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# Three equivalent definitions of the correlation dimension of a probability measure.

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**Abstract**

For any probability Borel measure $\mu$ on $\mathbb{R}^n$, we define three functions:

$$\begin{array}{lcll} \Omega\mu(a) & = & \frac{1}{a^n}\int d\mu(x)\ \mu(B(x,a)), & a>0, \\ \Gamma\mu(a) & = & \frac{1}{a^{2n}}\int dx\ \left[\mu(B(x,a))\right]^2, & a>0, \\ M\mu(T) & = & \frac{1}{T^n}\int_{|\xi|\le T} d\xi |\hat{\mu}(\xi)|^2, & T>0, \end{array}$$

where $B(x,a)$ is the ball of center $x\in\mathbb{R}^n$ and radius $a>0$. We show that

$$\begin{aligned} \lim_{a\to 0}\sup(\inf)\frac{\log\Gamma\mu(a)}{\log a} &= n-\lim_{T\to\infty}\inf(\sup)\frac{\log M\mu(T)}{\log T} \\ &= \lim_{a\to 0}\sup(\inf)\frac{\log\Omega\mu(a)}{\log a} = D^+\ (D^-), \end{aligned}$$

where $D^+$ and $D^-$ are respectively the upper and lower correlation dimension of $\mu$.

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# 1 Introduction

Recently it has been shown that certain fractal properties of the spectral measure of a Hamiltonian $\mathcal{H}$ are responsible for the long time behavior of the unitary evolution group $e^{it\mathcal{H}}$ it generates (see e.g. [Com92], [Gua89], [KPG92], to cite only a few results). More precisely if $d\mu_\phi$ is the spectral measure associated to the state $\phi$, then one is interested in quantifying the behavior of the time average

$$\Phi(T) = \int_0^T dt \, \left| \left\langle \phi \mid e^{it\mathcal{H}} \phi \right\rangle \right|^2$$

as $T$ gets large. A simple application of standard functional calculus shows that the above expression equals

$$\Phi(T) = \int_0^T d\xi \, |\widehat{\mu}_\phi(\xi)|^2$$

where the Fourier transform of the measure $\mu_\phi$ is defined as usual

$$\widehat{\mu}_\phi(\xi) = \int d\mu_\phi(x) e^{-i\xi x}.$$

So some heuristic arguments have been given in [KPG92] supporting the fact that the average $\Phi(T)$ exhibits a scaling behavior $\sim T^{1-D}$ where $D$ is the correlation dimension of the measure $\mu_\phi$. In [Hol94] some new fractal dimensions have been introduced by means of wavelet transforms. For these dimensions it has been shown that actually the heuristic argument is true and that the long time evolution of the averages $\Phi$ is governed by the upper respectively lower wavelet - correlation dimension.

In this paper we now want to show, that the wavelet correlation dimensions are actually identical with the usual correlation dimensions associated with a probability measure. Hence we prove that the long time evolution of the average $\Phi(T)$ is governed by the correlation dimension of the spectral measure. A partial result has been obtained before in [BCM95].

Note that we limit ourselves to the case where $\mu$ is an unsigned measure, that is a probability measure. Positivity will be a central argument in this paper. If one is interested in quantities like

$$\int_0^T dt \, \left| \left\langle \psi \mid e^{it\mathcal{H}} \phi \right\rangle \right|^2$$

with $\psi \neq \phi$ the spectral measure is in general a complex measure. In this case the correlation dimension responsable for the time evolution is not the usual correlation dimension based on averages with a priori unsigned functions (see below) since these averages might exhibit cancelations. Therefore we believe that for complex measures, the right notion is the wavelet correlation dimension introduced in [Hol94]

## 2 Statement of the problem and formulation of main result

Consider a probability Borel measure $\mu$ on $\mathbb{R}^n$. One common definition (see [Pes93]) of its correlation dimension is the following . Take $B(x,a)$ the closed ball of radius $a$ around $x$ in $\mathbb{R}^n$ and consider the function $x \mapsto \mu(B(x,a))$. This is a measurable function. Indeed, if we decompose $\mu$ into a purely ponctual part, say $\mu_{pp} = \sum a_i \delta_{b_i}$, and a continuous part, say $\mu_c$, we have that for all $x \in \mathbb{R}^n$,

