DUALITY AND SINGULAR CONTINUOUS SPECTRUM IN THE ALMOST MATHIEU EQUATION

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Abstract. We study the almost Mathieu operator $(h_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(\pi\alpha n + \theta)u(n)$ on $\ell^2(\mathbb{Z})$, and prove that the dual of point spectrum is absolutely continuous spectrum. We use this to show that for $\lambda = 2$ it has purely singular continuous spectrum for a.e. pairs (α, θ). The α's for which we prove this are explicit.

§1. Introduction

Our main goal in this paper is to study the almost Mathieu operator on $\ell^2(\mathbb{Z})$ defined by

$$(h_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(\pi\alpha n + \theta)u(n). \tag{1.1}$$

Our results in Section 2 on measurability of (normalizable) eigenfunctions may be of broader applicability. For background on (1.1), see [19,23].

Our main result here concerns (1.1) at the self-dual point $\lambda = 2$.

Theorem 1. *Let α be an irrational so there exist $q_n \to \infty$ and p_n in $\mathbb{Z}$ with*

$$q_n^2 \left|\alpha - \frac{p_n}{q_n}\right| \to 0 \tag{1.2}$$

as $n \to \infty$. Then for a.e. θ, $h_{\lambda=2,\alpha,\theta}$ has purely singular continuous spectrum.

Remarks. 1. (1.2) is used because for such α, Last [22] has proven that the spectrum, $\sigma_{\lambda,\alpha}$, of $h_{\lambda,\alpha,\theta}$ (which is θ independent [4]) has $|\sigma_{2,\alpha}| = 0$ (where $|\cdot|$ denotes

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Lebesgue measure). Our proof is such that for any other α with $|\sigma_{\lambda=2,\alpha}| = 0$ (presumably all irrational α), one has that $h_{\lambda=2,\alpha,\theta}$ has purely singular continuous spectrum for a.e. θ. Prior to [22], Helffer-Sjöstrand [17] have shown $|\sigma_{2,\alpha}| = 0$ for a set of α's having all quotients of their continued fraction expansion sufficiently large. While this set is nowhere dense and of zero Lebesgue measure, it includes some α's for which (1.2) doesn't hold.

2. The set of α's for which (1.2) holds is a dense G_δ whose complement has Lebesgue measure 0, that is, (1.2) is generic in both Baire and Lebesgue sense. Although we do not describe the a.e. θ explicitly, the set with singular continuous spectrum contains a dense G_δ [20], so it is also generic in both Baire and Lebesgue sense.

3. Delyon [11] proved that $h_{\lambda=2,\alpha,\theta}$ has no eigenfunctions belonging to ℓ^1 (for any α, θ). More recently, Chojnacki [8] has proven that $h_{\lambda=2,\alpha,\theta}$ must have at least some continuous spectrum (for all α's and a.e. θ). His result does not contradict, however, the possibility of mixed (overlapping continuous and p.p.) spectrum. We note that the absence of absolutely continuous spectrum is obvious whenever the spectrum has zero Lebesgue measure.

While Theorem 1 is our main result, we also prove

Theorem 2. *If α is irrational and λ is such that $h_{4/\lambda,\alpha,\theta}$ has only p.p. spectrum for a.e. θ, then $h_{\lambda,\alpha,\theta}$ has purely a.c. spectrum for a.e. θ.*

Remarks. 1. It is known that if α has good Diophantine properties, then for $\lambda < \frac{4}{15}$, $h_{4/\lambda,\alpha,\theta}$ has only p.p. spectrum [18] (see [27,14,16] for earlier results). For such α, λ, we conclude purely a.c. spectrum of $h_{\lambda,\alpha,\theta}$ for a.e. θ.

2. Existence of some a.c. spectrum for small λ and Diophantine α has been proven by Bellissard, Lima, and Testard [6], who applied ideas earlier developed by Dinaburg-Sinai [12] and Belokolos [7]. Such existence (but not necessarily purely a.c. spectrum) is now known for all $\lambda < 2$ and all α, θ [15,21]. Purely a.c. spectrum for (unspecified) small λ and Diophantine α has been proven by Chulaevsky and Delyon [9] using duality. Their proof uses detailed information from Sinai's proof of localization [27].

We also provide a new proof of

Theorem 3. *If α is irrational and $\lambda < 2$, then for a.e. θ, $h_{\lambda,\alpha,\theta}$ has no point spectrum.*

Remark. Delyon [11] has proven that there is no point spectrum for all θ, which is strictly stronger. Moreover, his proof is much simpler. Our proof has a certain methodological advantage in that we don't use the positivity of the Lyapunov exponent for the dual model.

