Relaxation to equilibrium for two dimensional disordered Ising systems in the Griffiths phase

F. Cesi [1], C. Maes [2], F. Martinelli [3]

[1] Dipartimento di Fisica, Università “La Sapienza”, P.le A. Moro 2, 00185 Roma, Italy.
e-mail: cesi@vaxrom.roma1.infn.it

[2] Instituut voor Theoretische Fysika, K.U. Leuven, Celestijnenlaan 200D B-3001 Leuven and Onderzoeksleider N.F.W.O., Belgium.
e-mail: Christian.Maes@fys.kuleuven.ac.be

[3] Dipartimento di Energetica, Università dell’ Aquila, Italy.
e-mail: martin@mat.uniroma3.it

Dedicated to the memory of Roland Dobrushin

Abstract. We consider Glauber–type dynamics for two dimensional disordered magnets of Ising type. We prove that, if in equilibrium the disorder–averaged influence of the boundary condition is sufficiently small, then the corresponding Glauber dynamics is ergodic with probability one and the disorder–averaged of time–autocorrelations decays like $e^{-m(\log t)^2}$. For the standard dilute Ising ferromagnet with i.i.d. random nearest neighbor couplings taking the values 0 or $J > 0$, our results apply even if the active bonds percolate and J is larger than the critical value for the corresponding pure Ising model. For this model we also rigorously prove the existence of a dynamical phase transition when J crosses the critical value J_c for the standard two dimensional Ising model.

Key Words: Dilute Ising model, Griffiths phase, Glauber dynamics.

Work partially supported by grant CHRX-CT93-0411 of the EEC
Mathematics Subject Classification: 82B44, 82C22, 82C44, 60K35

1. Introduction

Roland Dobrushin certainly was one of the pioneers in the theory of interacting particle systems and in the study of stochastic dynamics for Gibbs fields in particular. From the start (at the end of the 60's) his insights related important concepts in probability theory to those of statistical physics. The mathematical tools he developed have turned out to be extremely useful in several physically relevant problems. The present paper concerns a topic he was very much interested in and to which he contributed many ideas. The general question is the study of Gibbs fields with random interactions. In particular we want to study (see [GM1], [GZ1], [GZ2] and [CMM] for related results) the speed of convergence to equilibrium in a single spin flip, Glauber–type, dynamics, which has a reversible (random) Gibbs measure in the so–called *Griffiths' phase*, namely that region of the phase diagram between the paramagnetic (high–temperature) phase and the phase with long–range order (at least for ferromagnetic systems).

The simplest example of such a system is the dilute Ising ferromagnet below the percolation threshold. In this case the random couplings J_{xy} between nearest neighbor spins take only two values, $J_{xy} = 0$ and $J_{xy} = J$ with probability $1 - p$ and p respectively, with $J > J_c$ and $p < p_c$, J_c and p_c being the critical values for the "pure" Ising model and independent bond percolation in $\mathbb{Z}^d$ respectively. Since $p < p_c$, equilibrium truncated correlations decay exponentially fast (with probability one or averaged over the disorder), but the presence of arbitrarily large connected clusters of the pure system below its critical temperature destroys, for example, the global analyticity of the free energy as a function of the external field (see [Gr], [F] and also [BD], [DKP], [GM2], [GM3] for recent progresse in this direction for systems without dilution).

More interesting but much more difficult to analyze is the case $J > J_c$ and $p > p_c$, since now there is, with probability one, an infinite cluster of low-temperature bonds. It is remarkable that, using the random–cluster representation for the Ising model (see [ACCN] and [N]) and some older results relating averaged correlations to the usual type of correlations for a translation–invariant system (see [OPG]), one can still find, for all $p \geq p_c$, an explicit value $J(p) > J_c$ (see Corollary 4.2 below) such that for all $J_c \leq J < J(p)$ the equilibrium behaviour is qualitatively similar to that *below* the percolation threshold. Moreover the behaviour of $J(p)$ close to p_c is essentially optimal (see [ACCN]).

Much more pronounced are the differences between the relaxational behaviour of Glauber–type dynamics in the paramagnetic and in the Griffiths phase. A rather complete, though not completely rigorous, theory was developed some years ago in [DRS], [B1], [B2] for dilute systems *below* the percolation threshold (see also [RSP]). The result was a predicted asymptotic decay for the disorder–averaged spin autocorrelation $C(t)$ of the form

$$C(t) \approx \exp[-A(\log t)^{\frac{d}{d-1}}] \tag{1.1}$$

to be compared with the expected pure exponential decay in the paramagnetic phase. Notice that such a decay is much slower than a stretched exponential $e^{-t^{\alpha}}$, $\alpha < 1$ being the so–called Kohlrausch exponent, that has been argued to occur for several disordered systems (see [AAPS], [DLO] and [O]). Computer simulations (see [J]) suggest however that the asymptotic behaviour (1.1) can only be seen for very large times and that for intermediate times a stretched exponential decay is more appropriate.

In [CMM] a behaviour very close to (1.1) was established for a general class, not necessarily with dilution, of disordered discrete lattice spin systems, under a rather strong assumption (see assumption (H) in section 2 below) on the distribution of the disorder. Although such an assumption was proved to hold at least in part of the Griffiths phase of some natural models, it is an interesting question to check its validity for dilute models in that part of the Griffiths phase *above* the percolation threshold.