$$\mu(B(x,a)) = \sum a_i \chi_{B(b_i,a)}(x) + \mu_c(B(x,a)),$$

where the notation $\chi_A$ designates the characteristic function of a set $A \subset \mathbb{R}^n$. The first term is a step function, which is measurable; the second one is a continuous function. Hence, we can form the $\mu$-average:

$$\Omega\mu(a) = \frac{1}{a^n} \int d\mu(x)\ \mu(B(x,a)).$$

The quantities

$$\omega^+(\mu) = \limsup_{a \to 0} \frac{\log \Omega\mu(a)}{\log a},$$

$$\omega^-(\mu) = \liminf_{a \to 0} \frac{\log \Omega\mu(a)}{\log a},$$

are called respectively upper and lower correlation dimensions of $\mu$.

If we formally replace $\mu(x)$ by its mean value $a^{-n}\mu(B(x,a))$ on a ball of radius $a$, we obtain a slightly different version of $\Omega\mu$:

$$\Gamma\mu(a) = \frac{1}{a^{2n}} \int dx\ \ [\mu(B(x,a))]^2,$$

which has the advantage of being numerically computable. Note that we have implicitly supposed $x \mapsto \mu(B(x,a))$ to be square integrable. This is actually the case, as we shall prove in the sequel. We also introduce the function:

$$M\mu(T) = \frac{1}{T^n} \int_{|\xi| \le T} d\xi |\hat{\mu}(\xi)|^2,$$

where $\hat{\mu}$ is the Fourier transform of $\mu$ :

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} d\mu(x) e^{-i\xi x}. \tag{2.1}$$

The dimensions associated to $\Gamma\mu$ and $M\mu$ are respectively:

$$\gamma^+(\mu) = \limsup_{a \to 0} \frac{\log \Gamma\mu(a)}{\log a}, \qquad \gamma^-(\mu) = \liminf_{a \to 0} \frac{\log \Gamma\mu(a)}{\log a}$$

and

$$m^+(\mu) = \limsup_{T\to+\infty} \frac{\log M\mu(T)}{\log T}, \qquad m^-(\mu) = \liminf_{T\to+\infty} \frac{\log M\mu(T)}{\log T}.$$

In [Hol94], the dimensions $m^\pm(\mu)$ were shown to coincide, up to the sign, with the wavelet-correlation dimensions. In this note, we establish simple relations between the three pairs of dimensions $\omega^\pm(\mu), \gamma^\pm(\mu)$ and $m^\pm(\mu)$. We will prove that:

$$\gamma^+(\mu) = \omega^+(\mu) = n - m^-(\mu),$$

$$\gamma^-(\mu) = \omega^-(\mu) = n - m^+(\mu).$$

and thus show that the correlation dimension of a probability Borel measure can be defined via anyone of the three functions $\Omega\mu, \Gamma\mu$ or $M\mu$.

# 3 Some generalizations.

Let $\chi$ be the characteristic function of the unit ball $B(0,1)$ in $\mathbb{R}^n$, and $\chi_a$ its dilated version:

$$\chi_a(x) = \frac{1}{a^n}\chi(\frac{x}{a}).$$

The functions $\Omega\mu, \Gamma\mu$ and $M\mu$ can be more simply rewritten as :

$$\Omega\mu(a) = \int d\mu(x)\ \mu * \chi_a(x),$$

$$\Gamma\mu(a) = \int dx\ \ [\mu * \chi_a(x)]^2,$$

$$M\mu(T) = \int d\xi\ |\hat{\mu}(\xi)|^2 \chi_T(\xi).$$

A natural extension of these definitions is to replace $\chi$ by some positive function $g$ well localized around zero. More precisely, let $g$ be a positive Borel function, that is a function for which all preimages $g^{-1}(A)$ of Borel sets $A \subset \mathbb{R}^n$ are Borel sets too, and as before let $g_a(x) = a^{-n}g(x/a)$ be its dilation. Suppose in addition that $g$ is rapidly decreasing, that is it decays faster than every polynomial:

$$\sup_{|\alpha|\le N}\ \sup_{x\in\mathbb{R}^n} |x^\alpha g(x)| < \infty, \qquad N = 0, 1, 2....$$

Then it makes sense to replace $\chi$ by $g$, as shown in the following proposition.