Our proof of Theorems 1–3 depends on a precise version of Aubry duality [2,3]. Recall that one way to understand duality is to note the following: Suppose a_n solves

$$a_{n+1} + a_{n-1} + \lambda \cos(\pi\alpha n + \theta) a_n = E a_n \tag{1.3}$$

with $a_n \in \ell^1$. Define

$$\varphi(x) = \sum a_n e^{i(\pi\alpha n + \theta)x} \tag{1.4}$$

which is continuous on $\mathbb{R}$ with

$$\varphi\left(x+\frac{2}{\alpha}\right)=e^{2i\theta/\alpha}\varphi(x). \tag{1.5}$$

For any η, the sequence

$$u(n)=\varphi\left(n+\frac{\eta}{\pi\alpha}\right) \tag{1.6}$$

is seen to obey

$$u(n+1)+u(n-1)+\frac{4}{\lambda}\cos(\pi\alpha n+\eta)u(n)=\frac{2E}{\lambda}u(n) \tag{1.7}$$

by manipulating (1.4). Thus, nice enough normalizable eigenfunctions at (λ, E) yield Bloch waves at $(\frac{4}{\lambda}, \frac{2E}{\lambda})$ for h and, conversely, nice enough Bloch waves (regularity of φ implies decay of the Fourier coefficients a_n in (1.4)) yield normalizable eigenfunctions.

If we slough over what "nice enough" means, we have naive duality:

(D1) point spectrum at $\lambda \Rightarrow$ a.c. spectrum at $\frac{4}{\lambda}$
(D2) a.c. spectrum at $\lambda \Rightarrow$ point spectrum at $\frac{4}{\lambda}$
(D3) s.c. spectrum at $\lambda \Rightarrow$ s.c. spectrum at $\frac{4}{\lambda}$

where the last statement follows from the first two.

The surprise in Last [21] is that this naive expectation is false. There exist α (Liouville numbers) for which the spectrum for $\lambda > 2$ is purely singular continuous, but the spectrum for $\lambda < 2$ has an a.c. component. Thus, (D2) need not be true. In a sense, the main result of the paper is that (D1) is still true. More explicitly, we show that the dual of point spectrum is a.c. spectrum, in the sense that some p.p. spectrum for λ implies some a.c. spectrum for $\frac{4}{\lambda}$, and only p.p. spectrum for λ implies only a.c. spectrum for $\frac{4}{\lambda}$. This strengthens Chojnacki's result [8], which shows (in a more general context, though) that the dual of point spectrum is continuous spectrum (but not necessarily a.c. spectrum).

Thus, there is a kind of more precise duality for the almost Mathieu operator:

(D1$'$) point spectrum at $\lambda \Rightarrow$ a.c. spectrum at $\frac{4}{\lambda}$
(D2$'$) a.c. spectrum at $\lambda \Rightarrow$ point or s.c. spectrum at $\frac{4}{\lambda}$
(D3$'$) s.c. spectrum at $\lambda \Rightarrow$ a.c. spectrum (or s.c. spectrum) at $\frac{4}{\lambda}$.

It is unclear if there is any s.c. spectrum for $\lambda < 2$.

Note that while we prove (D1$'$) (and thus (D3$'$)) we do not prove (D2$'$). It follows, of course, from the known results for the almost Mathieu operator, but we want to point out that this implication fails in certain more general contexts [25].

In Section 2, we prove that if there is point spectrum, one can always choose the eigenvalues and eigenvectors to be measurable in θ. That allows us to represent the set of all eigenvalues of $h_{\lambda,\alpha,\theta}$ as a union of values of a measurable multivalued function along the trajectory of the rotation by α — an object first introduced by Sinai [27].

In Section 3, we use this representation and analyze duality to show point spectrum implies there are spectral measures for the dual problem (at coupling $\frac{4}{\lambda}$) so that $d\mu_\theta$ is independent of θ. Here we use some important ingredients from [9] and [25].

In Section 4, we show that results of Deift-Simon [10] imply that the singular components of the spectral measures $d\mu_\theta$ and $d\mu_{\theta'}$ are a.e. disjoint. Thus, the θ independence of Section 3 implies that $d\mu_\theta$ is purely absolutely continuous. We make this precise and prove Theorems 1–3 in Section 5.

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§2. Measurability of Eigenfunctions

To emphasize that "measurability" here means in Borel sense rather than up to sets of measure zero in the completed measure, we'll initially discuss a setup with no measure! At the end of this section, we'll link this to the almost periodic situation that is the main focus of this paper.