The main scope of the present work is to carry out this analysis for *two dimensional* systems. For Ising–type but not necessarily ferromagnetic models, we will prove that the main hypothesis behind the results of [CMM] holds if the average of the infinite volume two point function of the associated *ferromagnetic* model with couplings $|J_{xy}|$ decays exponentially fast. When applied to the standard dilute Ising model this result, combined with the bounds of [ACCN], [N] and [OPG], allows us to extend (1.1) to a non trivial part of the Griffiths phase above p_c. For this model we also rigorously establish the existence of a dynamical phase transition as soon as one crosses the critical temperature of the pure system (see [DRS] for earlier results in this direction). Our limitation on the dimension, $d = 2$, comes from the fact that we need in a essential way a recent beautiful extension to random systems (see [Be]) of the results of [MOS].

2. Definition of the model and review of previous results

In this section we will briefly review the models of disordered magnets and the results for the relaxation to equilibrium of an associated Glauber–type dynamics discussed in [CMM] in the so–called Griffiths phase. Although the situation considered in [CMM] was slightly more general, we will restrict ourselves here to Ising–like lattice spin systems with random nearest neighbor interactions and random magnetic field.

2.1. The model

We consider the d dimensional lattice $\mathbb{Z}^d$ whose vertices are called *sites* and the set $\hat{\mathbb{Z}}^d$ of unordered pairs $b = \{x, y\} = \{y, x\}$ of sites x, y in $\mathbb{Z}^d$, with Euclidean distance $d(x, y) \equiv |x - y| = 1$.

By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that for each i, $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. A finite subset Λ of $\mathbb{Z}^d$ is said to be a "multiple" of Q_L, if Λ is the union of a finite number of cubes $Q_L(x_i)$ where $x_i \in L\mathbb{Z}^d$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$ and we denote by $\hat{\Lambda}$ the set of all bonds $b = \{x, y\} \in \hat{\mathbb{Z}}^d$ such that either x or y belong to Λ. The cardinality of Λ is denoted by $|\Lambda|$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{-1, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \eta_V$ is the configuration on $U \cup V$ which is equal to σ on U and η on V. Moreover, for any bond $b = \{x, y\}$, we set $\sigma_b = \sigma(x)\sigma(y)$.

If f is a function on Ω, Λ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Λ_f}. f is called *local* if Λ_f is finite. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x.

The interaction and the Gibbs measures. We consider an abstract probability space $(\Theta, \mathcal{B}, \mathbb{P})$ and two sets of i.i.d real valued random variables indexed by the sites and bonds of $\mathbb{Z}^d$: $h = \{h_x\}_{x \in \mathbb{Z}^d}$ and $J = \{J_b\}_{b \in \hat{\mathbb{Z}}^d}$. We assume (see hypothesis (H3) in [CMM]) that

(A) there exists $J_\infty > 0$ and $h_\infty \geq 0$ such that $\mathbb{P}\{|J_b| > J_\infty \text{ or } |h_x| > h_\infty\} = 0$.

We denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}(\cdot)$. For each $V \subset\subset \mathbb{Z}^d$ we define the Hamiltonian $H_V : \Omega \mapsto \mathbb{R}$ by

$$H_V^{J,h}(\sigma) = -\sum_{b \in \hat{V}} J_b \sigma_b - \sum_{x \in V} h_x \sigma(x)$$

We also set, for $\sigma, \tau \in \Omega$

$$H_V^{J,h,\tau}(\sigma) = H_V^{J,h}(\sigma_V \tau_{V^c}) \tag{2.1}$$

and τ is called the *boundary condition.* For each $V \subset\subset \mathbb{Z}^d$ and $\tau \in \Omega$ we define the (finite volume) conditional Gibbs measure by

$$\mu_V^{J,h\tau}(\sigma) = \begin{cases} \left(Z_V^{J,h,\tau}\right)^{-1} \exp[\,-H_V^{J,h,\tau}(\sigma)\,] & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

where $Z_V^{J,h,\tau}$ is the proper normalization factor called partition function. Notice that we have absorbed the inverse temperature β in the Hamiltonian. We will sometimes drop the superscripts J, h if that does not generate confusion. Given a measurable bounded function f on Ω, $\mu_V^\tau(f)$ denotes its average and $\mu_V^\tau(f, g)$ stands for the covariance of f and g. Finally, given $\Delta \subset \Lambda \subset \mathbb{Z}^d$ we denote by $\mu_{\Lambda,\Delta}^\tau$ the projection of the Gibbs measure μ_Λ^τ over the set Δ that is

$$\mu_{\Lambda,\Delta}^\tau(\sigma_\Delta) \equiv \mu_\Lambda^\tau(\eta : \, \eta_\Delta = \sigma_\Delta)$$

The dynamics. The stochastic dynamics we want to study is determined by the Markov generators $L_V^{J,h}$, $V \subset\subset \mathbb{Z}^d$, defined by

$$(L_V^{J,h} f)(\sigma) = \sum_{x \in V} c_{J,h}(x, \sigma)(\nabla_x f)(\sigma) \tag{2.3}$$

The nonnegative real quantities $c_{J,h}(x, \sigma)$, $x \in \mathbb{Z}^d$, $\sigma \in \Omega$, are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Finite range.* If $\sigma(y) = \sigma'(y)$ for all y such that $|x - y| \le 1$, then $c_{J,h}(x, \sigma) = c_{J,h}(x, \sigma')$