**Proposition 1** *If $g$ is a rapidly decreasing function, then for $a > 0$, the functions*

$$\Omega_g\mu(a) = \int d\mu(x)\ \mu * g_a(x), \tag{3.1}$$

$$\Gamma_g\mu(a) = \int dx\ \ (\mu * g_a(x))^2, \tag{3.2}$$

$$M_g\mu(T) = \int d\xi\ |\hat{\mu}(\xi)|^2 g_T(\xi), \tag{3.3}$$

*are well defined and take finite values. If furthermore $g$ belongs to the Schwarz class $\mathcal{S}(\mathbb{R}^n)$, then the Parseval-Bessel equality for $\Omega_g\mu$ and $\Gamma_g\mu$ holds, namely:*

$$\Omega_g\mu(a) = \frac{1}{(2\pi)^n}\int d\xi|\hat{\mu}(\xi)|^2\hat{g}(a\xi), \tag{3.4}$$

*and*

$$\Gamma_g\mu(a) = \frac{1}{(2\pi)^n}\int d\xi|\hat{\mu}(\xi)|^2|\hat{g}(a\xi)|^2. \tag{3.5}$$

Note the average in (3.2) coincides with the one given in [Hol94] in relation with the wavelet-correlation dimension.

*Proof.* The proofs are elementary but we shall give them anyway for the convenience of the reader.

*Proof of (3.1) :* Since $g_a$ is mesurable, there exist a sequence of step functions $g_a^N$ converging pointly to $g_a$ as $N \to \infty$ (see e.g [Rud78]). By the same argument as above, each function $\mu * g_a^N$ is mesurable. Hence the limit function $\mu * g_a$ is also mesurable. Moreover, $g_a$ is bounded by $a^{-n}||g||_\infty$ and:

$$\Omega_g\mu(a) \leq a^{-n}||g||_\infty \int d\mu(x)\mu * 1(x) = a^{-n}||g||_\infty.$$

*Proof of (3.2) :* The function $g$ together with its dilatations $g_a$ are in $\mathcal{L}^1$ and by Fubinis theorem the same holds for $\mu * g_a$. Now clearly $\mu * g_a$ is in $\mathcal{L}^\infty$ and thus $\mu * g_a$ is in $\mathcal{L}^2$ by Hölders inequality.

*Proof of (3.3) :* The Fourier transform of a probability measure is a continous function and there is no problem in defining $M_g\mu$. Moreover, $|\hat{\mu}|^2$ is bounded by one and $|M_g\mu(a)| \leq ||g||_1$.

*Proof of (3.4) :* Choose a non negative function $\phi$ in $C_0^\infty$ such that $\phi = 1$ on the unit ball $B(0,1)$ and form the sequence of functions:

$$\Omega_g^N\mu(a) = \int d\mu(x)\ \mu * g_a(x)\phi(\frac{x}{N}).$$

These functions converge for each $a$ to $\Omega_g\mu(a)$ since

$$|\Omega_g^N\mu(a) - \Omega_g\mu(a)| \leq \int d\mu(x)\ |\mu * g_a(x)|\ |\phi(\frac{x}{N}) - 1| \leq O(1)\int_{|x|\geq N} d\mu(x) \to 0,\ N \to \infty.$$

As we may we take $\mu$ as a tempered distribution and hence by definition we have

$$\Omega_g^N\mu(a) = \frac{1}{(2\pi)^n}\int d\xi\ \overline{\hat{\mu}}(\xi)\ [\mu * g_a(\cdot)\phi(\frac{\cdot}{N})]\hat{}(\xi).$$

Here $\widehat{\mu}$ is a priori taken in the sense of distributions but since $\widehat{\mu}$ is a continuous function, it coincides with the definition 2.1. Now for $\eta \in \mathcal{S}'$ and $s \in \mathcal{S}$ the convolution theorem applies, namely $(\eta * s)\hat{} = \hat{\eta}\cdot\hat{s}$ and therefore we have $[\mu * g_a(\cdot)\phi(\frac{\cdot}{N})]\hat{}(\xi) = (\hat{\mu}\widehat{g_a}) * \hat{\phi}_{1/N}(\xi)$. All together we end up with

$$\Omega_g^N \mu(a) = \frac{1}{(2\pi)^n}\int d\xi \ \overline{\hat{\mu}}(\xi) \ (\hat{\mu}\widehat{g_a}) * \hat{\phi}_{1/N}(\xi).$$