Let $A < \infty$ be positive and fixed. Let $\Omega = [-A, A]^{\mathbb{Z}}$, that is, $\omega \in \Omega$ is a sequence $\{\omega_n\}_{n=-\infty}^{\infty}$ with $|\omega_n| \leq A$. Ω is a separable compact metric space with Baire $=$ Borel sets. We call these sets "measurable." For each $\omega \in \Omega$, define a self-adjoint operator on $\ell^2(\mathbb{Z})$ by

$$(h_\omega u)(n) = u(n+1) + u(n-1) + \omega_n u(n).$$

A critical fact we'll use below is that (normalizable) eigenvalues are always simple.

Given any normalized eigenvector u for h_ω, we define $j(u)$ to be the leftmost maximum for $|u|$, that is, that j with

$$\begin{aligned} |u(j)| &\geq |u(k)| \qquad \text{all } k \\ &> |u(k)| \qquad \text{all } k < j. \end{aligned}$$

We'll always fix u by requiring $u(j) > 0$. Since $\sum |u(k)|^2 = 1$, we have $u(k) \to 0$ as $|k| \to \infty$, so it has a leftmost maximum. If j is the leftmost maximum for u, we say that u is attached to j.

Let $\{u_n\}$ be the collection of eigenvectors of h_ω. Define $N_j(\omega)$ to be the number of eigenvectors of h_ω attached to j. One of our goals below will be to prove

Theorem 2.1. *$N_j(\omega)$ is a measurable function on Ω.*

Let $\Omega_{j,k} = \{\omega \mid N_j(\omega) \geq k\}$ for $k = 1, 2, \dots$. On $\Omega_{j,1}$, define $u_1(n; \omega, j)$ to be the eigenfunction attached to j with maximal value of $u_1(j)$. Let $e_1(\omega, j)$ be its eigenvalue. If there are multiple eigenfunctions attached to j with the same value of $u_1(j)$, pick the one with the largest energy. Since $\sum_n |u_n(j)|^2 \leq 1$ (by Parseval's inequality for δ_j), there are only finitely many u's with this maximum value of $u(j)$ and so we can pick the one with largest e.

Similarly, on $\Omega_{j,2}$ we can define $u_2(n;\omega,j)$ by picking the attached eigenvector with second largest $u(j;\omega,j)$ again breaking ties by choosing the largest energy. In this way, we define $u_l(n;\omega,j)$ and $e_l(\omega,j)$ on $\Omega_{j,l}$ so that

(1) $\{u_l(\,\cdot\,;\omega,j)\}_{l=1}^{N_j(\omega)}$ is the set of eigenvectors attached to j,
(2) $u_l(j;\omega,j) \geq u_{l+1}(j;\omega,j)$, and if equality holds, then $e_l(\omega,j) > e_{l+1}(\omega,j)$.

Extend u_l and e_l to all of Ω by setting to 0 on $\Omega\backslash\Omega_{j,l}$. Then we'll prove that

Theorem 2.2. *$e_l(\omega,j)$ and $u_l(n;\omega,j)$ are measurable functions on Ω for each fixed l,j (and n).*

Notice that

Proposition 2.3. *For each ω, l,l' and $j \neq k$,*

$$\sum_{n=-\infty}^{\infty} \overline{u_l(n;\omega,j)}\, u_{l'}(n;\omega,k) = 0. \tag{2.1}$$

This is true because the u's are distinct eigenfunctions (since they are attached to distinct points) and so orthogonal.

As a first preliminary to the proofs of Theorems 2.1, 2.2, we make several simplifying remarks:

(i) Without loss, we can take $j = 0$.
(ii) Instead of looking only at eigenfunctions attached to $j = 0$, we can look at eigenfunctions with $u(0) \neq 0$ normalizing by $u(0) > 0$. If we define $\tilde{N}_0(\omega)$ and $\tilde{u}_l(n;\omega,0)$ analogously by requiring the analog of (1) and (2) and prove measurability, we recover Theorems 2.1, 2.2 by noting that since $\tilde{u}_1(n;\omega,0)$ is measurable, $\{\omega \mid \tilde{u}_1 \text{ is attached to } 0\}$ is measurable, and similarly for $\tilde{u}_2,\dots$. Then define u_1 in the obvious way and see that it is measurable.
(iii) It will be convenient to deal with that multiple η of u with $\eta(0) = 1$. u and η are related, of course, by $\eta(n) = u(n)/u(0)$ and $u(n) = \eta(n)/\big(\sum_{j=-\infty}^{\infty} |\eta(j)|^2\big)^{1/2}$. By these relations, if we show η is measurable, so is u. Notice that since $u(0) = 1/\|\eta\|$, ordering by maximal $u(0)$ is the same as ordering by minimal $\|\eta\|$.

