Quasi linear flows on tori: regularity of their linearization

F. Bonetto•, G. Gallavotti*, G. Gentile†, V. Mastropietro⋆

• Matematica, Università di Roma, P.le Moro 2, 00185 Roma

* Fisica, Università di Roma, P.le Moro 2, 00185 Roma

† IHES, 35 Route de Chartres, 91440 Bures sur Yvette

⋆ Matematica, Università di Roma II, Via Ricerca Scientifica, 00133 Roma

ABSTRACT. *Under suitable conditions a flow on a torus $C^{(p)}$–close, with p large enough, to a quasi periodic diophantine rotation is shown to be conjugated to the quasi periodic rotation by a map that is analytic in the perturbation size. This result is parallel to Moser's theorem stating conjugability in class $C^{(p')}$ for some $p' < p$. The extra conditions restrict the class of perturbations that are allowed.*

1. Introduction

1.1. The perturbation of the Hamiltonian of a system of ℓ harmonic oscillators with frequencies $\frac{1}{2\pi}(\omega_{01},\ldots,\omega_{0\ell})$ is described by the Hamiltonian

$$\mathcal{H}_0(\mathbf{A},\boldsymbol{\alpha}) = \boldsymbol{\omega}_0\cdot\mathbf{A} + \varepsilon\mathbf{A}\cdot\mathbf{f}(\boldsymbol{\alpha}) + \varepsilon\,\mathbf{A}\cdot F(\mathbf{A},\boldsymbol{\alpha})\,\mathbf{A}\ , \tag{1.1}$$

where $\mathbf{A},\boldsymbol{\alpha}\in\mathbb{R}^\ell\times\mathbb{T}^\ell$ are the *action–angle* variables of the oscillators, $\cdot$ denotes the scalar product, $\mathbf{f}$ is a vector and F a matrix that describe the perturbation structure and ε is the *intensity* of the perturbation.

The Hamiltonian system (1.1) is not integrable in general (see for instance (4.10) in [G3]). Nevertheless, if the *unperturbed rotation vector* $\boldsymbol{\omega}_0$ of the oscillators verifies a *diophantine condition* and if the perturbation is analytic, it is possible to add to the Hamiltonian a suitable "*counterterm*" $\mathbf{A}\cdot\mathbf{N}_\varepsilon(\mathbf{A})$, analytic in the perturbative parameter ε and depending only on the action variables $\mathbf{A}$, so that the modified Hamiltonian $\mathcal{H}_0+\mathbf{A}\cdot\mathbf{N}_\varepsilon(\mathbf{A})$ is integrable.

This was conjectured in [G1] and proven in [E2], then in [GM2] with the techniques introduced in [G4,GM1], by exploiting the cancellation mechanisms operating order by order in the perturbative series defining the counterterm and the solution of the equations of motion for the modified Hamiltonian; a third method is in [EV].

Note that integrability of the problem (1.1) with $F=0$ is equivalent to the problem of linearizability of the flow on the torus generated by the differential equations

$$\frac{d\boldsymbol{\alpha}}{dt} = \boldsymbol{\omega}_0 + \varepsilon\mathbf{f}(\boldsymbol{\alpha})\ , \tag{1.2}$$

i.e. the problem of finding a change of coordinates, $\boldsymbol{\alpha}=\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi})$, on the torus $\mathbb{T}^\ell$ such that the equations (1.2) become the trivial quasi periodic linear flow $d\boldsymbol{\psi}/dt=\boldsymbol{\omega}_0$.

A previous partial result about the existence of the counterterm $N_\varepsilon(\mathbf{A})$ is in [DS,R,PF], where the one-dimensional Schrödinger equation with a quasi periodic potential is studied: the latter problem can be shown to be equivalent to the problem of the existence of a counterterm which makes integrable the Hamiltonian of a system of interacting oscillators, provided $F = 0$ and $\mathbf{f}$ has a very special form, (see [G2] and §5 below): restricted to this case, the proof of existence of a counterterm making integrable the Hamiltonian follows also from the analysis of [M1], (see Appendix A4 below).

But the "beginning" of the interest in the above problems goes back to a problem similar to (1.2) investigated by Moser in [M1]. There more general non linear ordinary differential equations are perturbatively studied under the hypothesis that the *characteristic number* (defining the linear part of the equations) verifies a *generalized diophantine condition* (see Appendix A4 for details): under the assumption that $\mathbf{f}$ is analytic it is proved that one can add to the equations counterterms depending (analytically) *only* on the perturbative parameter ε and that the equations so modified admit a solution analytic in ε for ε small.

The conjecture in [G1] envisages the possibility of introducing a counterterm depending (analytically) *also* on the actions in the Hamiltonian in such a way that the equations of motion admit an analytic solution: the two problems (the one studied by Moser and the one studied in [G2]) deal in general with different equations. Nevertheless, if the perturbation is linear in the action variables in (1.1), *i.e.* $F = 0$, then the counterterm $\mathbf{N}_\varepsilon(\mathbf{A})$ turns out to be $\mathbf{A}$–independent and analytic in ε if $\mathbf{f}$ is analytic on $\mathbb{T}^\ell$: in that case, as pointed out above, the existence and analyticity of $\mathbf{N}_\varepsilon$ is then implied by Moser's theorem [M1], Theorem 1. In [M1] it is also pointed out that the latter result is the "core" of the proof of the KAM theorem in the analytic case.

However Moser's theorem gives no analyticity result when the perturbation is non analytic. So, in this paper we consider (1.1) with $F = 0$, or, equivalently, (1.2) and $\varepsilon\mathbf{f}$ modified into $\varepsilon\mathbf{f} + \mathbf{N}_\varepsilon$, with the aim of proving the existence *and analyticity* of the counterterm $\mathbf{N}_\varepsilon$ if the perturbation $\mathbf{f}$ belongs to a class specified below (see §1.3), *i.e.* if it depends *non analytically* on the angles in a special way. The convergence of the perturbative series for the solution of the equations of motion and for the counterterm is obtained by taking into account some cancellations which include, besides the ones necessary to treat the analytic case as in [GM2] (the *infrared cancellations*), also new cancellations called *ultraviolet cancellations*).

The analyticity (of the solution of the equations of motion and of the counterterm) in the perturbative parameter ε *even though the interaction* $\mathbf{f}(\boldsymbol{\alpha})$ *is non analytic in* $\boldsymbol{\alpha}$ is a result that we believe would be difficult to obtain by other tecniques which one could use to face this problem, like the Moser-Nash smoothing tecnique, [M2].

The technical tools for the proof are inherited from [BGGM], where interaction potentials belonging to the same class of functions are considered to prove the analyticity in the perturbative parameter of the KAM invariant tori.

1.2. The paper is organized as follows. In the remaining part of this section we introduce the notations and state the result (Theorem 1.4). In §2 we reintroduce our diagrammatic formalism, referring for details to Appendix A1 (or [BGGM], §3, and simply outlining the differences with respect to it). In §3 the so called *infrared cancellations* are discussed: such cancellations allow us to solve the problem of small divisors, and they are sufficient to prove the convergence of the perturbative series in the analytic case. Since there are some differences with respect to [GM2], mostly the use of Siegel-Eliasson's lemma (see Lemma 3.4 below) instead of Siegel-Bryuno's lemma (see [GM2], Lemma 5.3), we provide a selfconsistent discussion although the problem is the same as the one treated in [GM2], §8. In §4, we study the *ultraviolet cancellations* which, together with the infrared ones, make convergent the perturbative series for the class of interactions introduced in §1.3. In §5, we note briefly that Theorem 1.4 for non analytic interactions does not give really more results than the corresponding theorem for the analytic case (see [GM2], Theorem 1.4), if applied to the one-dimensional Schrödinger equation with a non analytic quasi periodic potential; this is a little deceiving, but not quite unexpected (see comments in §5). As a byproduct of the proof, analyticity of eigenvalues and eigenfunctions in the perturbative parameter is proven under the assumption that the interaction potential is at least of class $C^{(p)}$, $p > 3\tau + 1$: this result is new with respect to [Pa].

Note that, with respect to [BGGM], the discussion of the infared cancellations appears remarkably less involved (and therefore more suitable for a first approach to the techniques employed). On the contrary the discussion of the ultraviolet cancellations is essentially unchanged with respect the one in [BGGM], notwithstanding the simplified expression of the Hamiltonian (1.3) below, and will be here repeated only for selfconsistence purposes.

1.3. The Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \mathbf{A} \cdot \mathbf{N}(\varepsilon) \tag{1.3}$$

with $\mathcal{H}_0 = \boldsymbol{\omega}_0 \cdot \mathbf{A} + \varepsilon \mathbf{A} \cdot \mathbf{f}(\boldsymbol{\alpha})$ given by (1.1), where
(1) $\mathbf{A} \in \mathbb{R}^\ell$ and $\boldsymbol{\alpha} \in \mathbb{T}^\ell$ are canonically conjugated variables (respectively action and angle variables), and $\cdot$ denotes the scalar product;
(2) $\boldsymbol{\omega}_0$ is a rotation vector satisfying the *diophantine condition*

$$C_0 |\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > |\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^\ell \,, \boldsymbol{\nu} \neq \mathbf{0} \,, \tag{1.4}$$

for some positive constants C_0 and τ (here and henceforth $|\boldsymbol{\nu}| = \sqrt{\boldsymbol{\nu} \cdot \boldsymbol{\nu}}$, while $\|\boldsymbol{\nu}\| = \sum_{j=1}^{\ell} |\nu_j|$, if $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_\ell)$);
(3) $\mathbf{f}$ has the form $\mathbf{f} = (f_1, \ldots, f_\ell)$, with each f_j of the class $\hat{C}^{(p)}(\mathbb{T}^\ell)$ introduced in [BGGM], for some p: namely $f_j(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}} f_{j\boldsymbol{\nu}}\, e^{i\boldsymbol{\nu} \cdot \boldsymbol{\alpha}}$, $\mathbf{f}_{\boldsymbol{\nu}} = \mathbf{f}_{-\boldsymbol{\nu}}$, with $f_{j\mathbf{0}} = 0$ and for $\boldsymbol{\nu} \neq \mathbf{0}$,

$$f_{j\boldsymbol{\nu}} = \sum_{n \geq p+\ell}^{N} \frac{c_n^{(j)} + d_n^{(j)} (-1)^{||\boldsymbol{\nu}||}}{|\boldsymbol{\nu}|^n} \,, \tag{1.5}$$

for some $N \geq p + \ell$ and some constants $c_n^{(j)}$, $d_n^{(j)}$; and
(4) $\mathbf{N}(\varepsilon)$, called a *counterterm*, and has to be fixed in order to make the equations of motion soluble for the model (1.3).
For instance we can choose $f_{j\boldsymbol{\nu}} = a_j |\boldsymbol{\nu}|^{-b}$, with $b = p + \ell$ and $a_j \in \mathbb{R}$; then define $\sup_{j=1,\ldots,\ell} |a_j| = a$ and $\mathbf{a} = (a_1, \ldots, a_\ell)$. In the following we shall deal explicitly with such a function: the proof can be trivially extended to the class of functions (1.5).

1.4. THEOREM. *Given the Hamiltonian (1.1), with $\boldsymbol{\omega}_0$ satisfying the diophantine condition (1.4) and $\mathbf{f} = (f_1, \ldots, f_\ell)$, with each $f_j \in \hat{C}^{(p)}(\mathbb{T}^\ell)$, there exist two positive constants ε_0 and $p_0 = 2 + 3\tau$, and a function $\mathbf{N}(\varepsilon)$ analytic in ε for $|\varepsilon| < \varepsilon_0$, such that the equations of motion corresponding to the Hamiltonian (1.3) admit solutions in $C^{(0)}(\mathbb{T}^\ell)$ analytic in ε for $|\varepsilon| < \varepsilon_0$.*

1.5. The equations of motion for the Hamiltonian (1.3) are given by

$$\begin{aligned} \frac{d\alpha_j}{dt} &= \boldsymbol{\omega}_{0j} + \varepsilon f_j(\boldsymbol{\alpha}) + N_j(\varepsilon) \,, \\ \frac{dA_j}{dt} &= -\varepsilon \mathbf{A} \cdot \partial_{\alpha_j} \mathbf{f}(\boldsymbol{\alpha}) \,. \end{aligned} \tag{1.6}$$

We look for solutions of the form

$$\begin{aligned}
&\boldsymbol{\alpha}(t)=\boldsymbol{\omega}_0 t+\mathbf{h}(\boldsymbol{\omega}_0 t)\ , \qquad \mathbf{h}(\boldsymbol{\psi})=\sum_{k=1}^{\infty}\sum_{\boldsymbol{\nu}\in\mathbb{Z}}\mathbf{h}^{(k)}_{\boldsymbol{\nu}}\,e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}},\varepsilon^k\ ,\\
&\mathbf{A}(t)=\mathbf{A}_0+\mathbf{H}(\boldsymbol{\omega}_0 t)\ , \qquad \mathbf{H}(\boldsymbol{\psi})=\sum_{k=1}^{\infty}\sum_{\boldsymbol{\nu}\in\mathbb{Z}}\mathbf{H}^{(k)}_{\boldsymbol{\nu}}\,e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\,\varepsilon^k\ ,
\end{aligned}\tag{1.7}$$

with $\mathbf{h}$ *odd* and $\mathbf{H}$ *even* in ψ so that the equations for $\mathbf{h}$ and $\mathbf{H}$ become

$$\begin{aligned}
(\boldsymbol{\omega}_0\cdot\partial_{\boldsymbol{\psi}})\,h_j(\boldsymbol{\psi})&=\varepsilon f_j(\boldsymbol{\psi}+\mathbf{h}(\boldsymbol{\psi}))+N_j\ ,\\
(\boldsymbol{\omega}_0\cdot\partial_{\boldsymbol{\psi}})\,H_j(\boldsymbol{\psi})&=-\varepsilon[\mathbf{A}_0+\mathbf{H}(\boldsymbol{\psi})]\cdot\partial_{\alpha_j}\mathbf{f}(\boldsymbol{\psi}+\mathbf{h}(\boldsymbol{\psi}))\ ,
\end{aligned}\tag{1.8}$$

where $\partial_{\boldsymbol{\alpha}}$ denotes derivative with respect to the argument, and $\mathbf{N}$ has to be so chosen that the right hand side of the first equation in (1.8) has vanishing average.

We see from (1.8) that the equation for $\mathbf{h}$ is closed, so that, as far as one is interested only in the function $\mathbf{h}$, one can confine ourself to study only the first equation in (1.8). This is the equation that one has to solve to linearize the flow generated by $d\boldsymbol{\alpha}/dt=\boldsymbol{\omega}_0+\varepsilon\mathbf{f}(\boldsymbol{\alpha})+\mathbf{N}_\varepsilon$: hence it is not surprising that the equation for $\mathbf{H}$ can be easily solved once $\mathbf{h}$ is known: see §2.3 below.

Note that, since $\mathbf{f}$ is supposed even, then we expect that $\mathbf{h}$ is odd and $\mathbf{H}$ is even: hence while the equation for $\mathbf{H}$ does not seem to hit any obvious compatibility problems we see that the equation for $\mathbf{h}$ does, *unless N_j is suitably chosen.* In fact the function $\varepsilon\mathbf{f}(\boldsymbol{\psi}+\mathbf{h}(\boldsymbol{\psi}))$, being even, has no *a priori* reason to have a vanishing integral over $\boldsymbol{\psi}$ (as it should, being equal to $(\boldsymbol{\omega}_0\cdot\partial_{\boldsymbol{\psi}})\mathbf{h}(\boldsymbol{\psi})$).

2. Formal solutions and graph representation

2.1. We study now the equations (1.8) with $f_{j\boldsymbol{\nu}}=a_j|\boldsymbol{\nu}|^{-b}$ replaced with $f_{j\boldsymbol{\nu}}\,e^{-\kappa|\boldsymbol{\nu}|}$. The parameter κ is taken $\kappa>0$, and, after computing the coefficients $\mathbf{h}^{(k)}_{\boldsymbol{\nu}}$ in (1.7), which will depend on κ, one will perform the limit $\kappa\to0$ (*Abel's summation*).