(2) *Detailed balance.* For all $\sigma \in \Omega$ and $x \in \mathbb{Z}^d$,

$$\exp\bigl[-H_{\{x\}}^{J,h}(\sigma)\bigr] c_{J,h}(x, \sigma) = \exp\bigl[-H_{\{x\}}^{J,h}(\sigma^x)\bigr] c_{J,h}(x, \sigma^x) \tag{2.4}$$

(3) *Positivity and boundedness.* There exist non–negative real numbers c_m and c_M such that for all J

$$c_m \le \inf_{x,\sigma,J,h} c_{J,h}(x, \sigma) \qquad \text{and} \qquad \sup_{x,\sigma} c_{J,h}(x, \sigma) \le c_M \tag{2.5}$$

Two cases one may want to keep in mind are

$$c_{J,h}(x, \sigma) = \min\bigl\{e^{-(\nabla_x H_{\{x\}})(\sigma)}, \, 1\bigr\} \tag{2.6}$$

$$c_{J,h}(x, \sigma) = \mu_{\{x\}}^{J,\sigma}(\sigma^x) = \Bigl[1 + e^{(\nabla_x H_{\{x\}})(\sigma)}\Bigr]^{-1} \tag{2.7}$$

corresponding to the Metropolis and heat–bath dynamics respectively.

We denote by $L_V^{J,h,\tau}$ the operator $L_V^{J,h}$ acting on $L^2(\Omega, d\mu_V^{J,h,\tau})$ (this amounts to choose τ as the boundary condition). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L_V^{J,h,\tau}$, and whose semigroup we denote by $\{T_V^{J,h,\tau}(t)\}_{t\geq 0}$. $L_V^{J,h,\tau}$ is actually a bounded selfadjoint operator on $L^2(\Omega, d\mu_V^{J,h,\tau})$. The process has a unique invariant measure given by $\mu_V^{J,h,\tau}$. Moreover $\mu_V^{J,h,\tau}$ is *reversible* with respect to the process, *i.e.* $L_V^{J,h,\tau}$ is self–adjoint on $L^2(\Omega, d\mu_V^{J,h,\tau})$.

Infinite volume dynamics. Let μ be a Gibbs measure for the interaction J, h. Since the transition rates are bounded, then the infinite volume generator $L^{J,h}$ obtained by choosing $V = \mathbb{Z}^d$ in (2.3) is well defined on the set of functions f in $L^2(\Omega, d\mu)$ (or $C(\Omega)$) such that $|||f||| \equiv \sum_{x\in\mathbb{Z}^d} \|\nabla_x f\|_\infty$ is finite. The closure of $L^{J,h}$ in $L^2(\Omega, d\mu)$ ($C(\Omega)$) is a Markov generator (see, for instance Theorems 3.9 in Chapter I and 4.1 in Chapter IV of [L]), which defines a Markov semigroup denoted by $T(t)$. Again $L^{J,h}$ is self–adjoint on $L^2(\Omega, d\mu)$.

2.2. Main results on the dynamics

In [CMM] one main result on the speed of relaxation of the dynamics to its equilibrium Gibbs measure has been proved under a key probabilistic assumption on the Gibbs measure with random interactions. In order to state this hypothesis we need first the following definition. Given $V \subset\subset \mathbb{Z}^d$, $n \in \mathbb{Z}_+$ and $\alpha > 0$, we say that the condition $SMT(V, n, \alpha)$ holds if for all local functions f and g on Ω such that $d(\Lambda_f, \Lambda_g) \geq n$ we have

$$\sup_{\tau\in\Omega} |\mu_V^\tau(f, g)| \leq |\Lambda_f||\Lambda_g| \|f\|_\infty \|g\|_\infty \exp(-\alpha d(\Lambda_f, \Lambda_g))$$

Then the main hypothesis of [CMM] can be formulated as follows:

(H) There exist $L_0 \in \mathbb{Z}_+$, $\alpha > 0$, $\vartheta > 0$ such that for all $L \geq L_0$

$$\mathbb{P}\{\, SMT(Q_L, L/2, \alpha)\,\} \geq 1 - e^{-\vartheta L}$$

Remark 1. It is not difficult to check that the main result of [CMM] (see theorem 2.1 below) follows even if in hypothesis (H) we replace "for all $L \geq L_0$ " with the slightly weaker "for all L multiple of L_0 ". It is actually this (apparently) weaker form of (H) that we will show to be implied by a simpler assumption (see theorem 3.4 below).

Remark 2. Some of the results of [CMM] were obtained under a more general assumption on the distribution of J and h than boundedness, namely $\mathbb{E}\{e^{|J_b|^{1+\delta}}\} < \infty$ and $\mathbb{E}\{e^{|h_x|^{1+\delta}}\} < \infty$ for some positive δ. Here we prefer to work directly in the bounded case since the latter is more relevant from the physical point of view and the results on the dynamics are more transparent.

The main theorem of [CMM] about the speed of relaxation to equilibrium for the model described above is as follows.