Note. Since the passage from η to u involves an infinite sum, weak continuity of η does not imply weak continuity of u, only weak measurability. The use of η below is critical because it, not u, is weakly continuous.

As a second preliminary, we note a few standard facts about Borel functions.

Lemma 2.4. *Let X be a complete metric space, $Y \subset X$ an arbitrary subspace, and B_X, B_Y their Borel subsets. Then*

$$B_Y = \{Y \cap A \mid A \in B_X\}. \tag{2.2}$$

Proof. Let $\tilde{B}_Y$ be the right side of (2.2). Then $B_Y \subset \tilde{B}_Y$ since $\tilde{B}_Y$ is clearly a sigma algebra containing the closed sets in Y since if C is closed in Y and D is its closure in X, then $D \cap Y = C$. Conversely, let $\tilde{B}_X = \{A \subset X \mid A \cap Y \in B_Y\}$. $\tilde{B}_X \supset B_X$ since $\tilde{B}_X$ is a sigma algebra containing the open sets. Thus, $\tilde{B}_Y \subset B_Y$. $\square$

Lemma 2.5. *Let X be a complete metric space, $Y \subset X$ with $Y \in B_X$, the Borel subsets of X. Then any $C \in B_Y$, the Borel subset of Y, is Borel as a subset of X.*

Proof. By Lemma 2.4, $C = Y \cap A$ with $A \in B_X$. Since $Y \in B_X$, so is C. $\square$

Proposition 2.6. *Let X be a complete metric and B_X its Borel subsets. Let $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in B_X$. Let $f : X \to \mathbb{R}$ be such that for each n, $f_n \equiv f \restriction X_n$ is a Borel function from X_n to X. Then f is Borel.*

Proof. Let $(a,b) \subset \mathbb{R}$. Then $f^{-1}(a,b) = \bigcup_{n=1}^{\infty} f_n^{-1}(a,b)$ is Borel by Lemma 2.5. $\square$

Example. Let $X = [0,1]$ and let $f = \chi_{[1/2,1]}$ the characteristic function of $[\frac{1}{2},1]$. Let $X_n = [\frac{1}{2},1] \cup [0, \frac{1}{2} - \frac{1}{n}]$. Then $f \restriction X_n : X_n \to \mathbb{R}$ is continuous, so f is Borel. This example shows that Proposition 2.6 is false if a Borel function is replaced by a continuous function. It is useful to keep in mind, given the continuous function argument we use below.

As a final preliminary, we note the following elementary fact:

Proposition 2.7. *Suppose $\omega^{(m)} \in \Omega$ and $\omega^{(m)} \to \omega^{(\infty)}$ and let $h_m \equiv h_{\omega^{(m)}}$. Suppose for each finite m, there is $\eta^{(m)}$ with*

(i) $h_m \eta^{(m)} = e_m \eta^{(m)}$
(ii) $\eta^{(m)}(0) = 1$
(iii) $\sup\limits_m \|\eta^{(m)}\| \equiv C < \infty$.

Then there exists a subsequence m_i, $\eta^{(\infty)}$ and e_∞ so that

(1) $h_\infty \eta^{(\infty)} = e_\infty \eta^{(\infty)}$
(2) $\eta^{(\infty)}(0) = 1, \qquad \|\eta^{(\infty)}\| \leq C$
(3) $e_{m_i} \to e_\infty$, $\eta^{(m_i)}(n) \to \eta^{(\infty)}(n)$ *for each n.*

Proof. $|e_m| \leq 2 + A$, so by compactness of the unit ball of $\ell^2(\mathbb{Z})$ in the weak topology, we can pick a subsequence so that (3) holds. (2) is obvious and (1) holds by taking pointwise limits in the equation (i). $\square$

In the proof below, we have to worry about three possibilities that destroy continuity of a function like $\tilde{N}_0$. First, $\sup\limits_m \|\eta^{(m)}\| = \infty$. We'll avoid this by looking at subsets with $\|\eta\| \leq k$, get measurability, and take k to infinity. Second, as $\omega^{(m)} \to \omega^{(\infty)}$, two distinct eigenvalues of h_m can converge to a single e so that h_∞ has fewer eigenvalues. We'll avoid this by looking at subsets where eigenvalues stay at least a distance $2^{-\ell}$ from each other. Then we'll take ℓ to infinity. Third, after we restrict to η's with $\|\eta\| \leq k$, in a limit a bunch of η's with $\|\eta\| \geq k$ can approach one with $\|\eta\| = k$ increasing N. We'll handle this by proving semicontinuity rather than continuity as the starting point of a proof of measurability.