The formal solubility of (1.8) with $f_{j\boldsymbol{\nu}}$ replaced with $f_{j\boldsymbol{\nu}}\,e^{-\kappa|\boldsymbol{\nu}|}$ follows from [GM2], §8.1, where more general interaction potentials are considered.

One has $h^{(k)}_{j\mathbf{0}}=H^{(k)}_{j\mathbf{0}}=0\ \forall k\ge1$, while, when $\boldsymbol{\nu}\ne\mathbf{0}$, for $k=1$,

$$h^{(1)}_{j\boldsymbol{\nu}}=\frac{f_{j\boldsymbol{\nu}_0}}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\ ,\qquad H^{(1)}_{j\boldsymbol{\nu}}=-\frac{i\nu_{0j}}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\Big(\mathbf{A}_0\cdot\mathbf{f}_{\boldsymbol{\nu}_0}\Big)\ ,\tag{2.1}$$

and, for $k\ge2$,

$$\begin{aligned}
h^{(k)}_{j\boldsymbol{\nu}}=&\frac{1}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\sum_{p>0}\frac{1}{p!}\sum_{\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\boldsymbol{\nu}}f_{j\boldsymbol{\nu}_0}\sum_{k_1+\ldots+k_p=k-1}\prod_{s=1}^{p}\Big(i\boldsymbol{\nu}_0\cdot\mathbf{h}^{(k_s)}_{\boldsymbol{\nu}_s}\Big)\ ,\\
H^{(k)}_{j\boldsymbol{\nu}}=&-\frac{1}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\sum_{p>0}\frac{1}{p!}\sum_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\boldsymbol{\nu}}(i\nu_{0j})\,\cdot\\
&\cdot\sum_{\tilde{k}+k_1+\ldots+k_p=k-1}\Big(\mathbf{H}^{(\tilde{k})}_{\tilde{\boldsymbol{\nu}}}\cdot\mathbf{f}_{\boldsymbol{\nu}_0}\Big)\prod_{s=1}^{p}\Big(i\boldsymbol{\nu}_0\cdot\mathbf{h}^{(k_s)}_{\boldsymbol{\nu}_s}\Big)
\end{aligned}$$

$$- \frac{1}{i\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} \sum_{p>0} \frac{1}{p!} \sum_{\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\boldsymbol{\nu}} (i\nu_{0j}) \cdot \tag{2.2}$$

$$\cdot \sum_{k_1+\ldots+k_p=k-1} \left(\mathbf{A}_0 \cdot \mathbf{f}_{\boldsymbol{\nu}_0}\right) \prod_{s=1}^{p} \left(i\boldsymbol{\nu}_0 \cdot \mathbf{h}^{(k_s)}_{\boldsymbol{\nu}_s}\right) ,$$

provided $N_j(\varepsilon) = \sum_{k=1}^{\infty} N_j^{(k)} \varepsilon^k$, with $N_j^{(k)}$ defined by $N_j^{(1)} = -f_{j\boldsymbol{\nu}_0}$ and, for $k \geq 2$,

$$N_j^{(k)} = -\sum_{p>0} \frac{1}{p!} \sum_{\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\mathbf{0}} f_{j\boldsymbol{\nu}_0} \sum_{k_1+\ldots+k_p=k-1} \prod_{s=1}^{p} \left(i\boldsymbol{\nu}_0 \cdot \mathbf{h}^{(k_s)}_{\boldsymbol{\nu}_s}\right) . \tag{2.3}$$

The equality (2.3) assures the formal solubility of (1.8). Note that, as $\mathbf{f}$ is even, then $\mathbf{h}$ is odd and $\mathbf{H}$ is even.

If $\mathbf{f}$ is analytic ($\kappa > 0$) the convergence of the series defining the functions $\mathbf{h}$ and $\mathbf{H}$ is a corollary of [GM2], Theorem 1.4, but the convergence radius is not uniform in κ (it shrinks to zero when $\kappa \to 0$). The aim of the present paper is to show that, if $\mathbf{f}$ belongs to the class of functions $\hat{C}^{(p)}(\mathbb{T}^\ell)$, then there are cancellation mechanisms assuring the convergence of the series and so the analyticity in ε of the solutions of the equations of motion.

2.2. We shall use a representation of (2.2) in terms of *Feynman graphs* following the rules in [BGGM], §3: the reader not familiar with [BGGM] can find in Appendix A1 below a brief but selfconsistent description of the graphs. See [BGM] for the terminology motivation. The only difference will be that that the "value" of a graph ϑ is now given by

$$\mathrm{Val}(\vartheta) = \prod_{v<r} \frac{(i\boldsymbol{\nu}_{v'} \cdot \mathbf{f}_{\boldsymbol{\nu}_v})}{i\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_v}} , \tag{2.4}$$

where

(1) v' is the node immediately following v in ϑ, and $\lambda_v = v'v$ is the line emerging from v and entering v';

(2) r is the *root* of the graph, and $i\boldsymbol{\nu}_r$ denotes the unit vector in the jth direction, $i\boldsymbol{\nu}_r = \mathbf{e}_j$, $j = 1, \ldots, \ell$;

(3) $\boldsymbol{\nu}_v$ is the *external momentum* associated to the *node* v, $\boldsymbol{\nu}_{\lambda_v} = \sum_{w \leq v} \boldsymbol{\nu}_w$ is the *momentum* flowing through the line λ_v, and $\boldsymbol{\nu}(\vartheta)$ is the momentum flowing through the line entering the root.

It can be convenient to introduce the notations

$$g_\lambda \equiv \frac{1}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda} , \qquad D(\vartheta) = \prod_{\lambda \in \vartheta} g_\lambda , \tag{2.5}$$

so that (2.3) becomes

$$\mathrm{Val}(\vartheta) = D(\vartheta) \prod_{v<r} \left(\boldsymbol{\nu}_{v'} \cdot \mathbf{f}_{\boldsymbol{\nu}_v}\right) ; \tag{2.6}$$

g_λ is called the propagator of the line λ. Let us denote by $\mathcal{T}(k, \boldsymbol{\nu})$ the set of non equivalent labeled graphs of order k with $\boldsymbol{\nu}(\vartheta) = \boldsymbol{\nu}$ and $i\boldsymbol{\nu}_r = \mathbf{e}_j$; then

$$h^{(k)}_{j\boldsymbol{\nu}} = \frac{1}{k!} \sum_{\vartheta \in \mathcal{T}(k,\boldsymbol{\nu})} \mathrm{Val}(\vartheta) = \frac{1}{k!} \sum_{\vartheta^0 \in \mathcal{T}^0(k)} W(\vartheta^0, \boldsymbol{\nu}) , \tag{2.7}$$

where $\vartheta = (\vartheta^0, \{\boldsymbol{\nu}_\lambda\})$, if ϑ is a labeled graph, while ϑ^0 is a graph bearing no external momentum labels, $\mathcal{T}^0(k)$ is the set of such graphs of order k, and

$$W(\vartheta^0, \boldsymbol{\nu}) \equiv \sum_{\{\boldsymbol{\nu}_x\}:\, \boldsymbol{\nu}(\vartheta)=\boldsymbol{\nu}} \mathrm{Val}(\vartheta) \; . \tag{2.8}$$

2.3. The function $H^{(k)}_{j\boldsymbol{\nu}}$ can be expressed in terms of the same graphs as in §2.2, but the value associated to a graph ϑ is no more given by (2.3). On the contrary one define

$$\begin{aligned} \mathrm{Val}^*(\vartheta) =& D(\vartheta)\,(-i\nu_{0j}) \sum_{\tilde v\in\vartheta} \Big[\prod_{v\notin \mathcal{C}(v_0,\tilde v)} (\boldsymbol{\nu}_{v'}\cdot \mathbf{f}_{\boldsymbol{\nu}_v}) \Big] \cdot \\ & \cdot \Big[\prod_{v\in\mathcal{C}(v_0,\tilde v)\setminus\tilde v} (-\mathbf{f}_{\boldsymbol{\nu}_v}\cdot\boldsymbol{\nu}_{v''}) \Big] (\mathbf{f}_{\boldsymbol{\nu}_{\tilde v}}\cdot \mathbf{A}_0) \; , \end{aligned} \tag{2.9}$$

where
(1) v_0 is the highest node in ϑ, *i.e.* $v_0' = r$,
(2) $\mathcal{C}(v_0,\tilde v)$ is the collection of vertices crossed by the connected path of branches in ϑ linking the node v_0 with a node $\tilde v \le v_0$, with $\mathcal{C}(v_0,v_0) = v_0$,
(3) v'' is the node on $\mathcal{C}(v_0,\tilde v)$ immediately preceding v.
With the above definition (2.8), one has

$$H^{(k)}_{j\boldsymbol{\nu}} = \frac{1}{k!} \sum_{\vartheta\in\mathcal{T}(k,\boldsymbol{\nu})} \mathrm{Val}^*(\vartheta) \; , \tag{2.10}$$

where $\mathcal{T}(k,\boldsymbol{\nu})$ is defined as after (2.6).

3. Infrared cancellations

The cancellations discussed in this section are cancellations that are sufficient to treat the problem in the analytic case. Hence they are not really characteristic of the problems that we address in this paper. Nevertheless they must be taken into account and their compatibility with the cancellations that are typical of the differentiable problem will have to be, eventually, discussed.

3.1. Let us define $\chi(x)$ as the characteristic function of the set $\{x\in\mathbb{R} : |x| \in [1/2,1)\}$, and $\chi_1(x)$ as the characteristic function of the set $\{x\in\mathbb{R}, |x|>1\}$. Then each propagator in (2.4) can be decomposed as

$$g_\lambda = \frac{\chi_1(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda)}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda} + \sum_{n=-\infty}^{0} \frac{\chi(2^{-n}\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda)}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda} = \sum_{n=-\infty}^{1} g^{(n)}_\lambda \; , \tag{3.1}$$

and, inserting the above decompositions in the definition of the value of a graph (2.3), we see that the value of each graph is naturally decomposed into various addends. We can identify the addends simply by attaching to each line λ a scale label $n_\lambda \le 1$.

3.2. Definition (Cluster). *Given a graph ϑ, a cluster of scale $n\le 1$ is a maximal set of nodes connected by lines of scale $\ge n$. A line λ which connects nodes both inside a* *cluster T is said to be internal to the cluster: $\lambda \in T$; the lines which connect a node inside with a node outside the cluster are called external to the cluster; if a line λ is internal or external to a cluster T, we say that λ intersect T: $\lambda \cap T \neq \emptyset$. A line is outside the cluster T if it is neither internal nor external to it.*

The nodes of a cluster V of scale n_V may be linked to other nodes by lines of lower scale. Such lines are called *incoming* if they point at a node in the cluster or *outgoing* otherwise; there may be several incoming lines (or zero) but at most one outgoing line, because of the tree structure of the graphs.

3.3. DEFINITION (RESONANCE). *We define a resonance a cluster V such that:*
(1) there is only one incoming line λ_V and one outgoing line λ'_V and $|\boldsymbol{\nu}_{\lambda_V}| = |\boldsymbol{\nu}_{\lambda'_V}|$;
(2) if n_V is the scale of the cluster and n_{λ_V} is the scale of the line λ_V, one has $n_V \geq n_{\lambda_V} + 3$.
If $\boldsymbol{\nu}_{\lambda_V} = \boldsymbol{\nu}_{\lambda'_V}$, the resonance is called a real resonance, if $\boldsymbol{\nu}_{\lambda_V} = -\boldsymbol{\nu}_{\lambda'_V}$, it is called a virtual resonance.

Then the following result holds (see [S,E,BGGM]):

3.4. LEMMA (SIEGEL-ELIASSON'S BOUND). *If we consider only graphs with no real resonances, then*

$$\prod_{\lambda \in \vartheta} \frac{1}{|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda|} \leq C^k \, \frac{\prod_{v \in \vartheta} |\boldsymbol{\nu}_v|^{\frac{\eta}{2}\tau}}{(\sum_{v \in \vartheta} |\boldsymbol{\nu}_v|)^\tau} \, , \tag{3.2}$$

for some positive constant C and $\eta = 6$.

3.5. Consider a graph ϑ and call $\hat{\vartheta}$ the graph obtained by deleting the infrared scale labels $\{n_\lambda\}$ and ϑ^0 the graph obtained by deleting the scale and external momentum labels: $\vartheta = (\hat{\vartheta}, \{n_\lambda\})$, or $(\vartheta^0, \{n_\lambda\}, \{\boldsymbol{\nu}_x\})$.

Suppose that the set of scales $\{n_\lambda\}$ is consistent with the existence of a fixed family $\mathbf{V}_1$ of *maximal real resonances*: *i.e.* of real resonances not contained in any larger real resonance. If $V \in \mathbf{V}_1$ we call $\lambda_V = v_V^b v_V^1$ the line incoming into the real resonance and n_{λ_V} its scale; likewise $\lambda'_V = v_V^0 v_V^a$ is the outgoing line, ($v_V^a, v_V^b \in V$ while v_V^0, v_V^1 are out of it).

We consider the graph values at fixed set of scales for the lines not in any $V \in \mathbf{V}_1$, and we say that such a set of scales is "compatible" with $\mathbf{V}_1$, denoting this property by $\{n_\lambda\}\&\mathbf{V}_1$.

We introduce the *momentum flowing on $\lambda_v \in V$ intrinsic to the cluster V* as $\boldsymbol{\nu}^0_{\lambda_v} = \sum_{v \geq w \in V} \boldsymbol{\nu}_w$, and define the *resonance path* Q_V as the totally ordered path of lines joining the line incoming into the real resonance V and the outgoing line and *not* including the latter two lines. Then

$$\sum_{\{n_\lambda\}} \mathrm{Val}(\hat{\vartheta}, \{n_\lambda\}) = \sum_{\mathbf{V}_1} \sum_{\{n_\lambda\}\&\mathbf{V}_1} \cdot \Big[\prod_{\substack{\lambda \cap \mathbf{V}_1 = \emptyset \\ \lambda = xy}} (\boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y}) \, \frac{\chi(2^{-n_\lambda} \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda)}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda} \Big] \cdot \tag{3.3}$$

$$\cdot \prod_{V \in \mathbf{V}_1} \Big\{ \Big[(\boldsymbol{\nu}_{v_0^V} \cdot \mathbf{f}_{\boldsymbol{\nu}_{v_V^a}}) \, (\boldsymbol{\nu}_{v_b^V} \cdot \mathbf{f}_{\boldsymbol{\nu}_{v_V^1}}) \, \frac{\chi(2^{-n_\lambda} \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V})}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V})^2} \Big] \mathcal{V}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V} | V, \{n_\lambda\}_{\lambda \in V}) \Big\} \, ,$$

where the *resonance value* $\mathcal{V}$ is defined by

$$\mathcal{V}(\zeta | V, \{n_\lambda\}_{\lambda \in V}) = \prod_{\substack{\lambda \in V \\ \lambda = xy}} (\boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y}) \, \frac{\chi(2^{-n_\lambda}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}^0_\lambda + \sigma_\lambda \zeta))}{\boldsymbol{\omega} \cdot \boldsymbol{\nu}^0_\lambda + \sigma_\lambda \zeta} \, , \tag{3.4}$$

with $\sigma_\lambda = 1$ if λ is on the resonance path Q_V, ($\lambda \in Q_V$), else $\sigma_\lambda = 0$.

