Michel theory of symmetry breaking

and pure gauge theories

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Summary. We extend Michel's theorem on the geometry of symmetry breaking [1] to the case of pure gauge theories, i.e. of gauge-invariant functionals defined on the space $\mathcal{A}$ of connections of a principal fiber bundle. Our proof follows closely the original one by Michel, using several known results on the geometry of $\mathcal{A}$.

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The Michel theorem on critical orbits of G-invariant functions on a G-manifold [1] allows to identify G-orbits which are critical for *any* G-invariant function; thus, it permits to *study spontaneous symmetry breaking in a model-independent way* [2].

The purpose of the present short note is to show that Michel's theory can be extended to the study of (pure) gauge theories, such as those defined by a Yang-Mills functional.

It is quite remarkable (and, to us, surprising) that one can extend not only Michel's results, but also the essence of Michel's construction and proof, from its original setting (M a finite dimensional manifold, G a compact Lie group) to the case of gauge theories (infinite dimensional, non-compact manifold and group).

It is also remarkable that a large part of the background results we need have been available for over fifteen years [3,4]; it seems that lack of communication between different mathematical communities (and between mathematicians and physicists) prevented such an extension to be obtained earlier.

In the present note, we limit ourselves to the main result, i.e. the direct extension of Michel's theorem; a more complete discussion of the topic will be given elsewhere.

Let B be an n-dimensional (n finite) riemannian manifold (e.g. $B = R^4$ or S^4); let G be a compact, connected and semisimple Lie group (e.g. $G = SU(N)$) acting through a given matrix representation; let P be a principal fiber bundle over B with projection $\pi : P \to B$ and fiber $\pi^{-1}(x) \approx G$. We denote by $\mathcal{G}$ the Lie algebra of G.

Let $\mathcal{C}$ be the set of connections on P. It is well known that $\mathcal{C}$ is an affine space modelled on the vector space $\mathcal{A} := \Lambda^1 (\mathrm{T}^* B, \mathcal{G})$ of the one-forms on B taking value in $\mathcal{G}$. With any connection $A \in \mathcal{C}$ we can associate a covariant derivative ∇^α and a one-form $\alpha \in \mathcal{A}$ that are given, in local coordinates $(x^1, ..., x^n)$ on B and with $A_\mu : B \to \mathcal{G}$, by

$$\alpha \ = \ A_\mu(x) \ \mathrm{d}x^\mu \ ; \tag{1}$$

$$\nabla^\alpha_\mu \ = \ \partial_\mu \ + \ A_\mu \ . \tag{2}$$

Choosing as reference point in $\mathcal{C}$ the null connection A^0 such that $A^0_\mu = 0$ ($\mu = 1, ..., n$), from now on we will say "the connection α" to mean "the connection $A = A^0 + \alpha$ whose associated connection form is α".

Let Γ denote the space of differentiable sections of P; Γ has the structure of a Hilbert-Lie group, and a section $\gamma \in \Gamma$ will be written in local coordinates as

$$\gamma \ = \ g(x) \ , \tag{3}$$

with $g : B \to G$. We say that Γ is a gauge group modelled on G.

The gauge group Γ acts naturally on the operator of covariant derivative associated to a connection $A \in \mathcal{C}$ by conjugation, i.e. $\gamma : \nabla^\alpha \to \gamma \nabla^\alpha \gamma^{-1}$; the action of Γ on $\mathcal{A}$ is given in local coordinates by

$$\gamma \ : \ A_\mu \ \Longrightarrow \ g(x) \cdot A_\mu(x) \cdot g^{-1}(x) \ - \ (\partial_\mu g)(x) \cdot g^{-1}(x) \ . \tag{4}$$

The action of $\gamma \in \Gamma$ on the connection form $\alpha \in \mathcal{A}$ will be denoted by $\gamma(\alpha)$.