Proof of Theorems 2.1, 2.2. For each pair of positive integers k, p, define $M_{k,p}(\omega)$ to be the maximum number of m's so that $\|\eta_m\| \leq k$ and $|e_m - e_{m'}| \geq 2^{-p}$, for all $m' \neq m$. Since Parseval's inequality implies that $\sum\limits_m 1/\|\eta_m\|^2 \leq 1$, there are at most k^2 m's with $\|\eta_m\| \leq k$ and so we can determine the maximum number with $|e_m - e_{m'}| \geq 2^{-p}$.

We claim that $S \equiv \{\omega \mid M_{k,p}(\omega) \geq l\}$ is closed so $M_{k,p}(\,\cdot\,)$ is measurable. For if $\omega^{(m)} \in S$ and $\omega^{(m)} \to \omega^{(\infty)}$, we can use Proposition 2.7 and find η's and e's for h_∞ by taking limits. Clearly, the limiting e's still obey the $|e_m - e_{m'}| \geq 2^{-p}$ condition. Thus, S is closed and $M_{k,p}(\,\cdot\,)$ is measurable.

Now define

$$M_{k,\infty}(\omega) = \#\text{ of } m\text{'s with } \|\eta_m\| \leq k.$$

Because of simplicity of the spectrum,

$$M_{k,\infty}(\omega) = \max_p M_{k,p}(\omega)$$

so

$$\{\omega \mid M_{k,\infty}(\omega) \geq l\} = \bigcup_{p=1}^{\infty} \{\omega \mid M_{k,p}(\omega) \geq l\}$$

is an F_σ, and so $M_{k,\infty}$ is measurable.

Next define

$$\Sigma_{k,l,p} = \{\omega \mid M_{k,\infty}(\omega) = M_{k,p}(\omega) = l\},$$

the set of ω's with exactly l eigenvalues with $\|\eta_m\| \leq k$ so that $|e_m - e_{m'}| \geq 2^{-p}$. $\Sigma_{k,l,p}$ is a Borel set. Let $e_1 > \cdots > e_l$ be the eigenvalues and $\eta_1, \ldots, \eta_l$ the eigenvectors (normalized by $\eta_i(0) = 1$). We claim that e_i and η_i are continuous on $\Sigma_{k,l,p}$. For if $\omega^{(m)} \to \omega^{(\infty)}$, $e_i^{(m)}$ has limit points which are distinct (as i varies) since $|e_i^{(m)} - e_j^{(m)}| \geq 2^{-p}$. By Proposition 2.7, these limit points must be eigenvalues of h_∞ with eigenvectors $\eta_i^{(\infty)}$ obeying $\|\eta_i^{(\infty)}\| \leq k$ and so these limit points must be the $e_i^{(\infty)}$ since h_∞ has only l such eigenvalues. That is, $e_i^{(\infty)}$ are the unique limits points of $e_i^{(m)}$, so $e_i^{(m)} \to e_i^{(\infty)}$. Similarly, the η's converge by Proposition 2.7 and the uniqueness of eigenvectors.

Now let

$$\Sigma_{k,l} = \{\omega \mid M_{k,\infty}(w) = l\} = \bigcup_p \Sigma_{k,p,l}.$$

By Proposition 2.6, e_i and η_i are measurable on $\Sigma_{k,l}$.

Now change the labeling so that instead of $e_1 > e_2 > \cdots > e_l$, we have $\|\eta_i\| \leq \|\eta_{i+1}\|$ with $e_i > e_{i+1}$ if $\|\eta_i\| = \|\eta_{i+1}\|$. This involves a permutation π so that

$$\begin{aligned} e_i^{(\text{new})} &= e_{\pi(i)}^{\text{old}} \\ \eta_i^{(\text{new})} &= \eta_{\pi(i)}^{\text{old}}. \end{aligned}$$

Because $\eta_i^{(\text{old})}$ and $e_i^{(\text{old})}$ are Borel functions, the set $\Sigma_{k,l}^{(\pi)}$ on which a given permutation π is used is a Borel set. $e_i^{(\text{new})}$ is built out of Borel functions on each $\Sigma^{(\pi)}$ and so we have measurability with the changed labeling.