Let $\chi^{(n,n')}(x)$ denote the characteristic function of the set $|x| \in [2^{n-1}, 2^{n'})$; then (3.3) becomes

$$\sum_{\{\boldsymbol{\nu}_\lambda\}} \mathrm{Val}(\hat{\vartheta}, \{\boldsymbol{\nu}_\lambda\}) = \sum_{\mathbf{V}_1} \sum_{\{\boldsymbol{\nu}_\lambda\}_{\lambda \notin \mathbf{V}_1} \& \mathbf{V}_1} X_1(\hat{\vartheta}, \{\boldsymbol{\nu}_\lambda\}) ,$$

where

$$\begin{aligned} X_1(\hat{\vartheta},\{n_\lambda\}) = \sum_{\mathbf{V}_1} \sum_{\{n_\lambda\}_{\lambda\in\mathbf{V}_1}\&\mathbf{V}_1} \cdot \Big[\prod_{\substack{\lambda \cap \mathbf{V}_1 = \emptyset \\ \lambda = xy}} (\boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y}) \frac{\chi(2^{-n_\lambda} \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda)}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda} \Big] \cdot \\ \cdot \prod_{V \in \mathbf{V}_1} \Big\{ \Big[(\boldsymbol{\nu}_{v_V^0} \cdot \mathbf{f}_{\boldsymbol{\nu}_{v_V^a}}) (\boldsymbol{\nu}_{v_V^b} \cdot \mathbf{f}_{\boldsymbol{\nu}_{v_V^1}}) \frac{\chi(2^{-n_\lambda} \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V})}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V})^2} \Big] \mathcal{V}'(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\lambda_V} | V, \{n_\lambda\}_{\lambda \in V}) \Big\} , \end{aligned} \tag{3.5}$$

with

$$\mathcal{V}'(\zeta | V, \{n_\lambda\}_{\lambda\in V}) = \prod_{\substack{\lambda \in V \\ \lambda = xy}} (\boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y}) \frac{\chi^{(n_{\lambda_V}+3,+\infty)}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\lambda^0 + \sigma_\lambda \zeta)}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\lambda^0 + \sigma_\lambda \zeta)^2} , \tag{3.6}$$

which can be rewritten, by using the Lagrange's interpolation formula,

$$\mathcal{V}'(\zeta | V, \{n_\lambda\}_{\lambda\in V}) = \mathcal{V}'(0 | V, \{n_\lambda\}_{\lambda\in V}) + \mathcal{V}_1'(\zeta | V, \{n_\lambda\}_{\lambda\in V}) ,$$

where

$$\begin{aligned} \mathcal{V}_1'(\zeta | V, \{n_\lambda\}_{\lambda\in V}) = \Big(\prod_{\substack{\lambda \in V/Q_V \\ \lambda = xy}} (\boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y}) \frac{\chi^{(n_{\lambda_V}+3,+\infty)}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda^0)}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda^0} \Big) \cdot \\ \cdot \zeta \int_0^1 dt_V \, \frac{\partial}{\partial t_V} \Big[\prod_{\substack{\lambda \in Q_V \\ \lambda = xy}} (\boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y}) \frac{\chi^{(n_{\lambda_V}+3,+\infty)}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda^0 + t_V \zeta)}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda^0 + t_V \zeta} \Big] . \end{aligned} \tag{3.7}$$

We can consider the value X_1 with the real resonance value corresponding to $V \in \mathbf{V}_1$ simply replaced by the expression defined in (3.7); this follows from the following result.

3.6 Lemma. *When all graphs in (3.3), with the real resonance value $\mathcal{V}'(\zeta | V, \{n_\lambda\}_{\lambda\in V})$ replaced with $\mathcal{V}'(0 | V, \{n_\lambda\}_{\lambda\in V})$, are summed together, they give a vanishing contribution.*

The proof is in Appendix A2.

3.7. We can perform explicitly the derivative in (3.7): we obtain (see also Remark after (5.6) in [BGGM])

$$\begin{aligned} X_1(\hat{\vartheta}, \{\boldsymbol{\nu}_\lambda\}) = \Big(\prod_{\substack{\lambda \in \hat{\vartheta} \\ \lambda = xy}} \boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y} \Big) \cdot \Big(\prod_{\lambda \in \hat{\vartheta}/\mathbf{V}_1} \frac{\chi(2^{-n_\lambda}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda))}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda} \Big) \cdot \\ \cdot \sum_{\lambda_V^0 \in Q_V} \sum_{z=0}^{1} \int_0^1 dt_V \, p(\lambda_V^0, z, t_V) \cdot \\ \cdot \prod_{V \in \mathbf{V}_1} \Big[\prod_{\lambda \in V} \frac{\chi^{(n_{\lambda_V}+3,+\infty)}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda(t_V))}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda(t_V)} \Big] , \end{aligned} \tag{3.8}$$

where, for $\lambda\in V$, we adopt the notation $\boldsymbol{\nu}_\lambda(t_V)=\boldsymbol{\nu}^0_\lambda+t_V\boldsymbol{\nu}_{\lambda_V}$, and

$$p(\lambda^0_V,z,t_V)=\begin{cases}-\frac{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_V}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda^0_V}(t_V)}\;, & \text{if } \; z=1\;,\\ \sum_{t^*_V}\delta(t_V-t^*_V)\;, & \text{if } z=0\;,\end{cases}\tag{3.9}$$

where t^*_V are the solutions to the equation $|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda^0_V}(t_V)|=2^{n_{\lambda_V}+2}$, if any (there are at most 2 solutions).

We then redecompose in (3.8) the characteristic functions of the lines inside the real resonances into individual scales from $n_{\lambda_V}+3$ up, so that (3.5) becomes

$$\begin{aligned}X_1(\hat\vartheta,\{\boldsymbol{\nu}_\lambda\})=&\sum_{\{n_\lambda\}_{\lambda\in\mathbf{V}_1}\&\mathbf{V}_1}\Big[\Big(\prod_{V\in\mathbf{V}_1}\int_0^1 dt_V\Big)\cdot\prod_{\substack{\lambda\in\hat\vartheta,\,\lambda=xy\\ \lambda\notin\{\lambda'_V\}_{V\in\mathbf{V}_1}}}\Big(\frac{\chi(2^{-n_\lambda}\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t}))}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t})}\Big)\cdot\\ &\cdot\Big(\prod_{\substack{\lambda\in\hat\vartheta\\ \lambda=xy}}\boldsymbol{\nu}_x\cdot\mathbf{f}_{\boldsymbol{\nu}_y}\Big)\cdot\Big(\prod_{V\in\mathbf{V}_1}\sum_{\lambda^0_V\in Q_V}\sum_{z_V=0}^{1}\frac{d^{z_V}_{\lambda^0_V}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda^0_V}(\mathbf{t})}\Big)\Big]\;,\end{aligned}\tag{3.10}$$

where $\mathbf{t}=\{t_V\}_{V\in\mathbf{V}_1}$ and we set $\boldsymbol{\nu}_\lambda(\mathbf{t})=\boldsymbol{\nu}^0_\lambda+t_V\boldsymbol{\nu}_{\lambda_V}$ if $\lambda\in Q_V$, and $\boldsymbol{\nu}_\lambda(\mathbf{t})=\boldsymbol{\nu}^0_\lambda\equiv\boldsymbol{\nu}_\lambda$ if $\lambda\notin\cup_{V\in V_1}Q_V$, and

$$d^1_{\lambda^0_V}=-1\;,\qquad d^0_{\lambda^0_V}=\frac{\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\lambda^0_V}(\mathbf{t})}{\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\lambda_V}}\sum_{t^*_V}\delta(t_V-t^*_V)\;,\tag{3.11}$$

where t^*_V is defined as in (3.9).

Each addend in (3.10), with fixed $\vartheta=(\hat\vartheta,\{n_\lambda\})$ and $\{\lambda^0_V,z_V,t_V\}_{V\in\mathbf{V}_1}$, is said to be *superficially renormalized* on the real resonances $\mathbf{V}_1$, on the line λ^0_V and on the choices z_V.

3.8. Remark. Note that the case $z_V=0$ is special as it forces $n_{\lambda^0_V}=n_{\lambda_V}+3$, so that the ratio in the definition (3.11) of $d^0_{\lambda^0_V}$ is bounded above by 2^4.

3.9. Having dealt with the maximal real resonances (*first generation* real resonances) we perform again the same operations: *i.e.* fixed $\hat\vartheta$, $\mathbf{V}_1,\{\lambda^0_V,\,z_V,t_V\}_{V\in\mathbf{V}_1}$ and the scales $\{n_\lambda\}$ for $\lambda\notin\cup_{V\in\mathbf{V}_1}V$, we identify the *second generation* real resonances as the maximal real resonances inside each $V\in\mathbf{V}_1$; call $\mathbf{V}_2$ the set of the real resonances of the first *and* second generations and proceed in a similar way to "renormalize" superficially the newly considered real resonances $W\in\mathbf{V}_2/\mathbf{V}_1$.

This means that we fix the scale labels of the lines outside the two generations of real resonances, and sum over the other scale labels $\{n_\lambda\}$ consistent with the elements of $\mathbf{V}_2$ being the first and second generation resonances.

We obtain that the product in (3.10) of the terms coming from the lines $\lambda\in W\in\mathbf{V}_2$, $W\subset V\in\mathbf{V}_1$ can be written in a form very close to (3.6), with the difference that the momenta flowing through the lines $\lambda\in Q_W\cap Q_V$ are $\boldsymbol{\nu}_\lambda(\mathbf{t})=\boldsymbol{\nu}^0_\lambda+t_W\,(\boldsymbol{\nu}^0_{\lambda_W}+t_V\boldsymbol{\nu}_{\lambda_V})$, and $n_{\lambda_V}+3$ is replaced by $n_{\lambda_W}+3$ in the characteristic functions.

We proceed to perform a Taylor expansion as above, in the variables $\zeta_W=\boldsymbol{\omega}_0\cdot(\boldsymbol{\nu}^0_{\lambda_W}+t_V\boldsymbol{\nu}_{\lambda_V})$ if $\lambda_W\in Q_V$ or $\zeta_W=\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}^0_{\lambda_W}$ otherwise. However this time we modify the renormalization procedure: if W contains the line λ^0_V we do nothing, while if $\lambda^0_V\notin W$ we write the first order remainder as above.

We then perform the derivatives with respect to the new interpolation parameters t_W, generated by the expression of the Taylor series remainders, hence redevelop the characteristic functions and rearrange, along the lines that generated (3.10), the various terms to simplify the notation and to get an expression very similar to (3.10), for a quantity that we could call X_2. Note that the order zero term of each Taylor expansion can be neglected, because of Lemma 3.5, which still hold if V is not a maximal resonance, but is contained in another resonance. The latter X_2 can subsequently be used, in the same way as the X_1 was already used to start the second renormalizations, for the superficial renormalization of the third generation of real resonances. Then we iterate step by step the procedure until there are no more real resonances inside the maximal real resonances found at the last step performed and all the n_λ have been fixed.

To write down the final expression we need some notations.
(1) Let us call $\mathbf{V}$ the collection of all real resonances selected along the iterative procedure. For each $V \in \mathbf{V}$ choose a line λ_V with the *compatibility condition* that if $V \subset Z \in \mathbf{V}$, $\lambda_Z^0 \in V$ implies $\lambda_V^0 = \lambda_Z^0$.
(2) Then if $\lambda_Z^0 \notin V$ we say that the line λ_V^0 is *new* and that the real resonance V is *new*, and define

$$\begin{aligned} &\pi_V(dt_V) = dt_V \;, \\ &d^1_{\lambda_V^0} = -1 \;, \qquad d^0_{\lambda_V^0} = \tfrac{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V^0}(\mathbf{t})}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V}} \textstyle\sum_{t_V^*} \delta(t_V - t_V^*) \;, \end{aligned} \tag{3.12}$$

where t_V^* are the solutions (at most 2) of the equation $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_V^0(\mathbf{t})| = 2^{n_{\lambda_V}} + 2$ for t_V, and the interpolated momenta $\boldsymbol{\nu}_\lambda(\mathbf{t})$ are defined as

$$\boldsymbol{\nu}_\lambda(\mathbf{t}) = \begin{cases} \boldsymbol{\nu}_\lambda \;, & \text{if } \lambda \text{ is not contained in any resonance paths,} \\ \boldsymbol{\nu}_\lambda \prod_{V\,:\lambda \in Q_V} t_V \;, & \text{otherwise} \;. \end{cases} \tag{3.13}$$

(3) If $\lambda_Z^0 \in V$ (so that $\lambda_Z^0 = \lambda_V^0$) we say that λ_V^0 and V are *old*, and define

$$\begin{aligned} &\pi_V(dt_V) = \delta(t_V - 1)\, dt_V \;, \\ &d^1_{\lambda_V^0} = 1 \;, \qquad d^0_{\lambda_V^0} = 0 \;. \end{aligned} \tag{3.14}$$

(4) Define

$$P_0(\vartheta) = \prod_{\lambda \notin \cup_V \lambda_V'} \frac{\chi(2^{-n_\lambda} \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda(\mathbf{t}))}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda(\mathbf{t})}, \qquad N(\vartheta) = \prod_{\substack{\lambda \in \vartheta \\ \lambda = xy}} \boldsymbol{\nu}_x \cdot \mathbf{f}_{\boldsymbol{\nu}_y} \;, \tag{3.15}$$

and denote by Λ the function $V \to \{\lambda_V^0, z_V\}$.
(5) Define

$$\mathcal{R}\,\mathrm{Val}(\vartheta) = N(\vartheta)\, \mathcal{R}D(\vartheta) \;, \tag{3.16}$$

where

$$\mathcal{R}D(\vartheta) = \sum_\Lambda \prod_{V \in \mathbf{V}} \int_0^1 \pi(dt_V)\, P_0(\vartheta) \prod_{V \in \mathbf{V}} \frac{d^{z_V}_{\lambda_V^0}}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V^0}(\mathbf{t}))^*} \;, \tag{3.17}$$

and the * means that $(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V^0}(\mathbf{t}))^* = \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V^0}(\mathbf{t})$ if the resonance is new, and $(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V^0}(\mathbf{t}))^* = \boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\lambda_V}$ if the resonance is old.

3.10. REMARK. By Definition 3.3, we have $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda(\mathbf{t})| \geq \frac{2}{3}|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda^0|$, uniformly in $\mathbf{t}$.

3.11. Then, from the iterative procedure and with the just introduced notations, we obtain

$$\sum_{\vartheta} \mathrm{Val}(\vartheta) = \sum_{\vartheta} \mathcal{R}\,\mathrm{Val}(\vartheta)\ , \tag{3.18}$$

as all terms discarded in each Taylor expansion add to zero when summed together (as a corollary of Lemma 3.6).

The number of terms thus generated is, at fixed $\mathbf{V}$, bounded by the product over $V\in\mathbf{V}$ of 2 times the number of pairs that are in $V/\cup_{W\subset V,W\in\mathbf{V}}W$ and therefore it is bounded by $2^k\prod_V k(V)^2$ if $k(V)$ is the number of nodes in V which are not in real resonances inside V. Hence this number is $\le (2^4)^k$. The number of families of real resonances in $\hat\vartheta$ (*hence at fixed* $\{\boldsymbol{\nu}_x\}$) is also bounded by 2^k.

3.12. After applying the $\mathcal{R}$ operations, we see that the contribution to the new "renormalized value" from the divisors in (3.10) will be bounded by the same product appearing in the non renormalized values of the graphs *deprived of the divisors due to the lines exiting resonances* times a factor

$$\begin{aligned} 2^{4k}\prod_{V\subset\vartheta}\frac{1}{\min_{\lambda\in V_0}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t})|} &\le C_1^k\prod_{V\subset\vartheta}\frac{1}{\min_{\lambda\in V_0}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}^0_\lambda|}\le\\ &\le C_1^k\prod_{V\subset\vartheta}\Big[\sum_{v\in V_0}|\boldsymbol{\nu}_v|\Big]^\tau, \end{aligned}\tag{3.19}$$

where the factor 2^{4k} arises from Remark 3.8 and $C_1=3/2$ from Remark 3.10, and V_0 is the set of nodes inside the real resonance V not contained in the real resonances internal to V.

Then, we can identify the real resonances $V\in\mathbf{V}$ of different generations. The set $\mathbf{V}_j$ of real resonances of the jth generation, $j\ge1$, just consists of the real resonances which are contained in $(j-1)$th generation real resonances (of lower scale) but not in any $(j+1)$th generation real resonances.