Theorem 2.1. *Assume (H). Then*

(a) *If $d \ge 1$ there exists a set $\bar\Theta \subset \Theta$ of full measure such that for each $J, h \in \bar\Theta$ there exists a unique infinite volume Gibbs measure $\mu^{J,h}$. Moreover there exists a constant k and, for each $J, h \in \bar\Theta$ and for any local function f there exists $t_0(J,h,f) < \infty$ such that for all $t \ge t_0$*

$$\|T^{J,h}(t)f - \mu^{J,h}(f)\|_\infty \le \exp\Big[-t\,\exp\big[-k\,(\log t)^{1-\frac{1}{d}}\,(\log\log t)^{d-1}\big]\Big] \tag{2.8}$$

(b) *Let $d \ge 2$. Then there exists a constant k and for any local function f there exists $t_0(f) < \infty$ such that, if $t \ge t_0(f)$ then*

$$\mathbb{E}\,\|T^{J,h}(t)f - \mu^{J,h}(f)\|_\infty \le \exp\big[-k\,(\log t)^{\frac{d}{d-1}}\,(\log\log t)^{-d}\big] \tag{2.9}$$

(c) *Let $h \equiv 0$ and let $\pi_0(\sigma) = \sigma(0)$. For each $d \ge 2$ there exist $J_1(d) > 0$ such that if $p_1 \equiv \mathbb{P}\{J_{xy} = J_1\} > 0$ and $p_2 \equiv \mathbb{P}\{|J_{xy}| \le 1/4\} > 0$ then we have, for all large enough t,*

$$\mathbb{E}\,\|T^J(t)\pi_0 - \mu^J(\pi_0)\|_{L^2(\mu^J)} \ge \exp\big[-k\,(\log t)^{\frac{d}{d-1}}\big] \tag{2.10}$$

for some k which depends on d, p_1 and p_2.

3. An alternative assumption to hypothesis (H) in two dimensions.

We present here, for *two dimensional* systems, an alternative assumption to the basic hypothesis (H) of [CMM]. The advantage of this new assumption is that in some concrete case, like the dilute Ising ferromagnet, it can be explicitly verified in interesting region of the phase diagram. The idea behind what follows is essentially based on a result obtained in [MOS] for two dimensional non–random systems and recently extended in [Be] to random ones. In order to formulate the main result of this section we first need some additional definitions.

Definition 3.1. *Given $C > 0$, $m > 0$ and $\Lambda \subset \mathbb{Z}^d$, we say that $WME(\Lambda, C, m)$ holds if for all $\Delta \subset \Lambda$*

$$\mathbb{E}\big\{\sup_{\tau,\tau'} \mathrm{Var}(\mu^\tau_{\Lambda,\Delta}, \mu^{\tau'}_{\Lambda,\Delta})\big\} \le C \sum_{x\in\Delta} e^{-m d(x,\Lambda^c)}$$

where Var *denotes the variation distance. We say that $WME(C,m)$ holds if $WME(\Lambda, C, m)$ holds for all $\Lambda \subset \mathbb{Z}^d$.*

Definition 3.2. *Given $C > 0$, $m > 0$ and $\Lambda \subset \mathbb{Z}^d$, we say that $SME(\Lambda, C, m)$ holds if for all $\Delta \subset \Lambda$ and all $y \in \Lambda^c$*

$$\mathbb{E}\big\{\sup_{\tau,\tau^y} \mathrm{Var}(\mu^\tau_{\Lambda,\Delta}, \mu^{\tau^y}_{\Lambda,\Delta})\big\} \le C \sum_{x\in\Delta} e^{-m|x-y|}$$

Using these notions we have the following key result [Be] (see also [MOS] for a similar statement in case of non–random systems).

Theorem 3.3. *Let $d = 2$. Then $WME(C, m)$ implies that there exists $L_0 \in \mathbb{Z}_+$, $m' > 0$ and $C' > 0$ such that $SME(\Lambda, C', m')$ holds for all sets Λ multiple of Q_{L_0}.*

Remark. Actually in [Be] the above result is proved under the hypothesis that $WME(\Lambda, C, m)$ holds for all rectangles Λ.

We are finally in a position to give our alternative to assumption (H).

Lemma 3.4. *Let $d = 2$ and let $\varepsilon \in (0, 1)$. Then $WME(C, m)$ implies that there exists $L_0 \in \mathbb{Z}_+$, $\alpha > 0$ and $\vartheta > 0$ such that for all integers L multiple of L_0*

$$\mathbb{P}\{SMT(Q_L, \varepsilon L, \alpha)\} \geq 1 - e^{-\vartheta L}$$

Proof. Let $l_0 < l_1 < L$ be three integers such that l_1 and L are both multiples of l_0 and suppose moreover that $L^{2/3} \leq l_1 \leq \varepsilon L/4$. Let us partition $\Lambda \equiv Q_L$ into disjoint rectangles as follows

$$\Lambda = \cup_{i=1}^{n} B_i\,, \qquad B_i = Q_{l_1}(x_i) \cap \Lambda; \qquad x_i \in l_1\mathbb{Z}^2$$

and let $\Lambda_I = \cup_{i \in I} B_i \quad \forall I \subset \{1 \dots n\}$. Finally we denote with $\mathcal{I}_\varepsilon$ the set of all pairs (I_1, I_2) with $I_1 \subset I_2 \subset \{1 \dots n\}$ such that $d(\Lambda_{I_1}, \Lambda \setminus \Lambda_{I_2}) \geq \varepsilon L/2$

Next, for a given $\alpha > 0$, we define $\Theta(\alpha, \varepsilon)$ as the set of those couplings such that for all pairs $(I_1, I_2) \in \mathcal{I}_\varepsilon$ one has

$$\sup_{y \in \Lambda \setminus \Lambda_{I_2}} \sup_{\tau} \mathrm{Var}(\mu^{\tau}_{\Lambda_{I_2}, \Lambda_{I_1}}, \mu^{\tau^y}_{\Lambda_{I_2}, \Lambda_{I_1}}) \leq e^{-2\alpha L} \tag{3.1}$$