Once we change labeling, e_i's and η_i's are defined consistently on $\Sigma_{k,l}$ as k, l vary, and so, using Proposition 2.6, on all Ω. $\square$

Note that although the set of all eigenvalues can be naively considered a nonmeasurable "function" of ω, since it is invariant but nonconstant, we have shown it admits a measurable selection.

As a final remark on the issue of this section, we want to rewrite (2.1) in a useful way. Taking $\omega_n(\theta) = \lambda\cos(\pi\alpha n + \theta)$ embeds the circle $\theta \in [0,\pi)$ into Ω, so we define $u_l(n;\theta) \equiv u_l(n;\theta,0)$, the eigenvectors with leftmost maximum at 0. Note that

$$u_l(\,\cdot\, - j; \theta + j\alpha\pi) = u_l(\,\cdot\,;\theta,j).$$

In particular, (2.1) becomes

$$\sum_n \overline{u_{l'}(n-j;\theta+j\alpha\pi)}\, u_l(n;\theta) = 0 \tag{2.3}$$

for all l, l' (even with $l = l'$) and $j \neq 0$. If we extend $u_l \equiv 0$ on the set where there aren't l eigenvectors with leftmost maximum at 0, then (2.3) holds for all θ.

§3. Duality

Fix α irrational throughout this section.

Let $\mathcal{H} = L^2(([0,2\pi),\frac{d\theta}{2\pi}) \times \mathbb{Z})$, that is, functions $\varphi : [0,2\pi) \times \mathbb{Z} \to \mathbb{C}$ with $\sum_n \int \frac{d\theta}{2\pi} |\varphi(\theta,n)|^2 < \infty$. Define Q_λ on $\mathcal{H}$ by

$$(Q_\lambda \varphi)(\theta,n) = \varphi(\theta,n+1) + \varphi(\theta,n-1) + \lambda\cos(\pi\alpha n + \theta)\varphi(\theta,n),$$

that is, Q_λ is the direct integral in θ of $h_{\alpha,\lambda,\theta}$.

Define $U : \mathcal{H} \to \mathcal{H}$ by the formal expression:

$$(U\varphi)(\eta,m) = \sum_n \int \frac{d\theta}{2\pi}\, e^{-i(\eta+\pi\alpha m)n} e^{-im\theta} \varphi(\theta,n). \tag{3.1}$$

In terms of the Fourier transform $\widehat{\varphi}(m,\eta)$ we have

$$(U\varphi)(\eta,m) = \widehat{\varphi}(m,\eta+\pi\alpha m), \tag{3.2}$$

which gives a precise definition even for cases where the sum in n may not converge absolutely and shows that U is unitary. Here is a precise version of Aubry duality [2,3]:

Theorem 3.1.

$$Q_\lambda U = \frac{\lambda}{2} U Q_{4/\lambda}$$

Proof. A straightforward calculation. For example, if $(T\varphi)(\theta,n) = \varphi(\theta,n+1)$, then $(U^{-1}TU\varphi)(\theta,n) = e^{-i(\pi\alpha n+\theta)}\varphi(\theta,n)$ so $(U^{-1}Q_\lambda U\varphi)(\theta,n) = 2\cos(\pi\alpha n + \theta)\varphi(\theta,n) + \frac{\lambda}{2}[\varphi(\theta,n+1) + \varphi(\theta,n-1)] = \frac{\lambda}{2}(Q_{4/\lambda}\varphi)(\theta,n)$. $\square$

Remark. Theorem 3.1 also provides a proof of duality for the integrated density of states first rigorously proven in [5]. For let

$$g(\theta,n) = \delta_{n0}.$$

Then, $Ug = g$. Moreover, if $k_\lambda(E)$ is the integrated density of states, then for any continuous function f,

$$\langle g, f(Q_\lambda)g\rangle = \int f(E)\, dk_\lambda(E).$$

Thus, Theorem 3.1 which implies

$$\begin{aligned}\langle g, f(Q_\lambda)g\rangle &= \langle Ug, Uf(Q_\lambda)U^{-1}Ug\rangle \\ &= \langle g, f(\tfrac{\lambda}{2}Q_{4/\lambda})g\rangle\end{aligned}$$

yields the duality of k.