If V is a real resonance in $\mathbf{V}_j$ with entering line $v^b_Vv^1_V$ and outgoing line $v^a_Vv^1_V$ with momentum $\boldsymbol{\nu}_{\lambda_V}$ we can construct a "V-contracted graph" by replacing the cluster V together with the incoming and outgoing lines by the single line $v^0_Vv^1_V$: *i.e.* by deleting the resonance V and replacing it by a line. We can also construct the "V-cut graphs" by deleting everything but the lines of the resonance V and its entering and outgoing lines and, furthermore, by deleting the outgoing line as well as the node v^V_a and attributing to the node v^1_V an external momentum equal to the momentum flowing into the entering line in the original graph ϑ: thus we get $p_{v^a_V}$ disconnected graphs.

We repeat the above two operations until we are left only with graphs ϑ_i, $i=1,2,\ldots$ without real resonances: by construction the product $\prod_{\lambda\in\vartheta}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t})|^{-1}$ is the same as the $\prod_i\prod_{\lambda\in\vartheta_i}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t})|^{-1}$.

Then we imagine to delete as well the lines of the various ϑ_j which were generated by the old entering lines (not all ϑ_i contain such lines, but some do) and we call ϑ^0_i the graphs so obtained. By doing so we change the momenta flowing into the lines of the graphs ϑ_i by an amount which is either $\mathbf{0}$ or the old momentum $\boldsymbol{\nu}_{\lambda_V}(\mathbf{t})$ entering a real resonance V, and from Remark 3.10, we have

$$\prod_{\lambda\in\hat\vartheta}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t})|^{-1}\le C_1^k\prod_i\prod_{\lambda\in\vartheta^0_i}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}^0_\lambda|^{-1}\prod_{\lambda\in\{\lambda'_V\}_{V\in\mathbf{V}}}|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda(\mathbf{t})|^{-1}\ , \tag{3.20}$$

where the last product can be bounded by (3.18), while Lemma 3.4 gives

$$\prod_i \prod_{\lambda\in\vartheta_i^0} |\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\lambda^0|^{-1} \le C^k \prod_i \frac{\prod_{v\in\vartheta_i^0}|\boldsymbol{\nu}_v|^{\frac{\eta}{2}\tau}}{(\sum_{v\in\vartheta_i^0}|\boldsymbol{\nu}_v|)^\tau}, \tag{3.21}$$

with $\eta=6$. Then (3.17)÷(3.21) imply

$$\Big|\mathcal{R}D(\vartheta)\Big| \le C^{2k}C_1^{2k}\prod_{v\in\vartheta}|\boldsymbol{\nu}_v|^{\frac{\eta}{2}\tau}\ , \qquad \eta=6\ . \tag{3.22}$$

.

4. Ultraviolet cancellations

The ultraviolet cancellations are characteristic of the linearization problems relative to (1.1), (1.2). They are quite different from the infrared ones discussed in §3 and the main technical problem, besides their identification, is their compatibility with the infrared cancellations. Exhibiting the two cancellations may not be possible simultaneously in the sense that the first cancellations may require grouping graphs in classes that are completely different from the groupings that are necessary to exhibit the second cancellations. If this happens one says that the cancellations are not independent and it is clear that one runs into serious problems.

Hence the analysis that follows will be mostly devoted to showing that, besides an obvious incompatibility that can be explicitly resolved, the two cancellations are in fact independent.

4.1. Given a graph ϑ, we can define the *scale* h_v of the node v to be the integer $h_v\ge 1$ such that $2^{h_v-1}\le|\boldsymbol{\nu}_v|<2^{h_v}$. We say that the labels $\{\boldsymbol{\nu}_x\}$ and $\{h_x\}$ are *compatible* if $|\boldsymbol{\nu}_v|\in[2^{h_v-1},2^{h_v})$ for all $v\in\vartheta$. The compatibility relationship between $\{\boldsymbol{\nu}_x\}$ and $\{h_x\}$ will be denoted $\{\boldsymbol{\nu}_x\}\,\mathrm{comp}\,\{h_x\}$.

Then we can write

$$\sum_\vartheta \mathcal{R}\mathrm{Val}(\vartheta) = \sum_{\vartheta^0}\sum_{\{\boldsymbol{\nu}_x\}}\mathcal{R}\mathrm{Val}(\vartheta^0,\{\boldsymbol{\nu}_x\}) = \sum_{\vartheta^0}\sum_{\{h_x\}}\sum_{\{\boldsymbol{\nu}_x\}\,\mathrm{comp}\,\{h_x\}}\mathcal{R}\mathrm{Val}(\vartheta^0,\{\boldsymbol{\nu}_x\})\ . \tag{4.1}$$

Set $\mathcal{Q}=\cup_{V\in\mathbf{V}}Q_V$, where $\mathbf{V}$ is the set of all resonances of ϑ and Q_V is the resonance path of the resonance V (see §3.5). Define $\mathcal{B}_v$ the subset of the nodes w among the p_v nodes immediately preceding v such that the branch vw is not on the resonance paths $\mathcal{Q}$.

Given a set of momenta and fixed a node $\bar v\in\vartheta^0$, we define the change of variables $U_{\bar v w}^{\sigma_w}:\mathbb{Z}^\ell\longleftrightarrow\mathbb{Z}^\ell$, where $w\in\mathcal{B}_{\bar v}$, by fixing a sign $\sigma_w=\pm1$ and defining $U_{\bar v w}^{\sigma_w}(\{\boldsymbol{\nu}_x\})=\{\boldsymbol{\nu}'_x\}$ as:

$$\begin{aligned}\boldsymbol{\nu}'_z =&\sigma_w\boldsymbol{\nu}_z, \qquad z\ge w\ ,\\ \boldsymbol{\nu}'_z =&\boldsymbol{\nu}_z, \qquad \text{for all other } z\ne\bar v\ ,\\ \boldsymbol{\nu}'_{\bar v} =&\boldsymbol{\nu}_{\bar v}+(1-\sigma_w)\sum_{z\le w}\boldsymbol{\nu}_z\equiv\boldsymbol{\nu}_{\bar v}+(1-\sigma_w)\boldsymbol{\nu}_{\lambda_w}\ ,\end{aligned} \tag{4.2}$$

so that, for any choice of the subset $\mathcal{B}_{1\bar{v}} \subseteq \mathcal{B}_{\bar{v}}$ of nodes immediately preceding $\bar{v}$, there are cancellations which allow us to write

$$\begin{aligned}
&\sum_{\{\sigma_w\}_{w\in\mathcal{B}_{1\bar{v}}}} \mathcal{R}\mathrm{Val}(\vartheta^0, \prod_{w\in\mathcal{B}_{1\bar{v}}} U^{\sigma_w}_{\bar{v}w}\{\boldsymbol{\nu}_x\}) \equiv \\
&\qquad \equiv \Big(\prod_{w\in\mathcal{B}_{1\bar{v}}} \int_1^0 dt_w\Big) \sum_{||\mathbf{m}_{\bar{v}}||=p_{\bar{v}}} \Big(\prod_{w\in\mathcal{B}_{1\bar{v}}} \frac{\partial}{\partial t_w}\Big)\Big(f_{j\boldsymbol{\nu}_{\bar{v}}(\mathbf{t}_{\bar{v}})}\,(\boldsymbol{\nu}_{\bar{v}}(\mathbf{t}_{\bar{v}}))^{\boldsymbol{\mu}_{\bar{v}}}\Big)\cdot \\
&\qquad \cdot \mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_{\bar{v}}}}\mathrm{Val}'(\vartheta^0)\Big\}\ ,
\end{aligned} \tag{4.3}$$

where:
(i) $\mathrm{Val}'(\vartheta^0)$ is a tensor containing all the other value factors relative to nodes v's different from $\bar{v}$.
(ii) The free indices of the $p_{\bar{v}}$-order tensor $\boldsymbol{\nu}_{\bar{v}}(\mathbf{t}_{\bar{v}})^{\mathbf{m}_{\bar{v}}}$ are contracted (by performing the $\sum_{\mathbf{m}_{\bar{v}}}$) with the ones that appear in the tensor $\mathrm{Val}'(\vartheta^0)$, and $\mathbf{m}_{\bar{v}}$ is a ℓ dimensional positive integer components vector (with $||\mathbf{m}_{\bar{v}}||$ denoting the sum of the components) and, given a vector $\mathbf{b}$, we put $\mathbf{b}^{\mathbf{m}_{\bar{v}}} = b_1^{m_{\bar{v}1}}\ldots b_\ell^{m_{\bar{v}\ell}}$; furthermore $\mathbf{t}_{\bar{v}} = (t_{w_1},\ldots,t_{w_{|\mathcal{B}_{1\bar{v}}|}})$ and $\boldsymbol{\nu}_{\bar{v}}(t_{w_1},\ldots,t_{w_{|\mathcal{B}_{1\bar{v}}|}}) \equiv \boldsymbol{\nu}_{\bar{v}}(\mathbf{t}_{\bar{v}})$ is defined as

$$\boldsymbol{\nu}_{\bar{v}}(\mathbf{t}_{\bar{v}}) = \boldsymbol{\nu}_{\bar{v}}(t_{w_1},\ldots,t_{w_{|\mathcal{B}_{1\bar{v}}|}}) = \boldsymbol{\nu}_{\bar{v}} + \sum_{w\in\mathcal{B}_{1\bar{v}}} 2t_w\,\boldsymbol{\nu}_{\lambda_w} = \boldsymbol{\nu}_{\bar{v}} + \sum_{w\in\mathcal{B}_{1\bar{v}}} t_w\,\Big(\sum_{z\le w} 2\boldsymbol{\nu}_z\Big)\ , \tag{4.4}$$

where $\boldsymbol{\nu}_{\bar{v}}(\mathbf{t}_{\bar{v}}) = \boldsymbol{\nu}_{\bar{v}}$ if $\mathcal{B}_{1\bar{v}} = \emptyset$.
(iii) The assumed form (1.5) of the $f_{\boldsymbol{\nu}}$ allows us to think that $f_{\boldsymbol{\nu}}$ is defined on $\mathbb{R}^\ell$ rather than on $\mathbb{Z}^\ell$ and hence to give a meaning to the derivatives of $f_{\boldsymbol{\nu}_v(\mathbf{t}_v)}$.
(iv) We use here and henceforth that $\mathcal{R}$ acts *only* on the product of propagators $D(\vartheta)$ (see (2.5) and (3.16)), and the fact that the definition of $\mathcal{B}_v$ after (4.1) yields that all real resonances remains such under the action of the change of variables (4.2).

4.2. Remark. The cancellations are expressed by the fact that the change of variables (4.2) leaves unchanged each factor $(\boldsymbol{\nu}_{v'}\cdot\mathbf{f}_{\boldsymbol{\nu}_v})[\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_v}]^{-1}$, except for the nodes w and $\bar{v}$, whose factors are modified in the following way:

$$\begin{aligned}
\frac{\boldsymbol{\nu}_{\bar{v}}\cdot\mathbf{f}_{\boldsymbol{\nu}_w}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_w}} &\quad\to\quad -\frac{(\boldsymbol{\nu}_{\bar{v}}+\boldsymbol{\zeta}_w)\cdot\mathbf{f}_{\boldsymbol{\nu}_w}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_w}}\ ,\\
\frac{\boldsymbol{\nu}_{\bar{v}'}\cdot\mathbf{f}_{\boldsymbol{\nu}_{\bar{v}}}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_{\bar{v}}}} &\quad\to\quad \frac{(\boldsymbol{\nu}_{\bar{v}'})\cdot\mathbf{f}_{\boldsymbol{\nu}_{\bar{v}}+\boldsymbol{\zeta}_w}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_{\bar{v}}}}\ ,
\end{aligned}$$

with $\boldsymbol{\zeta}_w = 2\boldsymbol{\nu}_{\lambda_w}$: then, if we set $\boldsymbol{\zeta}_w = \mathbf{0}$, the sum of the two graph values cancel exactly.

4.3. We can study the sum $S_k(\vartheta^0) = \sum_{\boldsymbol{\nu}} |\boldsymbol{\nu}|^s|\mathcal{R}W(\vartheta^0,\boldsymbol{\nu})|$, where $\mathcal{R}W(\vartheta^0,\boldsymbol{\nu})$ is defined as in (2.6) with $\mathrm{Val}(\vartheta)$ replaced with $\mathcal{R}\mathrm{Val}(\vartheta)$.[1] We have

$$\begin{aligned}
S_k(\vartheta^0) &= \sum_{\boldsymbol{\nu}} |\boldsymbol{\nu}|^s\Big|\mathcal{R}W(\vartheta^0,\boldsymbol{\nu})\Big|\\
&= \sum_{\boldsymbol{\nu}} |\boldsymbol{\nu}|^s\Big|f_{j\boldsymbol{\nu}_{v_1}}\,(\boldsymbol{\nu}_{v_1})^{\boldsymbol{\mu}_{v_1}}\,\mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}}\Big[\sum_{\boldsymbol{\nu}_{\lambda_w}} W(\vartheta^0_{v_1w},\boldsymbol{\nu}_{\lambda_w})\Big]\Big\}\Big|\ ,
\end{aligned} \tag{4.5}$$

[1] The reader familiar with [BGGM] can skip the following discussion, which is essentially identical to the one in [BGGM], §4, and leap directly to the final expression (4.27) in §4.7.

where v_1 is the highest node and $\boldsymbol{\nu}_{v_1} = \boldsymbol{\nu} - \sum_{w\in\mathcal{B}_{v_1}} \boldsymbol{\nu}_{\lambda_w}$.

Fixed $\boldsymbol{\nu}$ and $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ let $h_{v_1} = h_{v_1}(\boldsymbol{\nu}, \{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}})$ be *the scale of* $\boldsymbol{\nu}_{v_1}$: *i.e.* $\boldsymbol{\nu}_{v_1}$ is such that $2^{h_{v_1}-1} \le |\boldsymbol{\nu}_{v_1}| < 2^{h_{v_1}}$. Given w with $w' = v_1$ we say that w is *out of order* with respect to v if

$$2^{h_{v_1}} > 2^o p_{v_1} |\boldsymbol{\nu}_{\lambda_w}| \ , \qquad o = 5 \ , \tag{4.6}$$

where p_v is the number of branches entering v. We denote $\mathcal{B}_{1v_1} \equiv \mathcal{B}_{1v_1}(\boldsymbol{\nu}, \{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}) \subseteq \mathcal{B}_{v_1}$ the nodes $w \in \mathcal{B}_{v_1}$ which are out of order with respect to v_1. The number of elements in $\mathcal{B}_{1v_1}$ will be denoted $q_v = |\mathcal{B}_{1v_1}|$. The notion of w being out of order with respect to v_1 depends on $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{v_1}}$ and $\boldsymbol{\nu}$.

Given a set $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{v_1}}$ for all choices of $\sigma_w = \pm 1$ we define the transformation

$$U(\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{v_1}}) \equiv \{\sigma_w \boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{v_1}} \ , \tag{4.7}$$

and given a set $C \subseteq \mathcal{B}_{v_1}$ we call $\mathcal{U}(C)$ the set of all transformations U such that $\sigma_w = 1$ for $w \notin C$.

If $[2^{h-1}, 2^h)$ is a scale interval I_h, $h = 1, 2, \dots$ we call

- the first quarter of I_h the *lower part* $I_h^- = [2^{h-1}, \frac{5}{4}2^{h-1})$ of I_h,
- the fourth quarter of I_h the *upper part* $I_h^+ = [\frac{7}{8}2^h, 2^h)$ of I_h, and
- the remaining part the *central part* I_h^c.

We group the set of branch momenta $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ into collections by proceeding iteratively in the way described below. The collections will be built so that in each collection the cancellation discussed in Remark 4.2 above can be exhibited.