With any connection form α (i.e. with any connection A) we can associate a *gauge isotropy subgroup* Γ_α,

$$\Gamma_\alpha \ = \ \{ \gamma \in \Gamma \ : \ \gamma(\alpha) = \alpha \} \ . \tag{5}$$

It is well known [3] that, with $p_0 \in P$ any reference point in P and $x_0 = \pi(p_0)$,

$$\Gamma_\alpha \ = \ \{ \gamma \in \Gamma \ : \ \nabla^\alpha(\gamma) = 0 \ , \ g(x_0) \in C_G \left[H_\alpha(p_0) \right] \ \} \ , \tag{6}$$

where $H_\alpha(p_0)$ is the holonomy group of the connection ∇^α at p_0, and $C_G[H]$ the centralizer of H in G, i.e.

$$C_G[H] \ = \ \{ g \in G \ : \ [g,h] = 0 \ \ \forall h \in H \} \ . \tag{7}$$

Thus, Γ_α is isomorphic to a subgroup of the compact group G; notice that the isomorphism, given by $\nabla^\alpha \gamma = 0$, depends on A.

Given a connection α with isotropy group Γ_α we can consider: (i) the *fixed space* of Γ_α, i.e. the space of connections (associated with connection forms) which are left invariant by Γ_α,

$$\mathcal{F}(\alpha) \ = \ \{ \beta \in \mathcal{A} \ : \ \gamma(\beta) = \beta \ \ \forall \gamma \in \Gamma_\alpha \} \ = \ \{ \beta \ : \ \Gamma_\alpha \subseteq \Gamma_\beta \} \ ; \tag{8}$$

and (ii) the *isotropy type* of α, i.e. the space of connections (associated with connection forms) having isotropy subgroups which are Γ-conjugated to that of α,

$$\Sigma(\alpha) \ = \ \{ \beta \in \mathcal{A} \ : \ \exists \gamma \in \Gamma : \ \Gamma_\beta = \gamma \Gamma_\alpha \gamma^{-1} \ \} \ . \tag{9}$$

The orbit of α under Γ will be denoted by $\Gamma(\alpha)$ (or, for ease of notation, by ω_α). If we consider the gauge transformed of α, we have easily that $\Gamma_{\gamma(\alpha)} = \gamma \Gamma_\alpha \gamma^{-1}$, or in other words

$$\Gamma(\alpha) \ \subseteq \ \Sigma(\alpha) \ ; \tag{10}$$

thus, the equivalence classes under the relation of belonging to the same isotropy type consist of (necessarily, disjoint) unions of gauge orbits.

Actually, this equivalence relation leads to a *stratification* of $\mathcal{A}$ [3] (see [5] for more detail), pretty much as in the classical case of compact group action on finite dimensional manifolds; however, we have now a countable – rather than finite – set of strata [4,5].

The set $\Sigma(\alpha)$ will be called the *stratum* of α, and it can be shown to be a smooth manifold, and actually a principal bundle [5].

Let us now consider $\mathcal{A}$ in some more detail. We have already noticed that it is a linear space; using the G-invariant scalar product in $\mathcal{G}$, denoted by $\langle .,. \rangle$, we can define a scalar product in $\mathcal{A}$ by

$$(\alpha, \beta) \;=\; \int_B \left[\sum_{\mu=1}^{n} \langle A_\mu(x), B_\mu(x) \rangle \right] \mathrm{d}^n x \tag{11}$$

where $\alpha = A_\mu \mathrm{d}x^\mu$, $\beta = B_\mu \mathrm{d}x^\mu$. Let $|\alpha| = (\alpha, \alpha)^{1/2}$ denote the corresponding norm. Fixing a connection $C^0 \in \mathcal{C}$ and using the induced covariant derivative ∇^0 we define a Sobolev scalar product and a Sobolev norm of class k on $\mathcal{A}$ by

$$(\alpha, \beta)_k \;=\; \sum_{j=0}^{k} \left[\int_B \left(\left(\nabla^0\right)^j \alpha, \left(\nabla^0\right)^j \beta \right) d^n x \right] \tag{12}$$

and

$$||\alpha||_k \;=\; \left(\sum_{j=0}^{k} \int_B |(\nabla^0)^j)\alpha|^2 d^n x \right)^{1/2} \;. \tag{13}$$