We claim that for all $J \in \Theta(\alpha, \varepsilon)$ property $SMT(Q_L, \varepsilon L, \alpha)$ holds, provided that L is large enough depending only on α. Given in fact two arbitrary functions f and g, with $d(\Lambda_f, \Lambda_g) \geq \varepsilon L$, let us set

$$I_f = \{i \in \{1 \dots n\} : \ B_i \cap \Lambda_f \neq \emptyset\} \qquad I_f^c = \{1 \dots n\} \setminus I_f$$

and similarly for g. Notice that, because of our choice of l_1, we have

$$d(\Lambda_{I_f}, \Lambda_{I_g}) \geq \varepsilon L - 2l_1 \geq \varepsilon L/2 \tag{3.2}$$

Now we write, using the DLR equations

$$\mu^{\tau}_{\Lambda}(f, g) = \sum_{(\sigma, \sigma')} \mu^{\tau}_{\Lambda}(\sigma) \mu^{\tau}_{\Lambda}(\sigma') f(\sigma) [\mu^{\sigma}_{\Lambda_{I_f^c}}(g) - \mu^{\sigma'}_{\Lambda_{I_f^c}}(g)] \tag{3.3}$$

so that, using a standard interpolation argument (see e.g. the proof of proposition 4.3 in [CMM]),

$$\sup_\tau |\mu^\tau_\Lambda(f,g)| \le \|f\|_\infty \|g\|_\infty \sum_{y\in\Lambda_{I_f}} \sup_\tau \mathrm{Var}(\mu^\tau_{\Lambda_{I_f^c},\Lambda_{I_g}}, \mu^{\tau^y}_{\Lambda_{I_f^c},\Lambda_{I_g}}) \tag{3.4}$$

Notice that, because of (3.2), the pair (I_f^c, I_g) belongs to $\mathcal{I}_\varepsilon$. Thus, thanks to (3.1), we get

$$\text{RHS of (3.4) } \le \|f\|_\infty\|g\|_\infty L^2 e^{-2\alpha L} \le \|f\|_\infty\|g\|_\infty |\Lambda_f||\Lambda_g| e^{-\alpha d(\Lambda_f,\Lambda_g)} \tag{3.5}$$

namely $SMT(\Lambda, \varepsilon L, \alpha)$, for any $J \in \Theta(\alpha,\varepsilon)$ and any L large enough. In conclusion we have proved that, for any L large enough,

$$\mathbb{P}\big\{SMT(Q_L, \varepsilon L, \alpha) \text{ holds }\big\} \ge \mathbb{P}\big\{\Theta(\alpha,\varepsilon)\big\}$$

Let us estimate from above $\mathbb{P}\big\{\Theta(\alpha,\varepsilon)^c\big\}$. We write

$$\mathbb{P}\big\{\Theta(\alpha,\varepsilon)^c\big\} \le |\mathcal{I}_\varepsilon|^2 \sup_{(I_1,I_2)\in\mathcal{I}_\varepsilon} |\Lambda\setminus\Lambda_{I_2}| \sup_{y\in\Lambda\setminus\Lambda_{I_2}} \mathbb{P}\big\{\sup_\tau \mathrm{Var}(\mu^\tau_{\Lambda_{I_2},\Lambda_{I_1}}, \mu^{\tau^y}_{\Lambda_{I_2},\Lambda_{I_1}}) \ge e^{-2\alpha L}\big\} \tag{3.6}$$

Using the standard Chebyshev inequality we can bound from above the probability appearing in the r.h.s of (3.6) by

$$\mathbb{P}\big\{\sup_\tau \mathrm{Var}(\mu^\tau_{\Lambda_{I_2},\Lambda_{I_1}}, \mu^{\tau^y}_{\Lambda_{I_2},\Lambda_{I_1}}) \ge e^{-2\alpha L}\big\} \le e^{2\alpha L}\mathbb{E}\big\{\sup_\tau \mathrm{Var}(\mu^\tau_{\Lambda_{I_2},\Lambda_{I_1}}, \mu^{\tau^y}_{\Lambda_{I_2},\Lambda_{I_1}})\big\} \tag{3.7}$$

We are finally in a position to use our key hypothesis $WME(C,m)$. We know in fact, thanks to theorem 3.3, that $WME(C,m)$ implies that there exist L_0, C' and $m' > 0$ such that $SME(C',m')$ holds for all sets that are multiples of the square $Q_{L_0}(0)$. Thus, if we choose $l_0 = L_0$, we get that all sets of the form Λ_I for some $I \subset \{1\dots n\}$ are "multiple " of $Q_{L_0}(0)$ since l_1 and L are both multiple of l_0. Thus, using the definition of $SME(C',m')$, we get that

$$\text{RHS of (3.7) } \le L^2 e^{2\alpha L} C' e^{-m'\varepsilon L/2}$$

If we finally plug the above bound into the r.h.s of (3.6) and use the trivial bounds

$$|\mathcal{I}_\varepsilon| \le 2^{(L/l_1)^2} \le 2^{L^{2/3}}, \qquad \sup_I |\Lambda\setminus\Lambda_I| \le L^2$$

we obtain

$$\mathbb{P}\big\{\Theta(\alpha,\varepsilon)^c\big\} \le 2^{L^{2/3}} L^4 e^{2\alpha L} C' e^{-m'\varepsilon L/2} \tag{3.8}$$

The theorem now follows with e.g. $\vartheta = \alpha$ and $4\alpha = \varepsilon m'$. □

4. Applications

We have shown in lemma 3.4 that in $d = 2$ the hypothesis (H) of theorem 2.1 can be replaced by $WME(C, m)$. In this final section we will check condition $WME(C, m)$ for the dilute Ising model (ferromagnetic or not). In particular we show that, for the standard ferromagnetic case without external field h in which the random couplings J_b assume only two values, $J_b = J_0$ with probability p and $J_b = 0$ with probability $1-p$, the results on the dynamics explained in section 2.2 apply even when $p > p_c$ and J_0 is larger (but not too large) than the critical value J_c for the corresponding homogeneous system. Here $p_c = 1/2$ is the critical value for Bernoulli bond percolation in two dimensions.