Fixed $\boldsymbol{\nu}$ and h choose $\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ such that $|\boldsymbol{\nu}^1_{v_1}| \in I_h^c$: such $\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ is called a *representative*. Given the representative we define

- the *branch momenta collection* to be set of the $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ of the form

$$U(\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}), \qquad U \in \mathcal{U}(\mathcal{B}_{1v_1}(\boldsymbol{\nu}, \{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}})) \ ; \tag{4.8}$$

- the *external momenta collection* to be the set of momenta

$$\boldsymbol{\nu}^{1U}_{v_1} = \boldsymbol{\nu} - \sum_{w\in\mathcal{B}_{v_1}} \sigma_w \boldsymbol{\nu}^1_{\lambda_w}, \qquad \text{for } U \in \mathcal{U}(\mathcal{B}_{1v_1}(\boldsymbol{\nu}, \{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}})) \ . \tag{4.9}$$

Note that the elements of the above constructed external momenta collection need not be necessarily contained in I_h^c.

We consider then another representative $\{\boldsymbol{\nu}^2_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ such that $|\boldsymbol{\nu}^2_{v_1}| \in I_h^c$ *and* not belonging to the branch momenta collection associated with $\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$, if there are any left; and we consider the corresponding branch momenta and external momenta collections as above. We proceed in this way until all the representatives such that $\boldsymbol{\nu}_1$ is in I_h^c, for the given h, have been put into some collection of branch momenta.

We then repeat the above construction with the interval I_h^- replacing the I_h^c, always being careful not to consider representatives $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ that appeared as members of previously constructed collections. It is worth pointing out that not all the external momenta $\boldsymbol{\nu}^U_{v_1}$, $U \in \mathcal{U}(\mathcal{B}_{1v_1}(\boldsymbol{\nu}, \{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}))$, are in I_h^-, but they are all in the corridor $I^+_{h-1} \cup I_h^-$, by (4.6).

Finally we consider the interval I^+_{h-1}, (if $h = 1$ we simply skip this step). The construction is repeated for such intervals.

Proceeding iteratively in this way starting from $h=1$ and, after exhausting all the $h=1$ cases, continuing with the $h=2,3\ldots$ cases, we shall have grouped the sets of branch momenta into collections obtainable from a representative $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ by applying the operations $U\in\mathcal{U}(\mathcal{B}_{1v_1}(\boldsymbol{\nu},\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}))$ to it. *Note that, in this way, when the interval I^+_{h-1} is considered, all the remaining representatives are such that $|\boldsymbol{\nu}^U_{v_1}|\in I^+_{h-1}$ for all $U\in\mathcal{U}(\mathcal{B}_{1v_1}(\boldsymbol{\nu},\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}))$.*

Note that the graphs with momenta in each collection are just the graphs involved in the parity cancellation described in the previous section. In fact if U is generated by the signs $\{\sigma_w\}_{w\in\mathcal{B}_v}$, we have

$$\begin{aligned}\boldsymbol{\nu}^U_{v_1}&=\Big(\prod_{w\in\mathcal{B}_{1v_1}}U^{\sigma_w}_{v_1w}\{\boldsymbol{\nu}_x\}\Big)_{v_1}\,,\\(U(\{\boldsymbol{\nu}_{\lambda_{\tilde w}}\}_{\tilde w\in B_{v_1}}))_w&=\sum_{z\le w}\Big(\prod_{\tilde w\in\mathcal{B}_{1v_1}}U^{\sigma_{\tilde w}}_{v_1\tilde w}\{\boldsymbol{\nu}_x\}\Big)_z\,,\end{aligned}\tag{4.10}$$

where, given the sets $\{\boldsymbol{\nu}_x\}$ and $\{\boldsymbol{\nu}_{\lambda_{\tilde w}}\}$, $(\{\boldsymbol{\nu}_x\})_v$ denotes the external momentum in $\{\boldsymbol{\nu}_x\}$ corresponding to the node v and $(\{\boldsymbol{\nu}_{\lambda_{\tilde w}}\})_w$ denotes the branch momentum in $\{\boldsymbol{\nu}_{\lambda_{\tilde w}}\}$ corresponding to the branch λ_w.

4.4. REMARK. The complexity of the above construction is due to the necessity of avoiding overcountings. In fact it is possible that, for some $U\in\mathcal{U}(\mathcal{B}_{1v_1}(\boldsymbol{\nu},\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}})$, one has

$$\mathcal{B}_{1v_1}(\boldsymbol{\nu},U(\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}))\neq\mathcal{B}_{1v_1}(\boldsymbol{\nu},\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}})\,,\tag{4.11}$$

because the scale of $\boldsymbol{\nu}^U_{v_1}$ may be $h-1$, while that of $\boldsymbol{\nu}_{v_1}$ may be h; so that if one considered, for instance, I^+_{h-1} before I^-_h overcountings would be possible, and in fact they would occur.

4.5. A convenient way to rewrite (4.5) is the following:

$$\begin{aligned}&\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^s\Big|\sum_{h_{v_1}}\sideset{}{^*}\sum_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}}\sum_{U\in\mathcal{U}(B_{1v_1})}f_{j\boldsymbol{\nu}^U_{v_1}}\,(\boldsymbol{\nu}^U_{v_1})^{\boldsymbol{\mu}_{v_1}}\,\cdot\\&\qquad\cdot\,\mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}}W(\vartheta^0_{v_1w},\sigma_w\boldsymbol{\nu}_{\lambda_w})\Big\}\Big|\,,\end{aligned}\tag{4.12}$$

where $\sum^*_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}}$ means sum over the above defined representatives such that $\boldsymbol{\nu}_{v_1}$ is compatible with h_{v_1}; and we abridge $\mathcal{B}_{1v_1}(\boldsymbol{\nu},\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}})$ by $\mathcal{B}_{1v_1}$ in conformity with the notations introduced after (4.6). The explicit sum over the scales h_{v_1} is introduced to simplify the bounds analysis that we perform later, see §4.8. Note that $\boldsymbol{\nu}^U_{v_1}$ is, in general, not compatible with h_{v_1}, *i.e.* we are grouping together also terms with different scale label (but the difference in scale is at most one, see (4.16) below).

Noting that, by the parity properties of $\mathbf{f}$,

$$W(\vartheta^0_{v_1w},\sigma_w\boldsymbol{\nu}_{\lambda_w})=\sigma_wW(\vartheta^0_{v_1w},\boldsymbol{\nu}_{\lambda_w})\tag{4.13}$$

we have, from (4.12),

$$
\begin{aligned}
&\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^{s}\Big|\sum_{h_{v_1}}\sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}}\sum_{U\in\mathcal{U}(\mathcal{B}_{1v_1})} f_{j\boldsymbol{\nu}^U_{v_1}}\,(\boldsymbol{\nu}^U_{v_1})^{\boldsymbol{\mu}_{v_1}}\,\cdot \\
&\qquad\cdot\,\mathcal{R}\Big\{\Big(\prod_{w\in\mathcal{B}_{1v_1}}\sigma_w\Big)\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}}W(\vartheta^0_{v_1w},\boldsymbol{\nu}_{\lambda_w})\Big\}\Big|\,.
\end{aligned}
\tag{4.14}
$$

We can apply the interpolation in (4.3) to the node v and rewrite (4.14) as

$$
\begin{aligned}
&\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^{s}\Big|\sum_{h_{v_1}}\sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}}\Big[\sum_{||\boldsymbol{\mu}_{v_1}||=p_{v_1}}\Big(\prod_{w\in\mathcal{B}_{1v_1}}\int_1^0 dt_w\Big)\cdot \\
&\qquad\cdot\Big(\prod_{w\in\mathcal{B}_{1v_1}}\frac{\partial}{\partial t_w}\Big)\Big(f_{j\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})}\,\big(\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})\big)^{\boldsymbol{\mu}_{v_1}}\Big)\Big]\cdot \\
&\qquad\cdot\,\mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}}W(\vartheta^0_{v_1w},\boldsymbol{\nu}_{\lambda_w})\Big\}\Big|\,,
\end{aligned}
\tag{4.15}
$$

where if $\mathcal{B}_{1v_1}=\emptyset$ no interpolation is made; and we note that by (4.3), by the definition of nodes out of order and by the iterative grouping of the representatives:

$$
2^{h_{v_1}-2}\le|\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})|<2^{h_{v_1}}\,,
\tag{4.16}
$$

so that the interpolation formulae discussed in §4.1 *can be used* because no singularity arises in performing the $\mathbf{t}_{v_1}$-integrations.

By the definition of $W(\vartheta^0,\boldsymbol{\nu})$ we can write (4.15) as

$$
\begin{aligned}
&\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^{s}\Big|\sum_{h_{v_1}}\sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}}\Big[\sum_{||\boldsymbol{\mu}_{v_1}||=p_{v_1}}\Big(\prod_{w\in\mathcal{B}_{1v_1}}\int_1^0 dt_w\Big)\cdot \\
&\qquad\Big(\prod_{w\in\mathcal{B}_{1v_1}}\frac{\partial}{\partial t_w}\Big)\Big(f_{j\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})}\,\big(\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})\big)^{\boldsymbol{\mu}_{v_1}}\Big)\Big] \\
&\qquad\cdot\,\mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}}\sum_{[\{\boldsymbol{\nu}_x\}_{x\le w};\,\boldsymbol{\nu}_{\lambda_w}]}\mathrm{Val}(\vartheta^0_{v_1w},\{\boldsymbol{\nu}_x\}_{x\le w})\Big\}\Big|\,,
\end{aligned}
\tag{4.17}
$$

where the sum over $[\{\boldsymbol{\nu}_x\}_{x\le w};\,\boldsymbol{\nu}_{\lambda_w}]$ is a sum over the $\{\boldsymbol{\nu}_x\}_{x\le w}$ with $\sum_{x\le w}\boldsymbol{\nu}_x=\boldsymbol{\nu}_{\lambda_w}$.

If we use (see (4.3))

$$
\frac{\partial}{\partial t_w}\equiv\Big(2\boldsymbol{\nu}_{\lambda_w}\cdot\frac{\partial}{\partial\boldsymbol{\nu}}\Big)_{\boldsymbol{\nu}=\boldsymbol{\nu}_v(\mathbf{t}_v)}\equiv\Big(\sum_{z\le w}2\boldsymbol{\nu}_z\cdot\frac{\partial}{\partial\boldsymbol{\nu}}\Big)_{\boldsymbol{\nu}=\boldsymbol{\nu}_v(\mathbf{t}_v)}\,,
\tag{4.18}
$$

to compute differentiations with respect to t_w, we can write (4.15) as

$$
\begin{aligned}
&\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^{s}\Big|\sum_{h_{v_1}}\sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}}\Big\{\sum_{||\boldsymbol{\mu}_{v_1}||=p_{v_1}}\Big(\prod_{w\in\mathcal{B}_{1v_1}}\int_1^0 dt_w\Big)\cdot \\
&\qquad\cdot\Big(|\bar{\boldsymbol{\nu}}|^{-(p_{v_1}-q_{v_1})}\frac{\partial^{q_{v_1}}}{\partial\bar{\boldsymbol{\nu}}^{q_{v_1}}}f_{j\bar{\boldsymbol{\nu}}}\,(\bar{\boldsymbol{\nu}})^{\boldsymbol{\mu}_{v_1}}\Big)_{\bar{\boldsymbol{\nu}}=\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})}\Big\}\cdot \\
&\qquad\cdot\Big[\prod_{w\in\mathcal{B}_{1v_1}}\Big(\sum_{z\le w}2\boldsymbol{\nu}_z\Big)\Big]\Big[\prod_{w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}}|\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})|\Big] \\
&\qquad\mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}}\Big[\sum_{[\{\boldsymbol{\nu}_x\}_{x\le w};\,\boldsymbol{\nu}_{\lambda_w}]}\mathrm{Val}(\vartheta_{v_1w},\{\boldsymbol{\nu}_x\}_{x\le w})\Big]\Big\}\Big|\,,
\end{aligned}
\tag{4.19}
$$

where we recall that $q_{v_1}=|\mathcal{B}_{1v_1}|$ and the sum over $[\{\boldsymbol{\nu}_x\}_{x\le w};\,\boldsymbol{\nu}_{\lambda_w}]$ again denotes sum over the $\{\boldsymbol{\nu}_x\}_{x\le w}$ with $\sum_{x\le w}\boldsymbol{\nu}_x=\boldsymbol{\nu}_{\lambda_w}$; here the factor $|\bar{\boldsymbol{\nu}}|^{-(p_{v_1}-q_{v_1})}$ (which, computed for $\bar{\boldsymbol{\nu}}=\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})$, is identical to the inverse of $[\prod_{w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}}|\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})|]$) has been introduced so that a dimensional estimate of the factor in the second line of (4.19) can be taken proportional to $a\,2^{-h_{v_1}b}$ (see the end of §1.3 and (4.16)).

If $w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}$ we have

$$\begin{aligned}\prod_{w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}}|\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})| &= \prod_{w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}}(2^o p_{v_1})\,\tilde{\mathbf{x}}_{v_1w}(\mathbf{t}_{v_1})\cdot\boldsymbol{\nu}_{\lambda_w}\\ &= \prod_{w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}}(2^o p_{v_1})\,\tilde{\mathbf{x}}_{v_1w}(\mathbf{t}_{v_1})\cdot\sum_{z\le w}\boldsymbol{\nu}_z\;,\end{aligned}\tag{4.20}$$

where $\tilde{\mathbf{x}}_{v_1w}(\mathbf{t}_{v_1})$ is a suitable vector depending on $\boldsymbol{\nu}_{\lambda_w}$ *but not on the individual terms* $\boldsymbol{\nu}_z$, and such that $|\tilde{\mathbf{x}}_{vw}(\mathbf{t}_{v_1})|<1$.

We obtain, with the above notations (and taking $o=5$, see (4.6)),

$$\begin{aligned}S_k(\vartheta^0) = \sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^s\Big|\sum_{h_{v_1}}\sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}} & \Big\{\sum_{||\boldsymbol{\mu}_{v_1}||=p_{v_1}}\Big(\prod_{w\in\mathcal{B}_{1v_1}}\int_1^0 dt_w\Big)\cdot\\ \cdot\Big(\frac{Y_{v_1}(\mathbf{t}_{v_1})}{|\bar{\boldsymbol{\nu}}|^{p_{v_1}-q_{v_1}}}\frac{\partial^{|\mathcal{B}_{1v_1}|}}{\partial\bar{\boldsymbol{\nu}}^{|\mathcal{B}_{1v_1}|}} & f_{j\bar{\boldsymbol{\nu}}}\,(\bar{\boldsymbol{\nu}})^{\boldsymbol{\mu}_{v_1}}\Big)_{\bar{\boldsymbol{\nu}}=\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})}\Big\}\Big[\prod_{w\in\mathcal{B}_{v_1}}\Big(\sum_{z\le w}2\boldsymbol{\nu}_z\Big)\Big]\cdot\\ \mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}} & \Big[\sum_{\{\boldsymbol{\nu}_x\}_{x\le w};\boldsymbol{\nu}_{\lambda_w}}\mathrm{Val}(\vartheta_w,\{\boldsymbol{\nu}_x\}_{x\le w})\Big]\Big\}\Big|\;,\end{aligned}\tag{4.21}$$

where the tensor

$$Y_{v_1}(\mathbf{t}_{v_1})=\prod_{w\in\mathcal{B}_{v_1}\setminus\mathcal{B}_{1v_1}}2^4p_{v_1}\tilde{\mathbf{x}}_{v_1w}(\mathbf{t}_{v_1})\tag{4.22}$$

depends also on $\boldsymbol{\nu}$ and $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}$ (although this dependence is not shown, to simplify the notation), and has to be contracted with the external momenta $\boldsymbol{\nu}_z$, $z\le w\in\mathcal{B}_{1v_1}$.

4.6. Developing the sum $\sum_{z\le w}2\boldsymbol{\nu}_z$ in (4.21) $S_k(\vartheta^0)$ is given by a sum of terms corresponding to a collection of nodes lying on the paths $P(v_1,z(v_1,w))$ leading from v_1 to a node z: the collection is defined by the "choices" of one particular addend $2\boldsymbol{\nu}_z$ in the sum $\sum_{z\le w}2\boldsymbol{\nu}_z$, with $z=z(v_1,w)$, $w\in\mathcal{B}_{v_1}$. Therefore, in general, we can think that (4.21) corresponds to a sum over a collection of paths $P(v_1,z(v_1,w))$ for the $w\in\mathcal{B}_{v_1}$. The paths are regarded as totally ordered (and gapless) sequences of nodes on ϑ_0.

