1342

[HMR] B.Helffer,A.Martinez,D.Robert, *Ergodicité et limite semi-classique*, Commun. Math. Phys. **131** (1985) 493-520.

[Ke1] J. Keating, Ph.D. thesis University of Bristol (1989).

[Ke2] J. Keating, *The cat maps: quantum mechanics and classical motion*, Nonlinearity **4** (1991), 309-341.

[La] A. Laskshminarayan, *Semiclassical theory of the Sawtooth map*, preprint (1995).

[Li] C. Liverani, *Decay of correlations*, Annals of Mathematics **142** (1995), 239-301.

[LW] C. Liverani, M.P. Wojtkowski, *Ergodicity in Hamiltonian systems*, to appear in Dyn. Rep. **4**.

[OHT] P. O'Connor, R. Heller, S. Tomsovic, *Semi-classical dynamics in the strongly chaotic regime: breaking the log-time barrier*, Physica D **55** (1992), 340-357.

[R] D. Robert, *Autour de l'approximation semi-classique*, Birkhaüser Boston (1987).

[S] P. Sarnak, *Arithmetic quantum chaos*, Tel Aviv Lectures (1993).

[Sa] M. Saraceno, *Classical structures in the quantized Baker transformation*, Annals of Physics **199** (1990), 37-60.

[SV] M. Saraceno, A. Voros, *Towards a semiclassical theory of the quantum baker's map*, Physica D **79** (1994), 206-268.

[SVa] M. Saraceno, R.O. Vallejos, *The quantized D-transformation*, (preprint 1995).

[Va] S. Vaienti, *Ergodic properties of the discontinuous sawtooth map*, J. Stat. Phys. **67**, (1992), 251.

[Z1] S. Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. Jour., **55** (1987), 919-941.

[Z2] S. Zelditch, *Quantum transition amplitudes for ergodic and completely integrable systems*, Journ. Funct. Ana. **94** (1990) 415-436.

[Z3] S. Zelditch, *Quantum ergodicity of* C^* *dynamical systems*, to appear in Commun. Math. Phys. (1996).

[Z4] S. Zelditch, *Index and dynamics of quantized contact transformations*, (preprint 1996).

[ZZ] S. Zelditch, M. Zworski, *Ergodicity of eigenfunctions for ergodic billiards*, Commun. Math. Phys. **175**, (1996) 3, 673-682.