Theorem 4.1. *Assume*

(a) $h_x = 0$ *for all* $x \in \mathbb{Z}^d$

or

(b) $h_x \geq 0$ *and* $J_b \geq 0$ *for all* $x \in \mathbb{Z}^d$ *and all* $b \in \hat{\mathbb{Z}}^d$.

Then any of the following two conditions implies that there exist two positive constants C *and* m *such that* $WME(C, m)$ *holds.*

(i) $\mathbb{E}\{|J_{xy}|\} < J_c$

(ii) $\mathbb{E}\{(1 - e^{-|J_{xy}|})\} < p_c$

where J_c *and* p_c *are the critical inverse temperature of the ordinary Ising model* ($h_x \equiv 0$ *and* $J_b = J > 0$ *in (2.1)) and the critical bond percolation probability in* d*–dimensions respectively.*

Proof of Theorem 4.1.

(a) Using the Fortuin-Kasteleyn representation for the Ising model in absence of external field, one can prove the following very nice inequality (see [N])

$$\sup_{\tau,\tau'} \mathrm{Var}(\mu_{\Lambda,\Delta}^{J,\tau}, \mu_{\Lambda,\Delta}^{J,\tau'}) \leq 2 \sum_{x\in\Delta} \mu_\Lambda^{|J|,+}(\sigma(x)) \tag{4.1}$$

where $\mu_\Lambda^{|J|,+}$ denotes the Gibbs measure in Λ with plus boundary conditions and couplings $\{|J_b|\}_{b\in\hat{\Lambda}}$. Since the Gibbs measure $\mu_\Lambda^{|J|,+}$ is ferromagnetic, we can use at this point an inequality in Section 2 of [Hi] which states that

$$\mu_\Lambda^{|J|,+}(\sigma(x)) \leq \sum_{y\notin\Lambda} \sum_{\substack{z\in\Lambda \\ |z-y|=1}} \mu_\Lambda^{|J|,f}(\sigma(x)\sigma(z)) \tag{4.2}$$

where the superscript f in $\mu_\Lambda^{|J|,f}$ denotes free boundary conditions.

Thus, in order to prove the theorem, it is sufficient to show that both (i) or (ii) imply that there exist two positive constants C and m such that

$$\mathbb{E}\big\{\mu_\Lambda^{|J|,f}(\sigma(x)\sigma(z))\big\} \le Ce^{-m|x-z|} \tag{4.3}$$

Let us start with (i). In this case we can use an old result [OPG] that states that

$$\mathbb{E}\big\{\mu_\Lambda^{|J|,f}(\sigma(x)\sigma(z))\big\} \le \mu_\Lambda^{\mathbb{E}\{|J|\},f}(\sigma(x)\sigma(z)) \tag{4.4}$$

where $\mu^{\mathbb{E}\{|J|\},f}$ denotes standard Ising Gibbs measure in Λ with zero external field, free boundary conditions and coupling $\mathbb{E}\{|J_b|\}$. Thanks to the second Griffiths inequality

$$\text{R.H.S. of (4.4)} \le \mu^{\mathbb{E}\{|J|\}}(\sigma(x)\sigma(z)) \le Ce^{-m|x-z|} \tag{4.5}$$

for suitable C and m. Here $\mu^{\mathbb{E}\{|J|\}}$ denotes the infinite volume limit as $\Lambda \to \mathbb{Z}^d$ of $\mu_\Lambda^{\mathbb{E}\{|J|\},f}$ and in the last inequality in the r.h.s of (4.5) we used the fact that $\mathbb{E}\{|J|\} < J_c$ implies exponential decay of infinite volume correlations (see [ABF]).

Let us now consider case (ii). Here we can use another result on the FK representation for the random Ising model (see [ACCN] and also [N]) which states that

$$\mathbb{E}\big\{\mu_\Lambda^{|J|,f}(\sigma(x)\sigma(z))\big\} \le \text{Prob}_p(x \to z) \tag{4.6}$$

where $p \equiv \mathbb{E}\big\{(1 - e^{-\beta|J_{xy}|})\big\}$ and $\text{Prob}_p(x \to z)$ denotes the probability for independent bond percolation with occupation density p that there exists a path of occupied bonds starting in x and ending in z. Since by assumption $p < p_c$ this last probability is exponentially small in $|x - y|$ (see e.g. [Gri] and references therein) and (4.3) follows.

(b) In the ferromagnetic case $J_b \ge 0$ with non–negative external field h_x, it has been proved in section 2 of [Hi] that

$$\sup_{\tau,\tau'} \text{Var}(\mu_{\Lambda,\Delta}^{J,\tau}, \mu_{\Lambda,\Delta}^{J,\tau'}) \le 2 \sum_{x\in\Delta} \mu_\Lambda^{J,h=0,+}(\sigma(x)) \tag{4.7}$$

At this point we proceed exactly as before. $\square$

As a corollary to the above theorem we get the following result for the two dimensional dilute Ising ferromagnet.