We can call $\mathcal{P}_1$ the family of the possible collections of paths that arise when expanding the sums $\sum_{z\le w}$ in (4.21): each element $\mathbf{P}_1$ of $\mathcal{P}_1$ can be identified with one contribution to (4.21). And, by using the notation $\mathbf{t}_v=\{t_w\}_{w\in\mathcal{B}_{1v}}$ as in (4.3), the result is the following more explicit interpolation formula reexpressing the r.h.s. of (4.21)

$$\begin{aligned}\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^s\Big|\sum_{h_{v_1}}\sum_{\mathbf{P}_1\in\mathcal{P}_1}\sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}} & \Big\{\sum_{||\boldsymbol{\mu}_{v_1}||=p_{v_1}}\Big(\prod_{w\in\mathcal{B}_{1v_1}}\int_1^0 dt_w\Big)\cdot\\ \cdot\Big(\frac{Y_{v_1}(\mathbf{t}_{v_1})}{|\bar{\boldsymbol{\nu}}|^{p_{v_1}-q_{v_1}}}\frac{\partial^{q_{v_1}}}{\partial\bar{\boldsymbol{\nu}}^{q_{v_1}}} & f_{j\bar{\boldsymbol{\nu}}}\,(\bar{\boldsymbol{\nu}})^{\boldsymbol{\mu}_{v_1}}\Big)_{\bar{\boldsymbol{\nu}}=\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})}\Big\}\cdot\Big(\prod_{z:P(v_1,z)\in\mathbf{P}_1}2\boldsymbol{\nu}_z\Big)\cdot\\ \cdot\,\mathcal{R}\Big\{\frac{1}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}\prod_{w\in\mathcal{B}_{v_1}} & \Big[\sum_{\{\boldsymbol{\nu}_x\}_{x\le w};\boldsymbol{\nu}_{\lambda_w}}\mathrm{Val}(\vartheta^0_{v_1w},\{\boldsymbol{\nu}_x\}_{x\le w})\Big)\Big]\Big\}\Big|\;,\end{aligned}\tag{4.23}$$

where the interpolation is considered when $\mathcal{B}_{1v_1} \neq \emptyset$ (*i.e.* when it makes sense), and the indices have to be contracted suitably.

The above formula can be rewritten as

$$\begin{aligned}
&\sum_{\boldsymbol{\nu}} |\boldsymbol{\nu}|^s \Big| \sum_{h_{v_1}} \sum^{*}_{\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{v_1}}} \sum_{\mathbf{P}_1\in\mathcal{P}_1} \Big(\prod_{v\in[\mathbf{P}_1]} \sum_{\{\boldsymbol{\nu}_{\lambda_y}\}_{y\in\mathcal{B}_v}} \Big) \\
&\quad \mathcal{R}\Big\{ \sum_{||\boldsymbol{\mu}_{v_1}||=p_{v_1}} \Big(\prod_{w\in\mathcal{B}_{1v_1}} \int_1^0 dt_w \Big) \cdot \Big(\frac{Y_{v_1}(\mathbf{t}_{v_1})}{|\bar{\boldsymbol{\nu}}|^{p_{v_1}-q_{v_1}}} \frac{\partial^{q_{v_1}}}{\partial \bar{\boldsymbol{\nu}}^{q_{v_1}}} \frac{f_{j\bar{\boldsymbol{\nu}}}\,(\bar{\boldsymbol{\nu}})^{\boldsymbol{\mu}_{v_1}}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}} \Big)_{\bar{\boldsymbol{\nu}}=\boldsymbol{\nu}_{v_1}(\mathbf{t}_{v_1})} \\
&\quad \cdot \prod_{v\in[\mathbf{P}_1]} \sum_{||\boldsymbol{\mu}_v||=p_v} (2\boldsymbol{\nu}_v)^{\eta_v} \frac{\mathbf{f}^{\mathbf{1}}_{\boldsymbol{\nu}_v}\,(\boldsymbol{\nu}_v)^{\boldsymbol{\mu}_v+1}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\lambda_v}} \prod_{y\in\mathcal{B}_v/[\mathbf{P}_1]} W(\vartheta^0_{vy},\boldsymbol{\nu}_{\lambda_y}) \Big\} \Big| \,,
\end{aligned} \tag{4.24}$$

where

- $[\mathbf{P}_1] = \bigcup_{w\in\mathcal{B}_{v_1}} P(v_1, z(v_1,w))/\{v_1\}$,
- $f^{\mathbf{1}}_{\boldsymbol{\nu}_v} = f^{1_1}_{1\boldsymbol{\nu}_v} \dots f^{1_\ell}_{\ell\boldsymbol{\nu}_v}$, with $\|\mathbf{1}\| = 1$, is contracted with a factor in $(\boldsymbol{\nu}_{v'})^{\boldsymbol{\mu}_{v'}}$, and
- η_v is equal to 1 if $v = z(v_1,w)$ for some $w \in \mathcal{B}_{v_1}$ and 0 otherwise.

We are now in position to iterate the resummation done in the previous section leading from (4.5) to (4.21) and "concerning" the highest node v_1. For each $\tilde{v} \in \mathbf{P}_1$, $\tilde{v} < v_1$, let $h_{\tilde{v}} = h_{\tilde{v}}(\boldsymbol{\nu}_{\lambda_{\tilde{v}}}, \{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}})$ be *the scale of* $\boldsymbol{\nu}_{\tilde{v}}$, *i.e.* $\boldsymbol{\nu}_{\tilde{v}} = \boldsymbol{\nu}_{\lambda_{\tilde{v}}} - \sum_{w\in\mathcal{B}_{\tilde{v}}} \boldsymbol{\nu}_{\lambda_w}$ is such that $2^{h_{\tilde{v}}-1} \leq |\boldsymbol{\nu}_{\tilde{v}}| < 2^{h_{\tilde{v}}}$.

Given an immediate predecessor w of $\tilde{v}$ we say that w is *out of order* with respect to $\tilde{v}$ if

$$2^{h_{\tilde{v}}} > 2^5 p_{\tilde{v}} |\boldsymbol{\nu}_{\lambda_w}| \,, \tag{4.25}$$

where $p_{\tilde{v}}$ is the number of branches entering $\tilde{v}$. We denote $\mathcal{B}_{1\tilde{v}} \equiv \mathcal{B}_{1\tilde{v}}(\boldsymbol{\nu}_{\lambda_{\tilde{v}}}, \{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}) \subseteq \mathcal{B}_{\tilde{v}}$ the nodes $w \in \mathcal{B}_{\tilde{v}}$ which are out of order with respect to $\tilde{v}$.

Given a set $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{\tilde{v}}}$ for all choices of $\sigma_w = \pm 1$ we define

$$U(\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{\tilde{v}}}) \equiv \{\sigma_w \boldsymbol{\nu}_{\lambda_w}\}_{w\in B_{\tilde{v}}} \,, \tag{4.26}$$

and given a set $C \subseteq \mathcal{B}_{\tilde{v}}$ we call $\mathcal{U}(C)$ the set of all transformations such that $\sigma_w = 1$ for $w \notin C$.

We group the set of branch momenta $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$ and the external momenta into collections by proceeding, very closely following the preceding construction, with $\boldsymbol{\nu}_{\lambda_w}$ playing the role of $\boldsymbol{\nu}$, in the way described below.

Fixed $\boldsymbol{\nu}_{\lambda_{\tilde{v}}}$ and h we choose a $\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$ such that $|\boldsymbol{\nu}^1_{\tilde{v}}| \in I^c_h$ where $\boldsymbol{\nu}^1_{\tilde{v}} = \boldsymbol{\nu}_{\lambda_{\tilde{v}}} - \sum_{w\in\mathcal{B}_{\tilde{v}}} \boldsymbol{\nu}_{\lambda_w}$.

Then $\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$ is called a *representative*. For such representative we define the *branch momenta collection*, associated with it to be the set of the $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$ having the form $U(\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}})$ and the *external momenta collection* to be the set of momenta $\boldsymbol{\nu}^{1U}_{\tilde{v}} = \boldsymbol{\nu} - \sum_{w\in\mathcal{B}_{\tilde{v}}} \sigma_w \boldsymbol{\nu}^1_{\lambda_w}$, for $U \in \mathcal{U}(\mathcal{B}_{1\tilde{v}}(\boldsymbol{\nu}_{\lambda_{\tilde{v}}}, \{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}})/[\mathbf{P}_1])$. Note again that the above constructed external momenta collection is not necessarily contained in I^c_h.

We consider then another representative $\{\boldsymbol{\nu}^2_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$ such that $|\boldsymbol{\nu}^2_{\tilde{v}}| \in I^c_h$ and does not belong to the just constructed branch momenta collection associated with $\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$, if there is any; and then we consider the branch momenta collections and external momenta collections obtained from $\{\boldsymbol{\nu}^2_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde{v}}}$ by the corresponding U transformations. And, as previously done, we proceed in this way until all the representatives such that $\boldsymbol{\nu}^1_{\tilde{v}}$ is in I^c_h are in some external momenta collections.

The construction is repeated for the interval I_h^-, always being careful not to consider $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde v}}$ that have been already considered, and finally for the interval I_{h-1}^+, see §4.3.

Proceeding iteratively in this way and considering the same sequence of h's as in the previous case (*i.e.* the natural $h=1,2,\ldots$), at the end we shall have grouped the set of branch momenta into collections obtainable from a representative $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde v}}$ by applying the operations $U\in\mathcal{U}(\mathcal{B}_{1\tilde v}(\boldsymbol{\nu}_{\lambda_w},\{\boldsymbol{\nu}^1_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde v}})\setminus[\mathbf{P}_1])$ to it.

In other words the definition of the representatives $\{\boldsymbol{\nu}_{\lambda_w}\}_{w\in\mathcal{B}_{\tilde v}}$ is identical to the one for v_1 except that the collections are defined only by transformations changing the branch momentum of the lines emerging from the nodes in $\mathcal{B}_{1\tilde v}$ but not in $\mathbf{P}_1$.

We repeat the above construction for all $\tilde v\in\mathbf{P}_1$ until all the $\tilde v\in\mathbf{P}_1$ are considered starting from the $\tilde v$ with $\tilde v'=v$ and, after exhausting them, continuing with $\hat v$ with $\hat v'=\tilde v$ and so on. We call $\mathcal{B}_{\tilde v}(\mathbf{P}_1)$ the nodes w immediately preceding $\tilde v$ *but which are not on the union of the paths $P\in\mathbf{P}_1$*, and $\mathcal{B}_{1\tilde v}(\mathbf{P}_1)$ the nodes in $\mathcal{B}_{\tilde v}(\mathbf{P}_1)$ which are out of order with respect to $\tilde v$; the set of just described transformations will be denoted by $\mathcal{U}(\mathcal{B}_{1\tilde v}(\mathbf{P}_1))$.

Proceeding as we did for the highest node v_1 and by performing the analogues of the transformations leading from (4.15) to (4.24), we construct for each $\tilde v\in\mathbf{P}_1$ new paths $\mathbf{P}_2$ which, by construction, *will not have common branches* with those in $\mathbf{P}_1$; call $\mathcal{P}_2$ the collection of the pairs $\mathbf{P}_1,\mathbf{P}_2$. The crucial point is that the factors $\tilde{\mathbf{x}}_{vw}(\mathbf{t}_v)$ are the same for all the terms generated by the action of $U\in\mathcal{U}(\mathcal{B}_{1v}(\mathbf{P}_1))$, by (4.20). We iterate then this procedure.

Eventually we end up by constructing a *pavement* $\mathbf{P}$ of the graph with non overlapping paths (and the union of the paths does cover the graph); note that the paths are "ordered", in the sense that they are formed only by comparable lines.

We call $\mathcal{P}$ the collection of all such pavements; $\mathcal{B}_v(\mathbf{P})$, $\mathbf{P}\in\mathcal{P}$, will be the set of nodes w immediately preceding v and such that a path $P(v,z(v,w))\in\mathbf{P}$ starting from v passes through w, and $\mathcal{B}_{1v}(\mathbf{P})$ is the collection of nodes in $\mathcal{B}_v(\mathbf{P})$ out of order with respect to v. Note that in general $\mathcal{B}_v(\mathbf{P})\subseteq\mathcal{B}_v$ (unless v is the highest node v_1, when $\mathcal{B}_{v_1}(\mathbf{P})=\mathcal{B}_{v_1}$).

The set of "path head" nodes v, *i.e.* the upper end nodes of paths in $\mathbf{P}$, will be denoted $M_h(\mathbf{P})$: hence if $v\notin M_h(\mathbf{P})$ (*i.e.* if no path in $\mathbf{P}$ has v as path head) then $\mathcal{B}_v(\mathbf{P})=\emptyset$; likewise $M_e(\mathbf{P})$ will denote the set of "path end" nodes, *i.e.* the nodes z such that $P(v,z)$ is a path in $\mathbf{P}$.

4.7. Then we see that (4.22) leads the following *path expansion* for $S_k(\vartheta^0)$ summarizing our analysis

$$\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^s|W(\vartheta^0,\boldsymbol{\nu})|=\sum_{\boldsymbol{\nu}}|\boldsymbol{\nu}|^s\Big|\sum_{\{h_x\}}\sum_{\mathbf{P}\in\mathcal{P}}\sum_{\{\boldsymbol{\nu}_\lambda\}}^{*}\prod_{v\in M_h(\mathbf{P})}\tag{4.27}$$
$$\Big\{\Big(\prod_{w\in\mathcal{B}_{1v}(\mathbf{P})}\int_1^0 dt_w\Big)\sum_{||\mathbf{m}_v||=p_v}\mathcal{O}_v\Big(\mathbf{f}^{\mathbf{1}}_{\boldsymbol{\nu}_v(\mathbf{t}_v)}\left(\boldsymbol{\nu}_v(\mathbf{t}_v)\right)^{\mathbf{m}_v}Y_v(\mathbf{t}_v)\Big)\Big\}\Big|\mathcal{R}D(\vartheta^0)\,,$$

where

(1) $\mathbf{P}$ is a partial pavement of the graph with non overlapping *paths* such that: (1.1) a path $P(v,z)$ is a connected set of comparable lines connecting the node v to the node $z<v$; (1.2) the resonance paths are not contained in any path; (1.3) for any line λ which is not contained in any resonance path there is one path in $\mathbf{P}$ covering it: $\lambda\in P(v,z)$ for some $P(v,z)\in\mathbf{P}$;

(2) $M_h(\mathbf{P})$ is the collection of upper end nodes of the paths in $\mathbf{P}$, and $M_e(\mathbf{P})$ of lower end nodes;

(3) $\mathcal{B}_v(\mathbf{P})$ is the set of nodes immediately preceding v, and $\mathcal{B}_{1v}(\mathbf{P})$ is the set of nodes $w \in \mathcal{B}_v(\mathbf{P})$ which are *out of order with respect to* v, *i.e.* such that

$$2^{h_v} > 2^5 p_v |\boldsymbol{\nu}_{\lambda_w}| \; ; \tag{4.28}$$

(4) $Y_v(\mathbf{t}_v)$ is defined as

$$Y_v(\mathbf{t}_v) = \begin{cases} \prod_{w \in \mathcal{B}_v(\mathbf{P}) \setminus \mathcal{B}_{1v}(\mathbf{P})} (2^4 p_v \tilde{\mathbf{x}}_{vw}(\mathbf{t}_v)) \, , & \text{if } v \in M_h(\mathbf{P}) \, , \\ 1 \, , & \text{otherwise} \ \ , \end{cases} \tag{4.29}$$

if $\tilde{\mathbf{x}}_{vw}(\mathbf{t}_v)$ is the vector defined via the implicit relation

$$\boldsymbol{\nu}_v(\mathbf{t}_v) = 2^5 p_v \, \tilde{\mathbf{x}}_{vw}(\mathbf{t}_v) \cdot \boldsymbol{\nu}_{\lambda_w} \ , \tag{4.30}$$

so that $|\tilde{\mathbf{x}}_{vw}(\mathbf{t}_v))| \le 1$ and $\tilde{\mathbf{x}}_{vw}(\mathbf{t}_v))$ depends on $\boldsymbol{\nu}_{\lambda_w}$ but not on the individual external momenta which add to $\boldsymbol{\nu}_{\lambda_w}$;
(5) the operator $\mathcal{O}_v$ is defined as

$$\begin{aligned} & \mathcal{O}_v\Big((\boldsymbol{\nu}_v(\mathbf{t}_v))^{\mathbf{m}_v} \, Y_v(\mathbf{t}_v) \, \mathbf{f}_{\boldsymbol{\nu}_v(\mathbf{t}_v)} \Big) = \\ & = \Big(\frac{Y_v(\mathbf{t}_v)}{|\bar{\boldsymbol{\nu}}|^{|\mathcal{B}_v(\mathbf{P})|-|\mathcal{B}_{1v}(\mathbf{P})|}} \frac{\partial^{|\mathcal{B}_{1v}(\mathbf{P})|}}{\partial \bar{\boldsymbol{\nu}}^{|\mathcal{B}_{1v}(\mathbf{P})|}} \mathbf{f}^{\mathbf{1}}_{\bar{\boldsymbol{\nu}}} \, (\bar{\boldsymbol{\nu}})^{\mathbf{m}_v} \, (2\bar{\boldsymbol{\nu}})^{\eta_v} \Big)_{\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_v(\mathbf{t}_v)} \, , \end{aligned} \tag{4.31}$$

with $\eta_v = 1$ if $v \in M_e(\mathbf{P})$, and $\eta_v = 0$ otherwise, and $\mathbf{f}^{\mathbf{1}}_{\boldsymbol{\nu}}$ defined after (4.24);
(6) the sum over $\{\boldsymbol{\nu}_\lambda\}$ has the restriction that the external momentum configuration $\{\boldsymbol{\nu}_x\}$ is compatible with the scales $\{h_x\}$;
(7) $\mathcal{R}D(\vartheta^0)$ is the same for all graphs involved in the cancellations mechanisms, as the moduli of the momenta do not change under the action of the change of variables (4.2), and the signs are taken into account by the interpolation formula (4.3) (see Remark 4.2).