Since $\hat{\mu}\widehat{g_a}$ is continuous and $\hat{\phi}$ rapidly decaying, it follows from the approximation of the identity lemma that $(\hat{\mu}\widehat{g_a}) * \hat{\phi}_{1/N}$ converges pointwise towards $\hat{\mu}\widehat{g_a}$. Moreover, $(\hat{\mu}\widehat{g_a}) * \hat{\phi}_{1/N}$ is uniformly bounded by some $\mathcal{L}^1$ function, as shown in the following lemma:

**Lemma 3.1** *Consider the function:*

$$K^\alpha(x) = (1+|x|)^{-\alpha}.$$

*Then for all $\alpha > n$, we have:*

$$|K^\alpha_{1/N} * K^\alpha(x)| \leq O(1)K^\alpha(x)$$

*uniformly in $N \geq 1$.*

Recall that $K_{1/N}(\cdot) = N\ K(N\cdot)$.

*Proof.* For any $x \in \mathbb{R}^n$, we split the convolution integral in two domains:

$$K^\alpha_{1/N} * K^\alpha(x) = \left\{\int_{y\in B(0,|\frac{x}{2}|)} + \int_{y\notin B(0,|\frac{x}{2}|)}\right\} dy\ K^\alpha_{1/N}(x-y)K^\alpha(y) = I_1(x) + I_2(x).$$

For all $y \in B(0,|\frac{x}{2}|)$, we have $|x-y| \geq \frac{|x|}{2}$. Hence:

$$I_1(x) \leq K^\alpha_{1/N}(\frac{x}{2})\int_{y\in B(0,|\frac{x}{2}|)} dy\ |K^\alpha(y)| \leq ||K^\alpha||_1 K^\alpha_{1/N}(\frac{x}{2}).$$

On the other hand,

$$I_2(x) \leq K^\alpha(\frac{x}{2})\int_{y\notin B(0,|\frac{x}{2}|)} dy\ K^\alpha_{1/N}(x-y) \leq K^\alpha(\frac{x}{2})||K^\alpha||_1.$$

Now on every domain bounded away from zero, say $\mathbb{R}^n\backslash B(0,1)$, we have:

$$K^\alpha_{1/N}(x) \leq O(|x|^{-\alpha})O(N^{n-\alpha}) \leq O(1)K^\alpha(\frac{x}{2}).$$

and thus

$$K^\alpha_{1/N} * K^\alpha = I_1(x) + I_2(x) \leq O(1)K^\alpha(\frac{x}{2}) \leq O(1)K^\alpha(x). \tag{3.6}$$

Since $K^\alpha_{1/N} * K^\alpha$ is uniformly bounded by $||K^\alpha||_1||K^\alpha||_\infty$ on $B(0,1)$, (3.6) holds on the whole space $\mathbb{R}^n$ and the proof of lemma (3.1) is completed. □

As a result, we have that:

$$|(\hat{\mu}\widehat{g_a}) * \hat{\phi}_{1/N}| \leq O(1)K^\alpha \in \mathcal{L}^1(\mathbb{R}^n),$$

for all $\alpha > n$, uniformly in $N \geq 1$. We use the Lebuesgue dominated convergence theorem to conclude:

$$\Omega_g\mu(a) = \lim_{N\to\infty}\Omega_g^N\mu(a) = \int d\xi\ |\hat{\mu}(\xi)|^2\ \hat{g}(a\xi).$$

*Proof of 3.5 :* This is similar and we leave it to the reader. □

Thus we have defined more general classes of functions $\Omega_g\mu, \Gamma_g\mu$ and $M_g\mu$ parametrized by $g$. We can associate to them a set of exponents that are a natural generalization of the previous exponents:

$$\omega_g^+(\mu) = \limsup_{a\to 0} \frac{\log \Omega\mu(a)}{\log a}, \qquad \omega_g^-(\mu) = \liminf_{a\to 0} \frac{\log \Omega\mu(a)}{\log a},$$

$$\gamma_g^+(\mu) = \limsup_{a\to 0} \frac{\log \Gamma\mu(a)}{\log a}, \qquad \gamma_g^-(\mu) = \liminf_{a\to 0} \frac{\log \Gamma\mu(a)}{\log a}$$

and

$$m_g^+(\mu) = \limsup_{T\to +\infty} \frac{\log M\mu(T)}{\log T}, \qquad m_g^-(\mu) = \liminf_{T\to +\infty} \frac{\log M\mu(T)}{\log T}.$$

# 4 Main result

The main result we want to prove can be summarized in the following theorem.