Corollary 4.2. *In the same setting of theorem 4.1, let us consider the two dimensional case and let us assume that the couplings J_b take only the values $J_b = 0$ and $J_b = J_0$ with probability $1-p$ and p respectively (dilute Ising model). Then if*

$$J_0 < \begin{cases} \frac{J_c}{p} \vee \log(\frac{p}{2p-1}) & \text{if } p > 1/2 \\ \infty & \text{if } p \le 1/2 \end{cases}$$

hypothesis (H) holds and theorem 2.1 applies.

Proof of the Corollary. If J_0 is smaller than the stated value then either (i) or (ii) of theorem 4.1 holds since $p_c = 1/2$ for $d = 2$. Thus in two dimensions we can apply theorem 3.4 above and get the result. □

Remark. Using the Fortuin–Kesteleyn representation, it has also been shown in [ACCN] that the dilute Ising magnet exihibits the phenomenon of spontaneous magnetization provided that

$$\mathbb{E}\left(\frac{1-e^{-J_b}}{1+e^{-J_b}}\right) > p_c$$

so that the infinite volume Gibbs state is non–unique and the associated dynamics is no longer ergodic.

We conclude by observing that if J_0 is smaller than the critical value J_c, then, thanks to the Griffiths inequality,

$$\mu_\Lambda^{J,h=0,f}(\sigma(x)\sigma(z)) \le Ce^{-m|x-z|}$$

for *any* configuration of J. Thus, in this case, we can apply theorem 3.1 of [MO1] and get that the relaxation to equilibrium is purely exponential for *any* configuration of J and any value of p. In order to show that a dynamical phase transition occurs when J_0 crosses the critical value J_c, we need a lower bound like the one given in theorem 2.1 for *any* $J_0 > J_c$. This is actually our last result.

Proposition 4.3. *Under the same hypotheses of corollary 4.2 let us also suppose that $J_0 > J_c$. Let $f(\sigma) = \sigma(0)$. Then for any $\delta > 0$ there exists $t_0 < \infty$ such that for all $t \ge t_0$*

$$\mathbb{E}\,\|T^J(t)f - \mu^J(f)\|_{L^2(\mu^J)} \ge e^{-k(\delta,p,J_0)(logt)^2}$$

where $k(\delta,p,J_0) = \log(1/p)(J_0\tau(J_0)(1-3\delta))^{-2}$ with $\tau(J_0)$ the surface tension at inverse temperature J_0 in the direction of the coordinate axes for the standard ferromagnetic two dimensional Ising model.

Proof. We follow closely the proof of the lower bound of theorem 3.3 given in [CMM]. Since J_0 satisfies the hypothesis of corollary 4.2, there exists a unique Gibbs state μ^J with zero spontaneous magnetization. Therefore, in order to prove the result, it is sufficient to prove the required lower bound only for $\mathbb{E}\,\|T^J(t)f\|_{L^2(\mu^J)}$. For this purpose let, for $\Lambda = Q_L$, $\bar\Theta$ be the set of all interactions $J \in \Theta$ such that

(a) $J_{xy} = J_0$ for all $\{x, y\}$ such that $\{x, y\} \subset \Lambda$
(b) $J_{xy} = 0$ for all $\{x, y\}$ which intersect both Λ and Λ^c (the boundary edges)

and let

$$F_\Lambda = \{\sigma \in \Omega :\, m_\Lambda(\sigma) > \tfrac{1}{2}\}$$

where $m_\Lambda(\sigma)$ denotes the normalized magnetization in Λ of the configuration σ. We can then use the following estimate that was established in [CMM] (see formula (6.25)... (6.28) there)

$$\begin{aligned}&\mathbb{E}\,\|T^J(t)f\|_{L^2(\mu^J)} \geq \\ &\geq \mathbb{P}(\bar\Theta)\left[\frac{1}{2} - \frac{3}{2}\mu_\Lambda^{J_0,f}(F_\Lambda)^{-1}\left(k\,|\Lambda|\,t\,\mu_\Lambda^{J_0,f}\{|m_\Lambda(\sigma) - 1/2| \leq 1/(100)\} + e^{-k'|\Lambda|t}\right)\right]\end{aligned} \tag{4.8}$$

for suitable positive constants k, k'. Here $\mu_\Lambda^{J_0,f}$ denotes the ordinary Ising Gibbs measure in Λ with free boundary conditions and coupling J_0. We need at this point the following large deviation result (see [CGMS])

Theorem 4.4. *Let $J_0 > J_c$. Then for any $\delta > 0$ there exists $L_0(\delta)$ such that for all $L \geq L_0(\delta)$*

$$\mu_\Lambda^{J_0,f}\{\,|m_\Lambda(\sigma) - 1/2| \leq 1\,\} \leq e^{-J_0\tau(J_0)(1-\delta)L} \quad ; \quad \mu_\Lambda^{J_0,f}\{\,m_\Lambda(\sigma) \geq 1/2\,\} \geq 1/3$$

where $\tau(J_0)$ denotes the surface tension at inverse temperature J_0 in the direction of the coordinate axes for the ordinary two dimensional Ising model.