4.8. We can bound

$$\begin{aligned} & |\boldsymbol{\nu}|^s \prod_{v \in \vartheta^0} \|\mathcal{O}_v\Big(\mathbf{f}^{\mathbf{1}}_{\boldsymbol{\nu}_v(\mathbf{t}_v)} (\boldsymbol{\nu}_v(\mathbf{t}_v))^{\mathbf{m}_v} \, Y_v(\mathbf{t}_v) \Big)\| \le \\ & \quad \le \prod_{v \in \vartheta^0} D_1 D_2^{p_v} q_v! \, p_v^{p_v - q_v} \, 2^{h_v(1-b+s+\eta_v)} \le \prod_{v \in \vartheta^0} D_3 D_4^{p_v} p_v! \, 2^{h_v(1-b+s+\eta_v)} \ , \end{aligned} \tag{4.32}$$

for suitable constants D_j, and use $\prod_{v \in \vartheta^0} |\boldsymbol{\nu}_v|^{\frac{\eta}{2}\tau} \le \prod_{v \in \vartheta^0} 2^{h_v \frac{\eta}{2}\tau}$, so that

$$\begin{aligned} & \sum_{\{\boldsymbol{\nu}\}} |\boldsymbol{\nu}|^s \Big| \sum_{\vartheta^0} \mathcal{R}W(\vartheta^0, \boldsymbol{\nu}) \Big| \le \\ & \le \max_{\mathbf{P} \in \mathcal{P}} \Big\{ C_3^k \Big[\prod_{v \le v_0} p_v! \Big] \sum_{\{h_x\}} \prod_{v \le v_0} \Big[2^{h_v(p_v + s + \ell + \frac{\eta}{2}\tau - b)} \prod_{P(v,z) \in \mathbf{P}} 2^{(h_z - h_v)} \Big] \Big\} \, . \end{aligned} \tag{4.33}$$

Then, setting $b = 2 + s + \ell + \frac{\eta}{2}\tau + \mu$, with $\mu > 0$, and exploiting the identity

$$\sum_{v < v_1} h_v p_v = \sum_{v < v_1} h_{v'} \ , \qquad v_1' = r \ , \tag{4.34}$$

one obtains for (4.5) the bound

$$
\begin{aligned}
&\sum_{\{\boldsymbol{\nu}\}} |\boldsymbol{\nu}|^s \Big| \sum_{\vartheta^0} \mathcal{R}W(\vartheta^0,\boldsymbol{\nu}) \Big| \le \\
&\le C^k \prod_v p_v! \sum_{\{h_x\}} \Big[2^{-(2+\mu)h_{v_0}} \prod_{v<v_0} 2^{-(1+\mu)h_v} \prod_{vw\in\mathcal{Q}} 2^{h_v-h_w} \Big] ,
\end{aligned}
\tag{4.35}
$$

for a suitable constant C.

We see that there is at most one factor $2^{h_v-h_w}$ per node v, because the resonance paths are totally ordered, so that the factors 2^{-h_v} in $2^{-(1+\mu)h_v}$ compensate (when necessary) the factors 2^{h_v} in $2^{h_v-h_{v'}}$ (and $2^{-h_{v'}} \le 1$): then the sum over the scales can be performed.

There are $k!/\prod_v p_v!$ graphs with given p_v's and fixed shape (*Cayley's formula*, see [HP]), so that the sum over the graph orders weighed by ε^k can be performed if ε is small enough; in particular we obtain that $\mathbf{h} \in C^{(s)}(\mathbb{T}^\ell)$, if $f \in \hat{C}^{(2+s+\frac{\eta}{2}\tau+\mu)}(\mathbb{T}^\ell)$, with $\mu > 0$.

4.9. Then we can pass to the equation (2.10) for $\mathbf{H}$, with $\mathrm{Val}^*(\vartheta)$ defined in (2.9). In such a case we give the extra prescription not to apply the ultraviolet interpolation procedure to the path $\mathcal{C}(v_0,\tilde{v})$; equivalently, modify slightly the definition of the set $\mathcal{B}_v$ after (4.1): $\mathcal{B}_v$ is the the subset of the nodes w among the p_v nodes immediately preceding v such that the branch vw is neither on the resonance paths $\mathcal{Q}$ nor on $\mathcal{C}(v_0,\tilde{v})$.

Then we obtain again a formula like (4.17), with respect to which there are the following differences.

(1) $\mathbf{P}$ is the partial pavement such that, besides the conditions (1.1)÷(1.3) after (4.27), verifies the further condition: (1.4) there is no overlapping between $\mathcal{C}(v_0,\tilde{v})$ and any $P(v,z) \in \mathbf{P}$.

(2) If $v \in \mathcal{C}(v_0,\tilde{v})$, $\mathbf{f}^{\mathbf{1}}_{\boldsymbol{\nu}_v}$ has to be contracted with a factor in $(\boldsymbol{\nu}_{v''})^{\boldsymbol{\mu}_{v''}}$, where $v'' \in \mathcal{C}(v_0,\tilde{v})$ is the node on $\mathcal{C}(v_0,\tilde{v})$ immediately preceding v.

(3) If $v \in \mathcal{C}(v_0,\tilde{v})$, the factors $(\boldsymbol{\nu}_v)^{\boldsymbol{\mu}_v}$ arise from the $p_v - 1$ branches not contained in $\mathcal{C}(v_0,\tilde{v})$ and entering v and from the branch on $\mathcal{C}(v_0,\tilde{v})$ exiting from v and pointing to v'.

Since the bound in §4.8 is independent on the exact structure of the contractions, the bound (4.35) can be still obtained, so that also $\mathbf{H} \in C^{(s)}(\mathbb{T}^\ell)$, if $f \in \hat{C}^{(2+s+\frac{\eta}{2}\tau+\mu)}(\mathbb{T}^\ell)$, with $\mu > 0$.

Thus the proof of Theorem 1.4 is complete.

5. Comparison with the one-dimensional Schrödinger equation in a quasi periodic potential

5.1. From Theorem 1.4, one could deduce the existence of Bloch waves for the one-dimensional Schrödinger equation with a potential belonging to a certain class of non analytic quasi periodic functions, and one could be tempted to compare the result with [Pa], where the existence of Bloch waves is proven with the Moser-Nash techniques for quasi periodic potentials having $p > 2(\ell+1)$ continuous derivatives (if ℓ is the dimension of the frequency vector of the quasi periodic potential and τ is supposed to be $\tau > \ell - 1$), with no other restriction on the potential regularity. However, in order to perform a meaningful comparison between the two results, one has to consider carefully the exact form of the interaction potential.

5.2. The problem studied in [DS,R,P] is the Schrödinger equation

$$\Big[-\frac{d^2}{dx^2}+\varepsilon V(x)\Big]\psi(x)=E\psi(x) \tag{5.1}$$

where $V(x)$ is a quasi periodic function of the form

$$V(x)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^{\ell-1}} e^{i\boldsymbol{\omega}\cdot\boldsymbol{\nu}x}\,V_{\boldsymbol{\nu}}\ , \tag{5.2}$$

with $\boldsymbol{\omega}\in\mathbb{R}^{\ell-1}$ satisfying a Diophantine condition.

The problem to find eigenvalues and eigenfunctions of (5.1) can be easily seen to be equivalent to the problem to solve the equations of motion of the classical mechanics system described by the Hamiltonian

$$\mathcal{H}=\frac{p^2}{2}+\boldsymbol{\omega}\cdot\mathcal{B}+\frac{q^2}{2}\Big[E-\varepsilon V(\boldsymbol{\beta})\Big]\ , \tag{5.3}$$

with $(p,q)\in\mathbb{R}^2$ and $(\mathcal{B},\boldsymbol{\beta})\in\mathbb{R}^{\ell-1}\times\mathbb{T}^{\ell-1}$. In fact the evolution equation for the coordinate q is nothing but the eigenvalue equation (5.1).

Then it is possible to introduce a canonical transformation $\mathcal{C}:(p,q)\to(A_1,\alpha_1)$, [G2], such that

$$\mathcal{H}=\sqrt{E}A_1+\boldsymbol{\omega}\cdot\mathcal{B}+\varepsilon f(\alpha_1,\boldsymbol{\beta})\ ,\qquad f(\alpha_1,\boldsymbol{\beta})=-\frac{A_1}{\sqrt{E}}\left(\sin^2\alpha_1\right)V(\boldsymbol{\beta})\ , \tag{5.4}$$

which can be reduced to the form (1.3), with $\mathbf{A}=(A_1,\mathcal{B})\in\mathbb{R}^\ell$, $\boldsymbol{\alpha}=(\alpha_1,\boldsymbol{\beta})\in\mathbb{T}^\ell$, and $\varepsilon\mathbf{f}(\boldsymbol{\alpha})=(f(\boldsymbol{\alpha}),0,\ldots,0)$. For the proof of such an assertion, we refer to [G2].

Then the equations of motions for $\boldsymbol{\beta}$ gives $\boldsymbol{\beta}(t)=\boldsymbol{\beta}_0+\boldsymbol{\omega}t$, and the derivatives whose number can grow up indefinitely are those acting on the α_1 variable: *however the perturbation is always analytic in* α_1.

Thus the reason why in Theorem 1.4 we needed much stronger assumptions on the interaction potential, compared with (5.4), is simply that the one-dimensional Schrödinger equation can be reduced to a classical mechanics problem with Hamiltonian of the form (1.1), but the interaction term depends analytically on α_1, independently on the regularity of the quasi periodic potential.

In this case the existence of the counterterm can be proven without exploiting the ultraviolet cancellations, and the infrared cancellations are sufficient to give the convergence of the perturbative series, provided the quasi periodic potential is so regular to guarantee the summability on the Fourier components in the perturbative series: the analysis in §3,4 gives $p>1+3\tau$, see Appendix A3 for details.

Then, if $\tau>\ell-1$, one has $p>3\ell-2$. With respect to [Pa], the result is weaker for $\ell\geq5$, but stronger for $\ell\leq3$ (and equivalent for $\ell=4$). Nevertheless, as the result in [Pa] has been obtained by using the Moser-Nash techniques for KAM theory, and it is known that the class of differentiability of the perturbations of integrable systems can be raised from the Moser result $p>2(\ell+1)$ to (the perhaps optimal) $p>2\ell$, [P], *then one can conjecture that also for the Schrödinger equation the ideas in* [P] *would lead to* $p>2\ell$. Therefore our result $p>3\ell-2$ can be considered, for low ℓ, an improvement of [Pa].

Furthermore it is important to note also that with the techniques described in the present paper no other regularity condition is required for the perturbation other than that of being differentiable enough (*i.e.* no special class of functions as $\hat{C}^{(p)}(\mathbb{T}^\ell)$ has to be invoked), in order to obtain *analyticity* in the perturbative parameter of the eigenvalue E and the eigenfunction $\psi(x)$ in (5.1).

5.3. The situation is essentially identical if one consider the Schrödinger equation

$$\Big[-\frac{d^2}{dx^2}+U(x)+\varepsilon V(x)\Big]\psi(x)=E\psi(x) \tag{5.5}$$

where $U(x)$ is a periodic potential with frequency ω_2 and $V(x)$ a quasi periodic function of the form

$$V(x)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^{\ell-2}} e^{i\boldsymbol{\omega}\cdot\boldsymbol{\nu}x}\, V_{\boldsymbol{\nu}}\ , \tag{5.6}$$

with $\boldsymbol{\omega}\in\mathbb{R}^{\ell-2}$, such that ω_2 and $\boldsymbol{\omega}$ satisfy a Diophantine condition.

The Hamiltonian of the corresponding classical mechanics problem is

$$\mathcal{H}=\frac{p^2}{2}+\omega_2 B_2+\boldsymbol{\omega}\cdot\mathcal{B}+\frac{q^2}{2}\Big[E-U(\beta_2)-\varepsilon V(\boldsymbol{\beta})\Big]\ , \tag{5.7}$$

with $(p,q)\in\mathbb{R}^2$, $(B_2,\beta_2)\in\mathbb{R}^1\times\mathbb{T}^1$, and $(\mathcal{B},\boldsymbol{\beta})\in\mathbb{R}^{\ell-1}\times\mathbb{T}^{\ell-1}$. If $\varepsilon=0$, the Hamiltonian is integrable, [G2,C], so that (5.7) becomes

$$\begin{aligned}&\mathcal{H}=\omega_{01}A_1+\omega_{02}A_2+\boldsymbol{\omega}\cdot\mathcal{B}+\varepsilon f(\alpha_1,\alpha_2,\boldsymbol{\beta})\ ,\\ &f(\alpha_1,\alpha_2,\boldsymbol{\beta})=-A_1\,G(\alpha_0,\alpha_1)\,V(\boldsymbol{\beta})\ ,\end{aligned} \tag{5.8}$$

where $G(\alpha_1,\alpha_2)$ is a function which depends analytically on α_0, [C], §V,VI, independently on the regularity of U and V in (5.5). The fact that the interaction is proportional only to A_1 (*i.e.* independent on the other action variables) implies that the equations of motion for α_1 and $\boldsymbol{\beta}$ can be trivially integrated and give $\alpha_2=\alpha_{20}+\omega_{02}t$ and $\beta_j=\beta_{j0}+\omega_{0j}t$, $2\le j\le\ell$. Then we can reason as in §5.2, and the same conclusions hold.

Appendix A1. Graphs and graph rules

We lay down one after the other, on a plane, k pairwise distinct unit segments oriented from one extreme to the other: respectively the *initial point* and the *endpoint* of the oriented segment. The oriented segment will also be called *arrow*, *branch* or *line*. The segments are supposed to be numbered from 1 to k.