**Theorem 1** *Let $\mu$ be a probability Borel measure on $\mathbb{R}^n$ and $g$ a Borel function on $\mathbb{R}^n$. If $g$ is non-negative, rapidly decreasing, continous at zero and such that $g(0) > 0$, then we have:*

$$m_g^+ = m^+, \qquad m_g^- = m^-, \tag{4.1}$$

$$\gamma_g^+ = n - m^-, \qquad \gamma_g^- = n - m^+, \tag{4.2}$$

$$\omega_g^+ = n - m^-, \qquad \omega_g^- = n - m^+. \tag{4.3}$$

Upon choosing $g$ the characteristic function of $B(0,1)$ we have

**Corollary 1** *The following holds true*

$$\gamma^+(\mu) = \omega^+(\mu) = n - m^-(\mu),$$

$$\gamma^-(\mu) = \omega^-(\mu) = n - m^+(\mu).$$

For the proof of theorem 1, we need the following lemma:

**Lemma 1** *Let $s(t)$ be a strictly positive function defined for $t > 0$. Then we have :*

$$\liminf_{t\to 0} \frac{\log s(t)}{\log t} = \sup\{\ \alpha | s(t) \le O(t^\alpha), t \to 0\}$$

$$\limsup_{t\to 0} \frac{\log s(t)}{\log t} = \inf\{\ \alpha | t^\alpha \le O(s(t)), t \to 0\}$$

The proof of this lemma can be found in [Hol94].

*Proof.* We successively are going to prove equations (4.1), (4.2) and (4.3).

*Proof of (4.1).* Since $g(0) > 0$ and $g$ continous at zero, it is greater than some characteristic function, that is we can find $\lambda > 0$ such that:

$$\frac{1}{\lambda}\chi(\lambda x) \leq g(x),$$

in which case

$$\chi_T(x) \leq \lambda^{n+1} g_{\lambda T}(x),$$

and

$$M\mu(T) \leq \lambda^{n+1} M_g\mu(\lambda T). \tag{4.4}$$

Now let us fix $\epsilon > 0$ and split $M_g\mu$ in two terms:

$$M_g\mu(T) = \left\{ \int_{|\xi|\leq T^{1+\epsilon}} + \int_{|\xi|>T^{1+\epsilon}} \right\} d\xi \; |\hat{\mu}(\xi)|^2 g_T(\xi).$$

The first part can be related to $M\mu$:

$$\int_{|\xi|\leq T^{1+\epsilon}} d\xi \; |\hat{\mu}(\xi)|^2 g_T(\xi) \leq ||g||_\infty T^\epsilon M\mu(T^{1+\epsilon}),$$

and the queue of the integral can be easily bounded:

$$\left| \int_{|\xi|>T^{1+\epsilon}} d\xi \; |\hat{\mu}(\xi)|^2 g_T(\xi) \right| \leq \int_{|\xi|>T^\epsilon} d\xi \; |g(\xi)|.$$

On one hand $T^\epsilon M\mu(T^{1+\epsilon})$ is a positive function which decreases not faster than $T^{\epsilon-1}$ (this is an evident consequence of $\hat{\mu}(0) > 0$ and $\hat{\mu}$ continuous). On the other hand, $\int_{|\xi|>T^\epsilon} d\xi \; |g(\xi)|$ is a rapidly decaying function of $T$. Hence, we can regroup the two terms under the same bound:

$$M_g\mu(T) \leq O(1) T^\epsilon M\mu(T^{1+\epsilon}) \tag{4.5}$$

Inequalities (4.4) and (4.5) imply the following relations for any real $\alpha$:

$$\begin{array}{lcl} M_g\mu(T) \leq O(1)T^\alpha & \Rightarrow & M\mu(T) \leq O(1)T^\alpha, \\ T^\alpha \leq O(1)M\mu(T) & \Rightarrow & T^\alpha \leq O(1)M_g\mu(T), \\ M\mu(T) \leq O(1)T^\alpha & \Rightarrow & M_g\mu(T) \leq O(1)T^{\alpha(1+\epsilon)+\epsilon}, \\ T^\alpha \leq O(1)M_g\mu(T) & \Rightarrow & T^\alpha \leq T^\epsilon O(1) M\mu(T^{1+\epsilon}). \end{array}$$

Thanks to lemma 1, this leads to

$$(1+\epsilon)^{-1} \; m_g^-(\mu) \leq m^-(\mu) \leq m_g^-(\mu),$$

$$m_g^+(\mu) \leq m^+(\mu) \leq (1+\epsilon) \; m_g^+(\mu) + \epsilon.$$

Since this holds for all $\epsilon > 0$, it proves (4.1).