We need one more simple calculation:

Proposition 3.2. *For any $\varphi \in \mathcal{H}$, $l \in \mathbb{Z}$, define a unitary operator S_l by*

$$(S_l\varphi)(\theta, n) = \varphi(\theta + \pi\alpha l, n - l). \tag{3.3}$$

Then

$$(US_l\varphi)(\eta, m) = e^{-il\eta}(U\varphi)(\eta, m). \tag{3.4}$$

Proof. Let φ be such that there exists N_0 with $\varphi(n, \theta) = 0$ if $|n| > N_0$. Then (3.4) is a simple change of variables in the integral (3.1). Since such φ's are dense and S_l is bounded, (3.4) holds for all φ. □

Proposition 3.3. *Let $\varphi \in \mathcal{H}$ so that for all $l \neq 0$, $\langle S_l\varphi, \varphi\rangle = 0$. Then $\sum_m |(U\varphi)(\eta, m)|^2 = g(\eta)$ is a.e. independent of η.*

Proof. Note that since $U\varphi \in \mathcal{H}$, $g \in L^1([0, 2\pi), \frac{d\theta}{2\pi})$. We compute $\int e^{il\eta}g(\eta)\frac{d\eta}{2\pi} = \langle U\varphi, e^{il\eta}U\varphi\rangle = \langle U\varphi, US_{-l}\varphi\rangle = \langle \varphi, S_{-l}\varphi\rangle = 0$, $l \neq 0$, by hypothesis. By the weak-$*$ density of finite linear combinations of $\{e^{il\eta}\}_{l=-\infty}^{\infty}$ in L^∞, we conclude that $g(\eta)$ is constant. □

We come now to the main result of this section.

Theorem 3.4. *Fix λ and fix α irrational. Let $u_l(\,\cdot\,; \theta, j)$ be the measurable function described in Theorem 2.2 for the Hamiltonian $h_{4/\lambda,\alpha,\theta}$. Let $f(\theta)$ be an arbitrary function in $L^2([0, 2\pi), \frac{d\theta}{2\pi})$. Let $\varphi(\theta, n) = f(\theta)u_l(n; \theta, j)$ for some fixed l, j. For each η, let*

$$\psi_\eta(n) = (U\varphi)(\eta, n)$$

and let $d\mu_\eta(E)$ be the spectral measure for Hamiltonian $h_{\lambda,\alpha,\eta}$ and vector ψ_η. Then $d\mu_\eta$ is a.e. η independent.

Proof. By (2.3), $S_k\varphi$ is orthogonal (in $\mathcal{H}$) to φ for any $k \neq 0$. Moreover, since

$$\left(F\left(\frac{\lambda}{2}\,Q_{4/\lambda}\right)\varphi\right)(\theta, n) = F\left(\frac{\lambda}{2}\,e_l(\theta, j)\right)\varphi(\theta, n),$$

we have that $S_k\varphi$, $k \neq 0$, is orthogonal to $F(Q_{4/\lambda})\varphi$ for any continuous function F. As in Proposition 3.3, we conclude that

$$\sum_m \overline{(U\varphi)(\eta, m)} \left(UF\left(\frac{\lambda}{2} Q_{4/\lambda}\right)\varphi\right)(\eta, m) \tag{3.5}$$

is independent of η. But by Theorem 3.1, $UF(\frac{\lambda}{2}Q_{4/\lambda}) = F(Q_\lambda)U$. So (3.5) is just $\int F(E)\, d\mu_\eta(E)$. Since this is a.e. η independent for each continuous F (and the set of continuous F's is separable), we conclude that $d\mu_\eta(E)$ is a.e. η constant. $\square$

§4. A.E. Mutual Singularity of the Singular Parts

In this section we want to note a simple consequence of Deift-Simon [10]:

Theorem 4.1. *Let h_ω be an ergodic family of Jacobi matrices and let $d\mu_\omega^s$ be the singular part of a spectral measure for h_ω. Then for a.e. ω, ω', $d\mu_\omega^s$ and $d\mu_{\omega'}^s$ are mutually singular.*

Remark. This is an analog of the celebrated result of Aronszajn [1]-Donoghue [13] of mutual singularity under rank one perturbations; see [26].

Proof. Let $G_\omega(n, m; z)$ be the Green's function for h_ω (matrix elements of $(h_\omega - z)^{-1}$) and let $\mathfrak{S}_\omega$ be the set of E_0 in $\mathbb{R}$ so that

$$\overline{\lim_{\epsilon \downarrow 0}} \{\mathrm{Im}[G_\omega(0,0; E_0 + i\epsilon)] + \mathrm{Im}[G_\omega(1,1,E_0+i\epsilon)]\} = \infty.$$