Choose now $\delta < 1/4$. Using theorem 4.4, we see that if t is large enough and L is larger than $J_0\tau(J_0)(1-2\delta)^{-1}\log t$ then

$$\text{RHS of (4.8)} \geq \frac{1}{3}p^{L^2}(1-p)^{4L} \geq e^{-k(\delta,p,J_0)(\log t)^2} \quad \square$$

References

[AAPS] R. G. Palmer, D. L. Stein, E. Abrahams and P. W. Anderson: *Models of hierarchically constrained dynamics for glassy relaxation*. Phys. Rev. Lett. **53**, No 10, 958, (1984)

[ABF] M. Aizenman, D. J. Barsky and R. Fernandez: *The phase transition in a general class of Ising–type models is sharp*. Journ. Stat. Phys. **47**, No. 3/4, 343, (1987)

[ACCN] M. Aizenman, J. T. Chayes, L. Chayes and C. M. Newman: *The phase boundary in dilute and random Ising and Potts ferromagnets*. J.Phys.A: Math. Gen. **20**, L313, (1987)

[BD] L. A. Bassalygo and R. L. Dobrushin: *Uniqueness of a Gibbs field with random potential– an elementary approach*. Theory Prob. Appl. **31**, No 4, 572, (1986)

[B1] A. J. Bray: *Upper and lower bounds on dynamic correlations in the Griffiths phase*. J.Phys.A: Math. Gen. **22**, L81, (1989)

[B2] A. J. Bray: *Dynamics of dilute magnets above T_c* . Phys. Rev. Lett. **60**, No 8, 720, (1988)

[Be] J. van den Berg: *A constructive mixing condition for 2-D Gibbs measures with random interactions*. Preprint 1996

[CGMS] F. Cesi, G. Guadagni, F. Martinelli and R. H. Schonmann: *On the 2D stochastic Ising model in the phase coexistence region close to the critical point*. to appear in Journ. Stat. Phys.

[CMM] F. Cesi, C. Maes and F. Martinelli: *Relaxation of disordered magnets in the Griffiths phase*. Roma preprint 1996

[DLO] C. De Dominicis, H. Orland and F. Lainée: *Stretched exponential in systems with random free energies*. J.Physique Lett. **46**, L–463, (1985)

[DRS] D. Dhar, M. Randeria and J. P. Sethna: *Griffiths singularities in the dynamics of disordered Ising models*. Europhys. Lett., **5**, No 6, 485, (1988)

[DKP] H. Von Dreyfus, A. Klein and J. F. Perez: *Taming Griffiths singularities: infinite differentiability of quenched correlations functions*. Comm. Math. Phys. **170**, 21, (1995)

[F] J. Fröhlich: *Mathematical aspects of the physics of disordered systems*. in “Critical Phenomena, Random Systems, Gauge Theories” Eds. K.Osterwalder and R. Stora, Elsevier (1986)

[GM1] G. Gielis and C. Maes: *Percolation techniques in disordered spin flip dynamics: relaxation to the unique invariant measure*. Comm.Math.Phys. **177**, 83, (1995)

[GM2] G. Gielis and C. Maes: *Local analyticity and bounds on the truncated correlation functions in disordered systems*. Markov Processes Relat. Fields **1**, 459, (1995)

[GM3] G. Gielis and C. Maes: *The uniqueness regime of Gibbs fields with unbounded disorder*. J. Stat. Phys. **81**, No. 3/4, 829, (1995)

[Gr] R. Griffiths: *Non–analytic behaviour above the critical point in a random Ising ferromagnet.* Phys. Rev. Lett. **23**, 17, (1969)

[Gri] G. Grimmett: *Percolation.* Springer–Verlag, (1989)

[GZ1] A. Guionnet and B. Zegarlinski: *Decay to equilibrium in random spin systems on a lattice.* Preprint 1996

[GZ2] A. Guionnet and B. Zegarlinski: *Decay to equilibrium in random spin systems on a lattice II.* Preprint 1996

[J] S. Jain: *Anomalously slow relaxation in the diluted Ising model below the percolation threshold.* Physica A, **218**, 279, (1995)

[Hi] Y. Higuchi: *Coexistence of infinite (*)–clusters II:–Ising percolation in two dimension.* Prob. Th. Rel. Fields, **97**, 1, (1993)

[L] T. M. Liggett: *Interacting particles systems.* Springer–Verlag (1985)

[M] F. Martinelli: *On the two dimensional dynamical Ising model in the phase coexistence region.* J. Stat. Phys. **76**, No. 5/6, 1179 (1994)

[MO1] F. Martinelli and E. Olivieri: *Approach to equilibrium of Glauber dynamics in the one phase region I: The attractive case.* Commun. Math. Phys. **161**, (1994)

[MOS] F. Martinelli, E. Olivieri and R.H.Schonmann: *For Gibbs state of 2D lattice spin systems weak mixing implies strong mixing.* Commun. Math. Phys. **165**, 33 (1994)

[N] C. M. Newman: *Disordered Ising systems and random cluster representation.* "Probability and Phase Transition" edited by G.Grimmet, NATO ASI Series vol. 120 (1993)

[O] A. T. Ogielski: *Dynamics of three dimensional Ising spin glasses in thermal equilibrium.* Phys. Rev. B, **32**, No 11, 7384, (1985)

[OPG] E. Olivieri, F. Perez and S. Goulart–Rosa–Jr.: *Some rigorous results on the phase diagram of the dilute Ising model.* Phys. Lett. **94A**, No 6,7, 309, (1983)

[RSP] M. Randeria, J. P. Sethna and R. G. Palmer: *Low–frequency relaxation in Ising spin–glasses.* Phys. Rev. Lett. **54**, No. 12, 1321, (1985)