We will call $\mathcal{A}_k$ the completion of $\mathcal{A}$ with respect to this norm. Then $\mathcal{A}_k$ is an Hilbert space, and different choices of the connection $C^0 \in \mathcal{C}$ give rise to equivalent norms [6,7]. If we consider also the Sobolev completion Γ_k of the gauge group Γ, we have that, for $k > [(\dim(B)+1)/2]$, Γ_k is an infinite dimensional Hilbert-Lie group modelled on a separable Hilbert space. The action of Γ on $\mathcal{A}$ can be extended to a smooth action of Γ_k on $\mathcal{A}_{k-1}$ and the Γ_k orbits are closed in $\mathcal{A}_{k-1}$ [8-10]

¿From now on we will assume that all objects requiring Sobolev completion have been completed in appropriate norms and we will write again $\mathcal{A}$ and Γ instead of $\mathcal{A}_k$ and Γ_{k+1}, and (α, β) instead of $(\alpha, \beta)_k$.

We remark that the norm (13) induce a Γ-invariant distance d on $\mathcal{A}$ defined in the usual way, i.e. as $d\,(\alpha, \beta) = ||\alpha - \beta||$.

It is known [3,4,7,10] that the action of Γ on $\mathcal{A}$ admits a slice S_α at any point $\alpha \in \mathcal{A}$. The existence of a slice guarantees the existence of a tubular neighbourhood $\mathcal{U}_\alpha$ of the Γ orbit ω_α; this can be obtained by Γ-transporting S_α, i.e. $\mathcal{U}_\alpha = \Gamma\,(S_\alpha)$.

We introduce the following

Definition: A gauge orbit ω_α is *isolated in its stratum* if and only if $\mathcal{U}_\alpha \cap \Sigma(\alpha) = \omega_\alpha$.

We recall that a slice at α is a submanifold $S_\alpha \subseteq \mathcal{A}$ such that $\alpha \in S_\alpha$ and:
(i) S_α is transversal and complementary to the orbit ω_α in $\mathcal{A}$ at α;
(ii) S_α is trasversal to all the Γ orbits which meet S_α;
(iii) S_α is (globally) invariant under Γ_α;
(iv) For $\beta \in S_\alpha$ and $\gamma \in \Gamma$, $\gamma(\beta) \in S_\alpha$ implies $\gamma \in \Gamma_\alpha$, i.e. Γ_α is the maximal subgroup which leaves S_α globally invariant; this also implies $\Gamma_\beta \subseteq \Gamma_\alpha$.

For later reference, we rewrite (i) as

$$\mathrm{T}_\alpha \mathcal{A} = \mathrm{T}_\alpha S_\alpha \oplus \mathrm{T}_\alpha \omega_\alpha \ ; \tag{14}$$

we also denote, again for later reference,

$$S_\alpha^0 \ = \ S_\alpha \cap \mathcal{F}(\alpha) \ . \tag{15}$$

It follows from properties (iii) and (iv) of S_α, and the compactness of Γ_α, that

$$S_\alpha^0 \ = \ \Sigma(\alpha) \cap S_\alpha \ , \tag{16}$$

$$\Sigma(\alpha) \cap \mathcal{U}_\alpha \ = \ \bigcup_{\beta \in \omega_\alpha} S_\beta^0 \ ; \tag{17}$$

hence, from (10),(14) and (17), we have that

$$\mathrm{T}_\alpha \Sigma(\alpha) \ = \ \mathrm{T}_\alpha \omega_\alpha \oplus \mathrm{T}_\alpha S_\alpha^0 \ . \tag{18}$$

Let us now consider a Γ-invariant functional $f : \mathcal{A} \to R$ of class C^1 (a special case of this is the Yang-Mills functional), i.e. a functional such that

$$f\left(\gamma(\alpha)\right) \ = \ f(\alpha) \qquad \forall \gamma \in \Gamma \ , \ \forall \alpha \in \mathcal{A} \ ; \tag{19}$$

its differential at α, $\mathrm{d}f_\alpha : \mathrm{T}_\alpha \mathcal{A} \to R$, will be a linear and continuous operator; as $\mathcal{A}$ is a Hilbert space, this will correspond to an element $\phi_\alpha \in \mathrm{T}_\alpha \mathcal{A} \simeq \mathcal{A}$, such that $\mathrm{d}f_\alpha(\beta) = (\phi_\alpha, \beta)$.