*Proof of (4.2) :* The idea of the proof here is to bracket $g$ by two Schwarz functions $\varphi_1$ and $\varphi_2$ and to use expressions (3.5) for $\Gamma_{\varphi_1}\mu$ and $\Gamma_{\varphi_2}\mu$.
Since $g(0) > 0$ and g continous at zero, there exists a positive function $\varphi_1$ in $C_0^\infty$ such that $\varphi_1 \leq g$ with $\varphi_1(0) > 0$. On the other hand, we may find a Schwarz function $\varphi_2$ which majorize $g$. Therefore we have:

$$\Gamma_{\varphi_1}\mu(a) \leq \Gamma_g\mu(a) \leq \Gamma_{\varphi_2}\mu(a),$$

with $\varphi_i,\ i = 1, 2, \in \mathcal{S}(\mathbb{R}^n)$. Now, taking (3.5) into account, we obtain:

$$\int d\xi\ |\hat{\mu}(\xi)|^2|\hat{\varphi}_1(a\xi)|^2 \leq \Gamma_g\mu(a) \leq \int d\xi\ |\hat{\mu}(\xi)|^2|\hat{\varphi}_2(a\xi)|^2,$$

that is

$$a^n M_{|\hat{\varphi}_1|^2}\mu(\frac{1}{a}) \leq \Gamma_g\mu(a) \leq a^n M_{|\hat{\varphi}_2|^2}\mu(\frac{1}{a}).$$

The inequality is conserved when one takes the logarithm. Hence,

$$n - m^-_{|\hat{\varphi}_1|^2}(\mu) \leq \omega_g^+(\mu) \leq n - m^-_{|\hat{\varphi}_2|^2}(\mu),$$

$$n - m^+_{|\hat{\varphi}_1|^2}(\mu) \leq \omega_g^-(\mu) \leq n - m^+_{|\hat{\varphi}_2|^2}(\mu).$$

Now, since $\hat{\varphi}_i(0) = \int \varphi_i > 0$, the two functions $|\hat{\varphi}_i|^2$ fullfill the conditions of the first part of theorem 1, whose direct application then proves (4.2). □

*Proof of (4.3) :* The proof of (4.3) is similar to (4.2) but there is one more difficulty. Suppose we still have $\varphi_1 \leq g \leq \varphi_2$ with $\varphi_i$ positive Schwarz functions. If we use expression (3.4) of $\Omega_{\varphi_i}$, we have :

$$\int d\xi\ |\hat{\mu}(\xi)|^2\overline{\hat{\varphi}}_1(a\xi) \leq \Omega_g\mu(a) \leq \int d\xi\ |\hat{\mu}(\xi)|^2\overline{\hat{\varphi}}_2(a\xi),$$

but the $\overline{\hat{\varphi}}_i$ are not necessarly positive real functions. Therefore, we must choose the $\varphi_i$ more carefully. Assume $\varphi_1$ to be radial (this choice is always possible). Then the auto-convolution $\varphi_1 * \varphi_1$ is a $C_0^\infty$ function whose Fourier transform takes real non-negative values. Moreover, since $g(0) > 0$, we still can find some constant $c > 0$ such that:

$$\frac{1}{c}\varphi_1 * \varphi_1(cx) \leq g(x).$$

On the other hand, we can majorize $|\hat{\varphi}_2|$ by some positive Schwarz function $\phi_2$. Thus, setting $\phi_1(x) = \frac{1}{c^2}\hat{\varphi}_1^2(x/c)$, we have the new bracketing:

$$\int d\xi\ |\hat{\mu}(\xi)|^2\phi_1(a\xi) \leq \Omega_g\mu(a) \leq \int d\xi\ |\hat{\mu}(\xi)|^2\phi_2(a\xi),$$

with real positive functions $\phi_1$ and $\phi_2$ and we can conclude the proof as before. □

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