The rule is that after laying down the first segment, the *root branch*, with the endpoint at the origin and otherwise arbitrarily, the others are laid down one after the other by attaching an endpoint of a new branch to an initial point of an old one and by leaving free the new branch initial point. The set of initial points of the object thus constructed will be called the set of the graph *nodes* or *vertices*. A graph of *order* k is therefore a partially ordered set of k nodes with top point the endpoint of the root branch, also called the *root* (which is not a node); in general there will be several "bottom nodes" (at most $k-1$).

We denote by $\le$ the ordering relation, and say that two nodes v, w are "comparable" if $v<w$ or $w<v$.

With each graph node v we associate an *external momentum* or *mode* which is simply an integer component vector $\boldsymbol{\nu}_v\neq\mathbf{0}$; with the root of the graph (which is not regarded as a node) we associate a label $j=1,\dots,\ell$.

For each node v, we denote by v' the node immediately following v and by $\lambda_v \equiv v'v$ the branch connecting v to v', (v will be the initial point and v' the endpoint of λ_v). If v is the node immediately preceding the root r (*highest node*) then we shall write $v' = r$, for uniformity of notation (recall that r is not a node).

We consider "comparable" two lines λ_v, λ_w, if v, w are such.

If p_v is the number of branches entering the node v, then each of the p_v branches can be thought as the root branch of a *subgraph* having root at v: the subgraph is uniquely determined by v and one of the p_v nodes w immediately preceding v. Hence if $w' = v$ it can be denoted ϑ_{vw}.

We shall call *equivalent* graphs which can be overlapped by

(1) changing the angles between branches emerging from the same node, or

(2) permuting the subgraphs entering into a node v in such a way that all the labels match.

The number of (non equivalent numbered) graphs with k branches is bounded by $4^k k!$, [HP].

Appendix A2. Proof of Lemma 3.4

We consider all the graphs we obtain by detaching from each resonance the subgraph with root v_V^b, if v_V^b is the node in which the resonance line λ_V enters, then reattaching it to all the nodes $w \in V$. We call this set of contributions a *resonance family*. If one set $\zeta \equiv \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V} = 0$, no propagator changes inside the resonance, and the only effect of the above operation is that in the factor

$$\Big[(\boldsymbol{\nu}_{v_V^0} \cdot \mathbf{f}_{\boldsymbol{\nu}_{v_V^a}}) \, (\boldsymbol{\nu}_{v_V^b} \cdot \mathbf{f}_{\boldsymbol{\nu}_{v_V^1}}) \, \frac{\chi(2^{-n_\lambda} \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V})}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_V})^2} \Big] \tag{A2.1}$$

appearing in (3.3) the external momentum $\boldsymbol{\nu}_{v_V^b}$ assumes successively the values $\boldsymbol{\nu}_w$, $w \in V$. In this way, by summing over all the trees belonging to a given resonance family, we build a quantity proportional to $\sum_{w \in V} \boldsymbol{\nu}_w = \mathbf{0}$, by Definition 3.3 of real resonance.

Appendix A3. Regularity of the potential for the Schrödinger equation in a quasi periodic potential

The equations of motion of the Hamiltonian system (5.4), once a counterterm $A_1 N_1(\varepsilon)$ has been added, give for the angle variables

$$\begin{aligned} \frac{d\alpha_1}{dt} &= \sqrt{E} - \varepsilon \frac{1}{\sqrt{E}} (\sin^2 \alpha_1) V(\boldsymbol{\omega}_0 t) + N_1(\varepsilon) \; , \\ \frac{d\boldsymbol{\beta}}{dt} &= \boldsymbol{\omega}_0 \; , \end{aligned} \tag{A3.1}$$

which can be discussed as in §2. A formula in terms of graphs (2.6) can be still obtained, where $\mathrm{Val}(\vartheta)$, defined in (2.4), becomes

$$\mathrm{Val}(\vartheta) = \prod_{v<r} \Big(-\frac{1}{\sqrt{E}} \Big) \frac{\nu_{v'} s_{\nu_v} V_{\boldsymbol{\nu}_v}}{\sqrt{E} \nu_{\lambda_v} + \boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\lambda_v}} \; , \tag{A3.2}$$

where $\boldsymbol{\nu}_v, \boldsymbol{\nu}_{\lambda_v} \in \mathbb{Z}^{\ell-1}$ and $\nu_v \in \{-2,0,2\}$ $\forall v \in \vartheta$, and we decomposed

$$V(\boldsymbol{\beta}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^{\ell-1}} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\beta}} V_{\boldsymbol{\nu}} \; , \qquad \sin^2\alpha_1 = \sum_{\nu=0,\pm2} e^{i\nu\alpha_1} s_\nu \tag{A3.3}$$

Then one sees that no problem arises from the numerators, (as the only appearing external momenta are of the form ν_v, and $\sin^2\alpha_1$ is a trigonometric polynomial in α_1), while the small divisors can be dealt with through Lemma 3.4. This gives a factor $|\boldsymbol{\nu}_v|^{3\tau}$ per node, so that summability on the Fourier labels requires at least $V \in C^{(p)}(\mathbb{T}^{\ell-1})$, $p > 3\tau$.

The equations of motion for the action variables give

$$\begin{aligned} \frac{dA_1}{dt} &= \varepsilon A_1 \frac{\partial \sin^2\alpha_1}{\partial\alpha_1} V(\boldsymbol{\omega}_0 t) \; , \\ \frac{d\mathcal{B}}{dt} &= \varepsilon A_1 \sin^2\alpha_1 \frac{\partial V(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}\Big|_{\boldsymbol{\beta}=\boldsymbol{\omega}_0 t} \; , \end{aligned} \tag{A3.4}$$

so that we can reason as above, with the only difference that the highest node of the graph v_0 has a factor $\boldsymbol{\nu}_{v_0}$ which requires, to guarantee the summability on $\boldsymbol{\nu}_{v_0}$, $V \in C^{(p)}(\mathbb{T}^{\ell-1})$, with $p > 3\tau + 1$.

Then one has to require at least $V \in C^{(p)}(\mathbb{T}^{\ell-1})$, $p > 3\tau + 1$, in order to have $\alpha_1 \in C^{(1)}(\mathbb{T}^1)$, as it has to be for the Schrïodinger equation (5.1) to be meaningful, if one recalls that (1) the wave function $\psi(x)$ solving (5.1) has to be of class $C^{(1)}$ for $V \in C^{(0)}$, and (2) $\psi(x) = q(x)$, where q is the variable related with α_1 by the canonical transformation $\mathcal{C}$ defined before (5.4).

Appendix A4. Comparison between Moser's counterterms theorem and the counterterms conjecture in [G1].

A4.1. In [M1] a perturbation theory for quasi-periodic solutions of a nonlinear system of ordinary differential equations is developed. Up to a (trivial) coordinate transformation, the system can be written in the form

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \boldsymbol{\omega} + \varepsilon \mathbf{f}(\mathbf{x},\mathbf{y};\varepsilon) \; , \\ \frac{d\mathbf{y}}{dt} &= \Omega\mathbf{y} + \varepsilon \mathbf{g}(\mathbf{x},\mathbf{y};\varepsilon) \; , \end{aligned} \tag{A4.1}$$

where $\mathbf{x} \equiv (x_1,\ldots,x_n) \in \mathbb{R}^n$, $\mathbf{y} \equiv (y_1,\ldots,y_m) \in \mathbb{R}^m$, $\boldsymbol{\omega} \in \mathbb{R}^n$, Ω is a constant $m \times m$ matrix with eigenvalues $\Omega_1,\ldots,\Omega_m$, and $\mathbf{f}$ and $\mathbf{g}$ are functions with period 2π in $x_1,\ldots,x_n$ and analytic in $\mathbf{x},\mathbf{y}$ and ε (in suitable domains).

If the characteristic numbers $\omega_1,\ldots,\omega_n,\Omega_1,\ldots,\Omega_n$ verify the generalized Diophantine condition

$$C_0 \Big| i \sum_{j=1}^{n} \nu_j\omega_j + \sum_{i=1}^{m} \mu_i\Omega_i \Big| \ge \left(|\boldsymbol{\nu}|^\tau + 1\right)^{-1} \; , \tag{A4.2}$$

with $\boldsymbol{\nu} \equiv (\nu_1,\ldots,\nu_m) \in \mathbb{Z}^n$, $|\boldsymbol{\nu}| = \sum_{j=1}^n \nu_j$, and $(\mu_1,\ldots,\mu_m) \in \mathbb{Z}^m$ then there exists a unique analytic vector valued functions $\boldsymbol{\lambda}(\varepsilon)$ and $\boldsymbol{\mu}(\varepsilon)$ and a unique analytic matrix valued function $M(\varepsilon)$ such that the modified system

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \boldsymbol{\omega} + \varepsilon \mathbf{f}(\mathbf{x},\mathbf{y};\varepsilon) + \boldsymbol{\lambda}(\varepsilon) \;, \\ \frac{d\mathbf{y}}{dt} &= \Omega \mathbf{y} + \varepsilon \mathbf{g}(\mathbf{x},\mathbf{y};\varepsilon) + \boldsymbol{\mu}(\varepsilon) + M(\varepsilon)\mathbf{y} \;, \end{aligned} \tag{A4.3}$$

admits a quasi periodic solution with the same characteristic number as the unperturbed one, [M1], Theorem 1.

A4.2. Let us consider the case in which $m = n = \ell$, $\Omega = 0$, and there exists a function $\mathcal{H}_0 = \boldsymbol{\omega} \cdot \mathbf{y} + \varepsilon f(\mathbf{x},\mathbf{y};\varepsilon)$ such that $\mathbf{f} = \partial_{\mathbf{y}} f$ and $\mathbf{g} = -\partial_{\mathbf{x}} f$. Then the system (A4.1) becomes the system studied in [GM2], §8.

Under the same hypotheses, if moreover $f(\mathbf{x},\mathbf{y};\varepsilon) \equiv \varepsilon \mathbf{y} \cdot \mathbf{f}(\mathbf{x})$ for some function $\mathbf{f}$, (A4.1) and (A4.3) become the equations of motion of systems described by the Hamiltonians, respectively, (1.1) and (1.3). In fact the linearity in the action variables of the term added to the Hamiltonian $\mathcal{H}_0$ in (1.3) leads to a term independent of the action variables in the equations of motion, *i.e.* $\mathbf{N}(\varepsilon) \equiv \boldsymbol{\lambda}(\varepsilon)$, while the counterterms $\boldsymbol{\mu}(\varepsilon)$ and $M(\varepsilon)$ are identically vanishing as a consequence of the symplectic structure of the equations of motion (as one can argue *a posteriori* from Theorem 1.4 in §1).

In the general case in which the function $f(\mathbf{x},\mathbf{y};\varepsilon)$ appearing in the Hamiltonian $\mathcal{H}_0$ depends arbitrarily (but always analytically) on $\mathbf{y}$, the systems studied in [M1] (under the same hypotheses as above) and in [GM2] are no longer equal to each other, *i.e.* the modified system (A4.3) is *not* the system with Hamiltonian considered Eq. (1.10) of [GM2], so that Theorem 1.4 in [GM2] can not be reduced to the results of [M1]: in fact not only there will be no more a trivial relation between the counterterms $\mathbf{N}(\varepsilon)$ and $\boldsymbol{\lambda}(\varepsilon)$, but also the solutions of the equations of motion will be different from each other.

Note however that the result following from Moser's theorem applied to such a system (*i.e.* a Hamiltonian system with $\Omega = 0$) can be (trivially) reproduced with our techniques. Also an extension of our techniques to Hamiltonian systems (verifying the anisochrony condition) such that $\Omega \neq 0$[3] could been envisaged: an example in this direction is in [Ge], where has eigenvalues $\Omega_1 = \ldots = \Omega_{\ell-1} = 0$, $\Omega_\ell = g^2$, and the existence of a counterterm $M(\varepsilon)$ analytic in ε is proven (while $\boldsymbol{\lambda}(\varepsilon) \equiv \boldsymbol{\mu}(\varepsilon) \equiv \mathbf{0}$ again for the symplectic structure of the equations of motion).

Acknowledgements. This work is part of the research program of the European Network on: "Stability and Universality in Classical Mechanics", # ERBCHRXCT940460. One of us (G.Ge) acknowledges finantial support from EC program TMR and is indebted to IHES for hospitality.

References

[BGGM] F. Bonetto, G. Gallavotti, G. Gentile, V. Mastropietro: Lindstedt series, ultraviolet divergences and Moser's theorem, Preprint IHES/P/95/102 (1995).

[C] L.Chierchia: Absolutely continuous spectra of quasi periodic Schrödinger operators, *J. Math. Phys.* **28**, 2891–2898 (1987).

[DS] E.I. Dinaburg, Ya.G. Sinai: The one dimensional Schrödinger equation with a quasi periodic potential, *Funct. Anal. and its Appl.* **9**, 279–289 (1975).

[E1] L.H. Eliasson: Absolutely convergent series expansions for quasi-periodic motions, Report 2–88, Department of Mathematics, University of Stockholm (1988).

[3] The anisochronus case with $\Omega = 0$ is simply the KAM theorem.

[E2] L.H. Eliasson: Hamiltonian systems with linear normal form near an invariant torus, in *Nonlinear Dynamics*, Ed. G. Turchetti, Bologna Conference, 30/5 to 3/6 1988, World Scientific, Singapore (1989).

[EV] Ecalle, J., Vallet, B.: Prenormalization, correction, and linearization of resonant vector fields or diffeomorphisms, Preprint 95-32, Université de Paris Sud - Mathematiques, Paris (1995).

[G1] G. Gallavotti: A criterion of integrability for perturbed harmonic oscillators. "Wick Ordering" of the perturbations in classical mechanics and invariance of the frequency spectrum, *Comm. Math. Phys.* **87**, 365–383 (1982).

[G2] G. Gallavotti: Classical Mechanics and Renormalization Group, Lectures notes at the 1983 Erice summer school, Ed. A. Wightman & G. Velo, Plenum, New York (1985).

[G3] G. Gallavotti: Quasi-integrable mechanical systems, in *Critical Phenomena, Random Systems, Gauge Theories*, Les Houches, Session XLIII (1984), Vol. II, 539–624, Ed. K. Osterwalder & R. Stora, North Holland, Amsterdam (1986).

[G4] G. Gallavotti: Twistless KAM tori, *Comm. Math. Phys.* **164**, 145–156 (1994).

[Ge] G. Gentile: Whiskered tori with prefixed frequencies and Lyapunov spectrum, *Dynam. Stability of Systems* **10**, 269–308, (1995).

[GM1] G. Gentile, V. Mastropietro: KAM theorem revisited, *Physica D* **90**, 225–234 (1996).

[GM2] G. Gentile, V. Mastropietro: Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications, to appear in *Rev. Math. Phys.*

[GGM] G. Gallavotti, G. Gentile, V. Mastropietro: Field theory and KAM tori, *Mathematical Physics Electronic Journal* **1**, (5): 1–13 (1995). http://mpej@ math. utexas. edu

[HP] F. Harary, E. Palmer: *Graphical enumeration*, Academic Press, New York, 1973.

[M1] J. Moser: Convergent series expansions for quasi periodic motions, *Math. Ann.* **169**, 136-176 (1967).

[M2] J. Moser: On the construction of almost periodic solutions for ordinary differential equations, *Proceedings of the International Conference of functional analysis and related topics*, Tokyo, 60–67, (1969).

[Pa] I.S. Parasyuk: On instability zones of the Schrödinger equation with a quasi periodic potential, Ukr. Mat. Zh. **30**, 70-78 (1985).

[PF] L.A. Pastur, A. Figotin: *Spectra of random and almost-periodic operators*, Grundlehren der Mathematischen Wissenschaften **297**, Springer, Berlin (1992).

[P] J. Pöschel: Über invarianten Tori on differenzierbaren Hamiltonschen Systemen, *Bonn. Math. Schr.* **120**, 1-120 (1980).

[R] H. Rüssmann: One dimensional Schrödinger equation, *Ann. New York Acad. Sci.* **357**, 90–107 (1980).

[S] C.L. Siegel: Iterations of analytic functions, *Ann. of Math.* **43**, No.4, 607–612 (1943).