It is easy to see that the invariance of f implies

$$\phi_{\gamma(\alpha)} \ = \ \gamma(\phi_\alpha) \ . \tag{20}$$

This also implies that if $\gamma \in \Gamma_\alpha$, then $\mathrm{d}f_\alpha$ is invariant under γ; that is,

$$\phi_\alpha \ \in \ \mathrm{T}_\alpha \mathcal{F}(\alpha) \ \ . \tag{21}$$

On the other side, it is clear from (19) that $(\phi_\alpha, \xi) = 0$ for all $\xi \in \mathrm{T}_\alpha \omega_\alpha$; thus we conclude [see (14),(18)] that

$$\phi_\alpha \ \in \ \mathrm{T}_\alpha S_\alpha^0 \ \ . \tag{22}$$

It is clear from (19) that if $\alpha \in \mathcal{A}$ is a critical point for f, all the points $\beta \in \Gamma(\alpha)$ are also critical for f[1]; we introduce the

[1] It is then natural to consider as index of α its index as a critical point of the restriction of df_α to the slice S_α, or equivalently to S_α^0.

Definition: The Γ-orbit $\omega \subset \mathcal{A}$ is *critical* if points on ω are critical for any Γ-invariant functional.

The above discussion shows that if an orbit is isolated in its stratum, then it is critical.

We could also prove the converse, i.e. that if an orbit is critical, then it is isolated in its stratum; the proof of this would be just the same as the one given by Michel [1] for compact groups and finite dimensional manifolds, and thus is omitted.

¿From the above discussion we conclude that for functionals defined on the space of connections of a principal bundle (in Physics' language, pure gauge theories) we have – in the framework, with the definitions, and under the conditions introduced so far – the following extension of Michel's theorem:

Theorem. *A gauge orbit ω is critical if and only if it is isolated in its stratum.*

In the classical case (compact group acting on a finite dimensional manifold), it is well known that the Michel's theory and its symmetry-based approach can be extended to consider general equivariant dynamics rather than just variational problems; in this way one can obtain[2] [11] in particular the Equivariant Branching Lemma (EBL) [12] (and its extension to the Hopf case) and the Reduction Lemma (RL) [13]; most of the results in equivariant bifurcation theory are based on these lemmata [13].

The same holds here, i.e. one could obtain the corresponding results for equivariant vector fields on $\mathcal{A}$. Indeed, in this case equivariance means that

$$X(\gamma(\alpha)) \;=\; \gamma^* [X(\alpha)] \qquad \forall\gamma \in \Gamma,\ \forall\alpha \in \mathcal{A}\ , \tag{23}$$

where γ^* denote the action of Γ on $\mathrm{T}\mathcal{A}$ induced by the action of Γ on $\mathcal{A}$. Thus, for $\gamma \in \Gamma_\alpha$ and X an equivariant vector field on $\mathcal{A}$, we have

$$\gamma^* [X(\alpha)] = X(\gamma(\alpha)) = X(\alpha)\ . \tag{24}$$

We have thus proven the lemma below, from which the EBL and the RL can be obtained in the same way as for the finite dimensional case:

Lemma. *Let $X : \mathcal{A} \to \mathrm{T}\mathcal{A}$ be a vector field on $\mathcal{A}$, equivariant under the gauge group Γ; then, $X(\alpha) \in \mathrm{T}_\alpha \mathcal{F}(\alpha)$.*

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[2] These results were originally obtained with no use of Michel's theory.

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