By the theorem of de la Vallée-Poussin, $d\mu_\omega^s$ is supported by $\mathfrak{S}_\omega$. Deift-Simon [10] prove that for every $E_0 \in \mathbb{R}$, $\{\omega' \mid E_0 \in \mathfrak{S}_{\omega'}\}$ has measure 0. Thus, integrating on E with respect to $d\mu_\omega$, we see that

$$\mu_\omega(\mathfrak{S}_{\omega'}) = 0$$

for a.e. ω'. Thus for each fixed ω, $d\mu_{\omega'}^s$ is mutually singular to $d\mu_\omega$ for a.e. ω'. $\square$

§5. Putting It All Together

Theorem 5.1. *Fix λ and fix α irrational. Let $u_l(\,\cdot\,; \theta, j)$ be the measurable function described in Theorem 2.2 for the Hamiltonian $h_{4/\lambda,\alpha,\theta}$. Let $f(\theta)$ be an arbitrary function in $L^2([0,2\pi), \frac{d\theta}{2\pi})$. Let $\varphi(\theta, n) = f(\theta)u_l(n;\theta,j)$ for some fixed l, j. For each η, let $\psi_\eta(n) = (U\varphi)(\eta, n)$ and let $d\mu_\eta(E)$ be the spectral measure for Hamiltonian $h_{\lambda,\alpha,\eta}$ and vector ψ_η. Then $d\mu_\eta$ is purely a.c. for a.e. η.*

Proof. By Theorem 3.4, $d\mu_\eta$ is a.e. constant. By Theorem 4.1, this means that $d\mu_\eta^s$ is a.e. zero. $\square$

Theorem 5.2. *Fix λ and fix α irrational. If $h_{4/\lambda,\alpha,\theta}$ has point spectrum for a set of θ's of positive measure, then $h_{\lambda,\alpha,\theta}$ has some a.c. spectrum for a.e. θ.*

Remark. It then follows by [24] that $h_{\lambda,\alpha,\theta}$ has some a.c. spectrum for *all* θ.

Proof. $\{u_l(\,\cdot\,;\theta,j)\}_{l,j}$ span the point spectrum for $h_{4/\lambda,\alpha,\theta}$, so if there is point spectrum, some $u_1(\,\cdot\,;\,\cdot\,,j)$ is a non-zero function in $\mathcal{H}$. Thus by Theorem 5.1, $d\mu_\eta$ is a.e. purely absolutely continuous. Since $\int d\mu_\eta(E)\,d\eta = \sum\limits_n \int |u_1(n;\theta,j)|^2 \frac{d\theta}{2\pi} > 0$, we conclude that $\int d\mu_\eta(E) \neq 0$ for a set of η's of positive measure and so for a.e. η since $d\mu_\eta(E)$ is a.e. η independent. $\square$

Theorem 5.3. *Fix λ and fix α irrational. If $h_{4/\lambda,\alpha,\theta}$ has only point spectrum for a.e. θ, then $h_{\lambda,\alpha,\theta}$ has only a.c. spectrum for a.e. θ.*

Proof. Let $f_m(\theta)$ be an orthonormal basis for $L^2([0,2\pi),\frac{d\theta}{2\pi})$. By hypothesis for a.e. θ, $\{u_l(\,\cdot\,;\theta,j)\}_{l,j}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$ where we run over those l,j for which $u_l(\,\cdot\,;\theta,j) \neq 0$. It follows that if $\varphi_{m,l,j}(\theta,n) \equiv f_m(\theta)u_l(n;\theta,j)$, then $\{\varphi_{m,l,j}\}_{m,l,j}$ is a complete orthogonal set (but not necessarily normalized). By the unitarity of U, $\{U\varphi_{m,l,j}\}_{m,l,j}$ is also a complete orthogonal set. Thus for a.e. η, $\{U\varphi_{m,l,j}(\eta,\cdot)\}$ is a complete set. But these vectors lie in the a.c. spectral subspace by Theorem 5.1. $\square$

Proof of Theorem 1. Last [22] has shown for such α, the Lebesgue measure of the spectrum $\sigma_{\lambda=2,\alpha}$ is zero. It follows there is no a.c. spectrum for such α and $\lambda = 2$. By Theorem 5.2, there can't be any point spectrum a.e. since such point spectrum would imply a.c. spectrum. Thus for a.e. θ, the spectrum is purely singular continuous. $\square$

Proof of Theorem 2. This is just a restatement of Theorem 5.3. $\square$

Proof of Theorem 3. Let $\lambda < 2$. Then $h_{4/\lambda,\alpha,\theta}$ has no a.c. spectrum since $\frac{4}{\lambda} > 2$ [3,4]. Thus by Theorem 5.2, $h_{\lambda,\alpha,\theta}$ can have no point spectrum. $\square$

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